

Research Article

Bifurcations of a Homoclinic Orbit to Saddle-Center in Reversible Systems

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The bifurcations near a primary homoclinic orbit to a saddle-center are investigated in a 4-dimensional reversible system. By establishing a new kind of local moving frame along the primary homoclinic orbit and using the Melnikov functions, the existence and nonexistence of 1-homoclinic orbit and 1-periodic orbit, including symmetric 1-homoclinic orbit and 1-periodic orbit, and their corresponding codimension 1 or codimension 3 surfaces, are obtained.

1. Introduction

In recent years, there are much interest in the phenomenon of homoclinics and heteroclinics in reversible dynamical systems because of their extensive applications in mechanics, fluids, and optics [1–10]. For example, Klaus and Knobloch [4] considered a homoclinic orbit to saddle-center with two-parameter families of non-Hamiltonian reversible vector fields by Lin's method. They derived the occurrence of 1-homoclinic orbits to the center manifold. Liu et al. [6] studied a singular perturbation system with action-angle variable and the unperturbed system was assumed to possess a saddle-center equilibrium in a general system without reversible or Hamiltonian structure. Mielke et al. [7] investigated bifurcations of homoclinic orbit to saddle-center in 4-dimensional reversible Hamiltonian systems. By using the Poincaré map and a special normal form, they detected the existence of N -homoclinic orbits to the equilibrium, N -periodic orbits, and chaotic behavior near the primary homoclinic orbit. As for purely Hamiltonian system in \mathbb{R}^4 , similar results are known from Koltsova and Lerman [2, 5]. In all of these papers the underlying Hamiltonian structure was heavily considered. Especially it was used to detect multiround orbits. Note that, the Hamiltonian can make the dynamics constrict in three-dimensional manifold in a zero level

set. In this paper, we use the method originated by Zhu [9], by constructing a local moving coordinate system and Poincaré map near the primary homoclinic orbit, the existence of transversal homoclinic orbits and periodic orbits bifurcated from the primary homoclinic orbit are obtained in a 4-dimensional reversible system. It is worth to mention that a new kind of moving coordinates is introduced firstly in our paper in order to simplify and facilitate the reversible system.

The remainder of this paper is organized as follows. Section 2 contains the assumptions for the perturbed and unperturbed system. The local coordinate moving frame, cross sections, and Poincaré map are set up in Section 3. Finally, we obtain the existence and nonexistence of 1-homoclinic orbit and 1-periodic orbit, including symmetric 1-homoclinic orbit and 1-periodic orbit, and their corresponding surfaces with different condimensions in Sections 4 and 5.

2. The General Setup

Consider the following system

$$\dot{z} = f(z) + \epsilon g(z; \mu), \quad z \in \mathbb{R}^4, \quad \mu \in \mathbb{R}^l \quad (2.1)$$

and the corresponding unperturbed system

$$\dot{z} = f(z), \quad (2.2)$$

where $0 < \epsilon \ll 1$, $l \geq 1$, $f : \mathbb{R}^4 \rightarrow \mathbb{R}^4$ and $g : \mathbb{R}^4 \times \mathbb{R} \rightarrow \mathbb{R}^4$ are C^r ($r \geq 4$), $f(0) = g(0; \mu) = 0$ and μ is a parameter. Also, we need the following assumptions.

(A1) System (2.1) is reversible with respect to the linear involution R such that $\dim(\text{Fix } R) = 2$, $f(Rz) + Rf(z) = g(Rz; \mu) + Rg(z; \mu) = 0$ for all $z \in \mathbb{R}^4$ and $\mu \in \mathbb{R}^l$.

Note that, throughout the paper, we will denote R -symmetric orbit as symmetric orbit.

(A2) The origin O is a saddle-center equilibrium of (2.2), $O \in \text{Fix } R$. More precisely, the Jacobian matrix $A \triangleq Df(0)$ has a pair of purely imaginary eigenvalues $\pm i\omega$ and two nonzero real eigenvalues, that is, $\sigma(A) = \{\pm i\omega, \pm\lambda\}$ with $\omega, \lambda > 0$.

(A3) System (2.2) has a symmetric homoclinic orbit $\Gamma = \{z = r(t) : t \in \mathbb{R}\}$, where $r(\pm\infty) = O$.

Note that, O is a symmetric equilibrium, and the eigenvalues of the Jacobian matrix A are symmetric with respect to the imaginary axis. Thus, assumption (A2) describes a scenario that is structurally stable. Furthermore, the saddle-center O has one-dimensional stable manifold $W_{\epsilon, \mu}^s(O)$ and one-dimensional unstable manifolds $W_{\epsilon, \mu}^u(O)$ (abbr. W^s and W^u as $\epsilon = 0$), and a two-dimensional center manifold $W_{\epsilon, \mu}^c(O)$ (abbr. W^c as $\epsilon = 0$) for ϵ close to 0. All of them are C^r . Confined to $W_{\epsilon, \mu}^c(O)$, O is a center. The reversibility implies that $W_{\epsilon, \mu}^u(O) = RW_{\epsilon, \mu}^s(O)$ and $W_{\epsilon, \mu}^c(O) = RW_{\epsilon, \mu}^c(O)$, and hence, the homoclinic orbit Γ is symmetric, that is, $R\Gamma = \Gamma$.

3. Local Moving Frame and Poincaré Map

Suppose the neighborhood U_0 of O is small enough, we can firstly straighten the center-stable manifold, the center-unstable manifold, subsequently, then the stable manifold and

the unstable manifold in U_0 by using the method introduced in Zhu [9]. According to the invariance and symmetry of these manifolds, we can deduce that system (2.1) has the following form in U_0 :

$$\begin{aligned} \dot{x} &= x(\lambda(\epsilon, \mu) + O(1)), \\ \dot{y} &= y(-\lambda(\epsilon, \mu) + O(1)), \\ \dot{u} &= v(\omega(\epsilon, \mu) + O(1)) + u(O(x) + O(y) + O(u)) + O(xy), \\ \dot{v} &= u(-\omega(\epsilon, \mu) + O(1)) - v(O(x) + O(y) + O(v)) - O(xy), \end{aligned} \tag{3.1}$$

where $\lambda(0, \mu) = \lambda$, $\omega(0, \mu) = \omega$, $O(1) = O(x) + O(y) + O(u) + O(v)$, the system is C^{r-2} , and the corresponding involution acts as $R(x, y, u, v) = (y, x, v, u)$.

In fact, by a linear transformation, system (2.1) takes the form in a small neighborhood of U_0 as follows:

$$\begin{aligned} \dot{x} &= \lambda(\epsilon, \mu)x + O(2), \\ \dot{y} &= -\lambda(\epsilon, \mu)y + O(2), \\ \dot{u} &= \omega(\epsilon, \mu)v + O(2), \\ \dot{v} &= -\omega(\epsilon, \mu)u + O(2), \end{aligned} \tag{3.2}$$

and $R(x, y, u, v) = (y, x, v, u)$. By the invariant manifold theorem, we know that there exist a local C^r center-stable manifold $W_{\epsilon, \mu}^{cs} = \{z = (x, y, u, v) : x = x_{\epsilon, \mu}^{cs}(y, u, v), x_{\epsilon, \mu}^{cs}(0, 0, 0) = 0, Dx_{\epsilon, \mu}^{cs}(0, 0, 0) = 0, z \in U_0\}$, a local C^r center-unstable manifold $W_{\epsilon, \mu}^{cu} = \{z = (x, y, u, v) : y = y_{\epsilon, \mu}^{cu}(x, u, v), y_{\epsilon, \mu}^{cu}(0, 0, 0) = 0, Dy_{\epsilon, \mu}^{cu}(0, 0, 0) = 0, z \in U_0\}$ and $RW_{\epsilon, \mu}^{cs} = W_{\epsilon, \mu}^{cu}$.

By the straightening coordinate transformation which is similar to that of [1, 6, 9], now we straighten the local manifolds $W_{\epsilon, \mu}^{cs}$ and $W_{\epsilon, \mu}^{cu}$ such that $W_{\epsilon, \mu}^{cs} = \{z \in U_0 : x = 0\}$, $W_{\epsilon, \mu}^{cu} = \{z \in U_0 : y = 0\}$.

Notice that the invariance of $W_{\epsilon, \mu}^{cs}$ and $W_{\epsilon, \mu}^{cu}$ implies the local invariance of $\{z \in U_0 : x = 0\}$ and $\{z \in U_0 : y = 0\}$, respectively, which produces that, in U_0 ,

$$\begin{aligned} \dot{x} &= x(\lambda(\epsilon, \mu) + O(1)), \\ \dot{y} &= y(-\lambda(\epsilon, \mu) + O(1)). \end{aligned} \tag{3.3}$$

Now the system is C^{r-1} and still reversible. By using a similar procedure to straighten the local C^{r-1} stable manifold $W_{\epsilon, \mu}^s$ and unstable manifold $W_{\epsilon, \mu}^u$, and the invariance and symmetry of these two local manifolds (that means the transformation is also symmetric), we get system (3.1). Clearly, corresponding to system (3.1), the center manifold $W_{\epsilon, \mu}^c$ is locally in the u - v plane, and the stable manifold $W_{\epsilon, \mu}^s$ (resp., unstable manifold $W_{\epsilon, \mu}^u$) is locally the y -axis (resp., x -axis) when they are confined in U_0 .

Define $r(-T) = (\delta, 0, 0, 0)^*$, $r(T) = (0, \delta, 0, 0)^*$ for $T \gg 1$, where $\delta > 0$ is small enough such that $\{(x, y, u, v)^* : |x|, |y|, |u^2 + v^2|^{1/2} < 2\delta\} \subset U_0$.

Consider the linear variational system

$$\dot{z} = Df(r(t))z, \quad (3.4)$$

and its adjoint system

$$\dot{\phi} = -(Df(r(t)))^* \phi. \quad (3.5)$$

Based on the invariance and symmetry of manifolds W^{cs} and W^{cu} , it is easy to know that system (3.4) has a fundamental solution matrix $Z(t) = (z_1(t), z_2(t), z_3(t), z_4(t))$ satisfying $z_1(t) \in (T_{r(t)}W^{cu})^c$, $z_2(t) = -\dot{r}(t)/|\dot{r}(T)| \in T_{r(t)}(W^s \cap W^u)$, $z_3(t), z_4(t) \in T_{r(t)}W^{cu}$ and

$$Z(-T) = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad Z(T) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \quad (3.6)$$

Actually, in the resulting coordinates, $W^s \cap U_0$ and $W^u \cap U_0$ are y -axis and x -axis, respectively, combining with the symmetry, it follows that

$$z_2(T) = (0, 1, 0, 0)^*, \quad z_2(-T) = (-1, 0, 0, 0)^*. \quad (3.7)$$

On the other hand, in a small tubular neighborhood of the homoclinic loop Γ , the center-unstable manifold W^{cu} (resp., center-stable manifold W^{cs}) can be foliated into a family of leaves, each is a 2-dimensional surface and asymptotic to W^c as the base point $z = r(t) \in \Gamma$ tends to O as $t \rightarrow -\infty$ (resp., $+\infty$). Notice that the limit of the linearization (3.4) of system (2.2) with respect to Γ as $t \rightarrow \pm\infty$ is

$$\dot{x} = \lambda x, \quad \dot{y} = -\lambda y, \quad \dot{u} = \omega v, \quad \dot{v} = -\omega u. \quad (3.8)$$

When confined on W^c , it becomes the following subsystem

$$\dot{u} = \omega v, \quad \dot{v} = -\omega u, \quad (3.9)$$

which is reversible with the involution $(u, v) \rightarrow (v, u)$. Obviously, for any $a > 0$ and $t_0 \in \mathbb{R}$, $u = u(t) = a \sin \omega(t - t_0)$, $v = v(t) = a \cos \omega(t - t_0)$ is a solution of (3.9), which defines a closed orbit $\hat{\Gamma}$ on W^c . More precisely, its tangent vectors $u_1(t) = \dot{u}(t) = a\omega \cos \omega(t - t_0)$, $v_1(t) = \dot{v}(t) = -a\omega \sin \omega(t - t_0)$ and its normal vectors $u_2(t) = -\dot{v}(t) = a\omega \sin \omega(t - t_0)$, $v_2(t) = \dot{u}(t) = a\omega \cos \omega(t - t_0)$ are the solutions of (3.9). Choose some appropriate a and t_0 such that $a\omega = 1$, $\omega(-T - t_0) = 2k\pi$ for some $k \in \mathbb{Z}$, then $(u(-T), v(-T)) = (0, a) = (0, \omega^{-1})$, $\bar{z}_3(-T) = (u_1(-T), v_1(-T)) = (1, 0)$, $\bar{z}_4(-T) = (u_2(-T), v_2(-T)) = (0, 1)$. Based on the reversibility, we have $(u(T), v(T)) = (a, 0) = (\omega^{-1}, 0)$, $\bar{z}_3(T) = (u_1(T), v_1(T)) = (0, -1)$, $\bar{z}_4(T) = (u_2(T), v_2(T)) = (1, 0)$ (see Figure 1 for details). Thus, if we take solutions $z_3(t)$ and $z_4(t)$ in $T_{r(t)}W^{cu}$ satisfying $z_3(-T) = (0, 0, 1, 0)^*$, $z_4(-T) = (0, 0, 0, 1)^*$, then, restricted

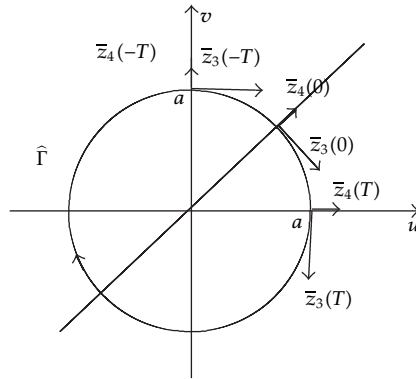


Figure 1: The geometry of vectors on the center manifold.

to the u - v plane, $z_3(-T)$ is the unit tangent direction of the closed orbit $\hat{\Gamma}$ at $(u, v) = (0, a) = (0, \omega^{-1})$. In addition, the restriction of $z_4(-T)$ is its unit exterior normal direction at the same point. By the reversibility, it is easy to obtain $z_3(T) = (0, 0, 0, -1)^*$, $z_4(T) = (0, 0, 1, 0)^*$.

Finally, if we choose a solution $z_1(t) \in (T_{r(t)}W^{cu})^c$ with $z_1(-T) = (0, 1, 0, 0)^*$, then the symmetry says that $z_1(T) = (1, 0, 0, 0)^*$.

Therefore, we have demonstrated the existence of the fundamental matrix $Z(t)$ with the specified properties.

Remark 3.1. In the following, we will regard $(z_1(t), z_2(t), z_3(t), z_4(t))$ as a moving coordinate in a small tubular neighborhood of Γ . This new kind of moving frame is firstly introduced for the homoclinic orbit to a saddle-center, which is the extension of the corresponding coordinates built in [1, 6, 9] for the homoclinic orbit to a saddle. The explicit advantage is that, these coordinate vectors inherit and exhibit the geometrical and dynamical properties of those invariant manifolds. As mentioned above, they will greatly simplify the original reversible system.

Let $\Sigma^u = \{(x, y, u, v) \mid x = \delta, |y|, |u^2 + v^2|^{1/2} \leq \delta/2\} \subset U_0$ and $\Sigma^s = \{(x, y, u, v) \mid y = \delta, |x|, |u^2 + v^2|^{1/2} \leq \delta/2\} \subset U_0$ be the cross sections of Γ at $t = -T$ and $t = T$, respectively.

Now we turn to seek the new coordinates of $q_0 \in \Sigma^u$ and $q_1 \in \Sigma^s$ (see Figure 2 for details) under the transformation $z(t) = r(t) + Z(t) \cdot N$, where $N = (n_1, 0, n_3, n_4)^*$. Take

$$\begin{aligned} q_0 &\triangleq (x_0, y_0, u_0, v_0) = r(-T) + Z(-T) \cdot (n_1^0, 0, n_3^0, n_4^0)^*, \\ q_1 &\triangleq (x_1, y_1, u_1, v_1) = r(T) + Z(T) \cdot (n_1^1, 0, n_3^1, n_4^1)^*, \end{aligned} \tag{3.10}$$

which are solved by

$$\begin{aligned} x_0 &= \delta, & n_1^0 &= y_0, & n_3^0 &= u_0, & n_4^0 &= v_0; \\ y_1 &= \delta, & n_1^1 &= x_1, & n_3^1 &= -v_1, & n_4^1 &= u_1. \end{aligned} \tag{3.11}$$

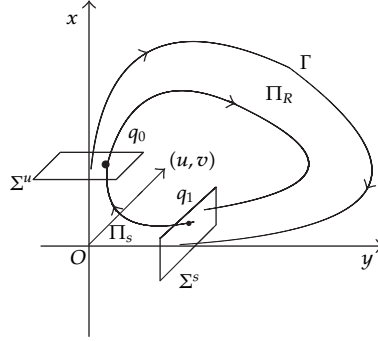


Figure 2: Setup of cross sections and Poincaré map.

Putting the transformation $z(t) = r(t) + Z(t) \cdot N$ into (2.1), we get

$$\begin{aligned} \dot{r}(t) + \dot{Z}(t) \cdot N + Z(t) \cdot \dot{N} &= f(r(t) + Z(t) \cdot N) + \varepsilon g(r(t) + Z(t) \cdot N, \mu) \\ &= f(r(t)) + Df(r(t))Z(t) \cdot N + \varepsilon g(r(t), \mu) + O((\varepsilon + N)^2). \end{aligned} \quad (3.12)$$

That implies $Z(t) \cdot \dot{N} = \varepsilon g(r(t), \mu) + O((\varepsilon + N)^2)$, which is C^{r-3} . Then multiplying the resulting equation by $Z^{-1}(t) = (\phi_1^*, \phi_2^*, \phi_3^*, \phi_4^*)^*$ and integrating it on both sides from $-T$ to T , we have the regular map $\Pi_R : \Sigma^u \rightarrow \Sigma^s$:

$$n_i(T) = n_i(-T) + \varepsilon M_i(T, \mu) + O((\varepsilon + N(-T))^2), \quad i = 1, 3, 4, \quad (3.13)$$

where $M_i(T, \mu) = \int_{-T}^T \phi_i^* g_i(r(t), \mu) dt$ are the Melnikov functions.

For conciseness, we will denote $\lambda(\varepsilon, \mu) \triangleq \lambda, \omega(\varepsilon, \mu) \triangleq \omega$. Now we consider the local map Π_S induced by the flow (3.1) in U_0 , where

$$\Pi_S : \Sigma^s \rightarrow \Sigma^u, \quad q_1 = (n_1^1, 0, n_3^1, n_4^1) \rightarrow q_0 = (n_1^0, 0, n_3^0, n_4^0). \quad (3.14)$$

Let τ be the flying time from q_1 to q_0 , by variation of constants formula, we can get the following expression:

$$\begin{aligned} x_1 &= x(T) = x_0 e^{-\lambda \tau} (1 + \rho_1), \\ y_0 &= y(T + \tau) = y_1 e^{-\lambda \tau} (1 + \rho_2), \\ u_0 &= u(T + \tau) = [u_1 \cos(\omega \tau) + v_1 \sin(\omega \tau)] (1 + \rho_3) + \rho_5, \\ v_0 &= v(T + \tau) = [-u_1 \sin(\omega \tau) + v_1 \cos(\omega \tau)] (1 + \rho_4) + \rho_6, \end{aligned} \quad (3.15)$$

where $\rho_i = \rho_i(x_0, y_1, u_1, v_1, \tau, \varepsilon, \mu)$, $\rho_i \in C^0$, $i = 1, \dots, 6$, and $\rho_i = O(\delta)$ for $i = 1, 2, 3, 4$, $\rho_j = O(\delta^2 e^{-\lambda \tau})$ for $j = 5, 6$.

Denote $s = e^{-\lambda\tau}$, then we get $\Pi_S : \Sigma^s \rightarrow \Sigma^u$ defined by

$$\begin{aligned} n_1^0 &= \delta s(1 + \rho_2), \\ n_3^0 &= \left[n_4^1 \cos\left(\frac{\omega}{\lambda} \log s\right) + n_3^1 \sin\left(\frac{\omega}{\lambda} \log s\right) \right] (1 + \rho_3) + \rho_5, \\ n_4^0 &= \left[n_4^1 \sin\left(\frac{\omega}{\lambda} \log s\right) - n_3^1 \cos\left(\frac{\omega}{\lambda} \log s\right) \right] (1 + \rho_4) + \rho_6. \end{aligned} \quad (3.16)$$

Combining the maps (3.13) and (3.16), we obtain the Poincaré map $\Pi = \Pi_R \circ \Pi_S : \Sigma^s \rightarrow \Sigma^s$ defined as

$$\begin{aligned} n_1(T) &= \delta s(1 + \rho_2) + \epsilon M_1(T, \mu) + h. o. t., \\ n_3(T) &= \left[n_4^1 \cos\left(\frac{\omega}{\lambda} \log s\right) + n_3^1 \sin\left(\frac{\omega}{\lambda} \log s\right) \right] (1 + \rho_3) + \epsilon M_3(T, \mu) + \rho_5 + h. o. t., \\ n_4(T) &= \left[n_4^1 \sin\left(\frac{\omega}{\lambda} \log s\right) - n_3^1 \cos\left(\frac{\omega}{\lambda} \log s\right) \right] (1 + \rho_4) + \epsilon M_4(T, \mu) + \rho_6 + h. o. t. \end{aligned} \quad (3.17)$$

4. Existence of 1-Homoclinic Orbit and 1-Periodic Orbit

Let $G(q_1) \triangleq \Pi(q_1) - q_1$ be the displacement function. Based on (3.11), (3.17), and the new coordinate of $q_1 = (n_1^1, 0, n_3^1, n_4^1)$, we see the small zero point (s, u_1, v_1) of G will satisfy the following equations:

$$\begin{aligned} \delta s(\rho_2 - \rho_1) + \epsilon M_1(T, \mu) + O(|s, \epsilon|^2) &= 0, \\ v_1 + \left[u_1 \cos\left(\frac{\omega}{\lambda} \log s\right) - v_1 \sin\left(\frac{\omega}{\lambda} \log s\right) \right] (1 + \rho_3) + \epsilon M_3(T, \mu) + \rho_5 + h. o. t. &= 0, \\ u_1 - \left[u_1 \sin\left(\frac{\omega}{\lambda} \log s\right) + v_1 \cos\left(\frac{\omega}{\lambda} \log s\right) \right] (1 + \rho_4) - \epsilon M_4(T, \mu) - \rho_6 + h. o. t. &= 0. \end{aligned} \quad (4.1)$$

Due to the coordinate transformations introduced in U_0 at the beginning of Section 3, the unstable manifold and the stable manifold are locally x -axis and y -axis, respectively, so it is evident that, near Γ , system (2.1) has a symmetric 1-homoclinic orbit to O if and only if (4.1) have a solution (s, u_1, v_1) with $s = u_1 = v_1 = 0$, and system (2.1) has an 1-homoclinic orbit to a periodic orbit on the center manifold and an 1-periodic orbit if and only if (4.1) have a solution (s, u_1, v_1) with $s = 0, u_1^2 + v_1^2 > 0$ and $s > 0, u_1^2 + v_1^2 \geq 0$, respectively. Clearly, system (4.1) is C^1 in (s, u_1, v_1) as $0 \leq s \ll 1, u_1^2 + v_1^2 \ll 1$, and C^3 in (ϵ, μ) , and has a solution $s = u_1 = v_1 = 0$ as $\epsilon = 0$, thus we can rescale $s = \epsilon^2 \bar{s}, u_1 = \epsilon^2 u, v_1 = \epsilon^2 v$, such that system (4.1) is reformulated as

$$\begin{aligned} M_1(T, \mu) + O(\epsilon) &= 0, \\ M_3(T, \mu) + O(\epsilon) &= 0, \\ M_4(T, \mu) + O(\epsilon) &= 0, \end{aligned} \quad (4.2)$$

where $O(\epsilon)$ depends on ϵ, \bar{s}, u, v and μ .

Now the following results are verified directly by the implicit function theorem.

Theorem 4.1. *Assume (A1)–(A3) are satisfied, $M_1(T, \mu_0) = M_3(T, \mu_0) = M_4(T, \mu_0) = 0$ and $\text{Rank}(\partial(M_1, M_3, M_4)/\partial\mu)(T, \mu_0) = 3$ for $l \geq 3$ and $\mu_0 \in \mathbb{R}^l$, then, for any given $r_0 > 0$, $\bar{s}_0 > 0$ and $\epsilon_0 > 0$ small enough, there exist a single-parameter family of codimension 3 surface Σ_ϵ and a four-parameter family of codimension 3 surface $\Sigma_{\epsilon uv\bar{s}}$ near μ_0 such that $\Sigma_0 = \Sigma_{0uv\bar{s}} = \{\mu \mid \mu = \mu_0\}$, $\lim_{\bar{s} \rightarrow 0} \Sigma_{\epsilon uv\bar{s}} = \Sigma_{\epsilon uv0}$ for $0 < \epsilon < \epsilon_0$, $0 \leq u^2 + v^2 < r_0^2$, $0 < \bar{s} < \bar{s}_0$, and system (2.1) has a symmetric 1-homoclinic orbit to O as $\mu \in \Sigma_\epsilon$, a 1-periodic orbit with approximate period $2T - \lambda^{-1} \log e^2 \bar{s}$ as $\mu \in \Sigma_{\epsilon uv\bar{s}}$, and a 1-homoclinic orbit to a periodic orbit on the center manifold with $u_1^2(t) + v_1^2(t) = \epsilon^4(u_0^2 + v_0^2)$ as $\mu \in \Sigma_{\epsilon u_0 v_0 0}$, $\Sigma_{\epsilon u_0 v_0 0} - \{\mu_0\} \neq \emptyset$, $0 < u_0^2 + v_0^2 < r_0^2$. Moreover, these codimension 3 surfaces are either coincident, or tangent to each other at $\mu = \mu_0$, and they have a common 3-dimensional normal space spanned by $M_{1\mu}(T, \mu_0)$, $M_{3\mu}(T, \mu_0)$, and $M_{4\mu}(T, \mu_0)$ as $l > 3$.*

In order to study the existence of the 1-periodic orbit bifurcation surface with lower codimension, we reconsider (4.1). Let $s = e^{-(\lambda/\omega)(2k\pi - \alpha)}$, $\alpha \in [-(\pi/2), (3\pi/2))$ and H be the vector defined by the left hand side of the last two equations of (4.1), then, by using

$$W \triangleq \det \left[\frac{\partial H}{\partial(u_1, v_1)} \Big|_{\substack{u_1=v_1=\epsilon=0 \\ k=\infty}} \right] \tag{4.3}$$

$$= -[2 + \rho_3 + \rho_4 + \rho_3\rho_4 - (2 + \rho_3 + \rho_4) \sin \alpha] \Big|_{\substack{u_1=v_1=\epsilon=0 \\ k=\infty}}$$

it follows that, for $\delta > 0$ so small that $|\rho_3 + \rho_4| < 0.1$, $|\rho_3 + \rho_4 + \rho_3\rho_4| < 0.1$, we have $W \neq 0$ for $\alpha \in [-(\pi/4), (5\pi/4)]$. Therefore, there is a unique pair (u_1, v_1) which solves the last two equations as $k > -(\omega/\lambda\pi) \log \epsilon$ and $\epsilon > 0$ is small enough.

Substituting this solution (u_1, v_1) into the first equation of (4.1), and using $s = O(\epsilon^2)$ for $k > -(\omega/\lambda\pi) \log \epsilon$, it produces

$$M_1(T, \mu) + O(\epsilon) = 0, \tag{4.4}$$

where the term $O(\epsilon)$ depends on ϵ, k, α and μ .

At this stage, we have actually demonstrated the following existence theorem.

Theorem 4.2. *Suppose (A1)–(A3) are satisfied, $M_1(T, \mu_0) = 0$, $(\partial M_1/\partial\mu)(T, \mu_0) \neq 0$ as $l \geq 1$ and $\mu_0 \in \mathbb{R}^l$, then for $\epsilon_0 > 0$, $\delta > 0$ small enough, $0 < \epsilon < \epsilon_0$, $k > -(\omega/\lambda\pi) \log \epsilon$ and $\alpha \in [-(\pi/4), (5\pi/4)]$, there exists a 3-parameter (resp., 2-parameter) family of codimension 1 surface $\Sigma_{\epsilon k\alpha}$ (resp., $\Sigma_{\epsilon\alpha} = \Sigma_{\epsilon\infty\alpha}$) near μ_0 satisfying $\Sigma_{0\infty\alpha} = \{\mu_0\}$, such that system (2.1) has one 1-periodic orbit $L(\epsilon, k, \alpha)$ (resp., 1-homoclinic orbit $L(\epsilon, \infty, \alpha)$) near Γ as $\mu \in \Sigma_{\epsilon k\alpha}$ (resp., $\mu \in \Sigma_{\epsilon\alpha}$), and $L(\epsilon, k, \alpha)$ has an approximate period $2T + ((2k\pi - \alpha)/\omega)$. In addition, if $M_1(T, \mu) \neq 0$ for any $\mu \in \mathbb{R}^l$, then system (2.1) has not any 1-periodic orbit near Γ .*

Remark 4.3. When confined in a neighborhood of the local center-manifold, the (u, v) -coordinates of 1-periodic or 1-homoclinic orbit guaranteed by Theorem 4.1 as $\mu \in \Sigma_{\epsilon uv\bar{s}}$ or $\mu \in \Sigma_{\epsilon u_0 v_0 0}$ satisfying $u_1^2 + v_1^2 = O(\epsilon^4)$, whereas, these coordinates of the corresponding orbits given in Theorem 4.2 as $\mu \in \Sigma_{\epsilon k\alpha}$ or $\mu \in \Sigma_{\epsilon\alpha}$ have the scale $O(\epsilon^2)$ in case $M_4^2(T, \mu_0) + M_3^2(T, \mu_0) \neq 0$ and $k > -(\omega/\lambda\pi) \log \epsilon$.

5. Existence of Symmetric 1-Periodic Orbit

In this section, we turn to seek the existence of symmetric 1-periodic orbit. Note that, a 1-periodic orbit L near Γ is symmetric if and only if $R(L \cap \Sigma^s) = L \cap \Sigma^u$, equivalently $L \cap \Sigma^s = (x_1, \delta, u_1, v_1)$ and $L \cap \Sigma^u = (\delta, y_0, u_0, v_0)$ should satisfy

$$x_1 = y_0, \quad u_1 = v_0, \quad v_1 = u_0. \quad (5.1)$$

Due to (3.15), it is equivalent to

$$\begin{aligned} \rho_1 - \rho_2 &= 0, \\ u_1 - \left[u_1 \sin\left(\frac{\omega}{\lambda} \log s\right) + v_1 \cos\left(\frac{\omega}{\lambda} \log s\right) \right] (1 + \rho_4) - \rho_6 &= 0, \\ v_1 - \left[u_1 \cos\left(\frac{\omega}{\lambda} \log s\right) - v_1 \sin\left(\frac{\omega}{\lambda} \log s\right) \right] (1 + \rho_3) - \rho_5 &= 0. \end{aligned} \quad (5.2)$$

Using (5.2) and rescaling $u_1 = \epsilon u$, $v_1 = \epsilon v$, $s = e^{-(\lambda/\omega)(4k\pi+2\theta)} = O(\epsilon^2)$ for $k > -(\omega/2\lambda\pi) \log \epsilon$, $\theta \in [0, 2\pi)$, system (4.1) reads as

$$\begin{aligned} M_1(T, \mu) + O(\epsilon) &= 0, \\ 2v + M_3(T, \mu) + O(\epsilon) &= 0, \\ M_4(T, \mu) + O(\epsilon) &= 0. \end{aligned} \quad (5.3)$$

Solving the second equation of (5.3), we have

$$v = -\frac{1}{2}M_3(T, \mu) + O(\epsilon). \quad (5.4)$$

On the other hand, $L \cap \Sigma^s$ and $L \cap \Sigma^u$ are symmetric if and only if their middle point $(x(T + (1/2)\tau), y(T + (1/2)\tau), u(T + (1/2)\tau), v(T + (1/2)\tau))$ lies on the plane $\text{Fix}(R)$, that is, (5.1) is equivalent to

$$x\left(T + \frac{1}{2}\tau\right) = y\left(T + \frac{1}{2}\tau\right), \quad u\left(T + \frac{1}{2}\tau\right) = v\left(T + \frac{1}{2}\tau\right), \quad (5.5)$$

where

$$\begin{aligned}x\left(T + \frac{1}{2}\tau\right) &= \delta\sqrt{s}(1 + \rho_{11}), \\y\left(T + \frac{1}{2}\tau\right) &= \delta\sqrt{s}(1 + \rho_{21}), \\u\left(T + \frac{1}{2}\tau\right) &= \left[u_1 \cos\left(\frac{\omega\tau}{2}\right) + v_1 \sin\left(\frac{\omega\tau}{2}\right)\right](1 + \rho_{31}) + \rho_{51}, \\v\left(T + \frac{1}{2}\tau\right) &= \left[-u_1 \sin\left(\frac{\omega\tau}{2}\right) + v_1 \cos\left(\frac{\omega\tau}{2}\right)\right](1 + \rho_{41}) + \rho_{61},\end{aligned}\tag{5.6}$$

and $\rho_{i1} = \rho_{i1}(x_1, y_1, u_1, v_1, \tau, \epsilon, \mu)$ for $i = 1, \dots, 6$, $\rho_{i1} = O(x_1/\sqrt{s}) + O_i(y_1) + O(\sqrt{u_1^2 + v_1^2}) = O(\delta)$, $O_i(y_1) = O(y_1)$ for $i = 1, \dots, 4$, $\rho_{j1} = O(x_1 y_1) = O(\delta^2 s)$ for $j = 5, 6$.

Explicitly, (5.5) can be formulated as

$$\begin{aligned}\rho_{11} - \rho_{21} &= 0, \\u_1 \left[\cos\left(\frac{\omega\tau}{2}\right) + \sin\left(\frac{\omega\tau}{2}\right)\right] - v_1 \left[\cos\left(\frac{\omega\tau}{2}\right) - \sin\left(\frac{\omega\tau}{2}\right)\right] + O(\delta) + O(\delta^2 s) &= 0.\end{aligned}\tag{5.7}$$

Due to (5.4) and $(1/2)\omega\tau = 2k\pi + \theta$, $\theta \in [0, 2\pi)$, we can uniquely solve the second equation in (5.7) by

$$u_1 = -\frac{\epsilon(\cos\theta - \sin\theta)}{2(\cos\theta + \sin\theta)} M_3(T, \mu)(1 + O(\delta)) + O(\epsilon^2)\tag{5.8}$$

as $\theta \neq (3/4)\pi, (7/4)\pi$, and δ, ϵ small enough.

Up to now, there are three equations

$$\begin{aligned}\rho_{11} - \rho_{21} &= 0, \\M_1(T, \mu) + O(\epsilon) &= 0, \\M_4(T, \mu) + O(\epsilon) &= 0\end{aligned}\tag{5.9}$$

left which should be fulfilled, where $v_1 = \epsilon v$ and u_1 are given by (5.4) and (5.8).

Applying the implicit function theorem to (5.9), where $O(\epsilon)$ depends on ϵ, θ, k and μ , we derive the following result.

Theorem 5.1. *Suppose that (A1)–(A3) hold, $M_1(T, \mu_0) = 0$, $M_4(T, \mu_0) = 0$ and*

$$\text{Rank} \left\{ \frac{\partial(\rho_{11} - \rho_{21})}{\partial\mu} \Big|_{\substack{\epsilon=0, \mu=\mu_0 \\ k=\infty}}, \frac{\partial(M_1, M_4)}{\partial\mu}(T, \mu_0) \right\} = 3\tag{5.10}$$

as $\mu_0 \in \mathbb{R}^l (l \geq 3)$, there exist $\epsilon_0 > 0$, $\theta_0 > 0$ small enough, and a 3-parameter family of codimension 3 surface $\Sigma_{\epsilon\theta k}$ near μ_0 , such that, system (2.1) has a symmetric 1-periodic orbit L near Γ with approximate period $2T + (1/\omega)(4k\pi + 2\theta)$ as $\mu \in \Sigma_{\epsilon\theta k}$ for ϵ, θ, k satisfying $0 < \epsilon < \epsilon_0$, $\theta \in [0, (3/4)\pi - \theta_0) \cup ((3/4)\pi + \theta_0, (7/4)\pi - \theta_0) \cup ((7/4)\pi + \theta_0, 2\pi)$, $k > -(\omega/2\lambda\pi) \log \epsilon$. Moreover, if there is no $\mu_0 \in \mathbb{R}^l$ satisfying

$$M_1^2(T, \mu_0) + M_4^2(T, \mu_0) + (\rho_{11} - \rho_{21})^2 \Big|_{\substack{\epsilon=0, \mu=\mu_0 \\ k=\infty}} = 0, \quad (5.11)$$

then system (2.1) has no symmetric 1-periodic orbit in the small neighborhood of Γ .

Remark 5.2. Similarly, if we take $s \triangleq e^{-\lambda\tau} = u_1 = v_1 = 0$ in (3.15), then $x_1 = y_0 = u_0 = v_0 = u_1 = v_1 = 0$ (i.e., condition (5.1)), it means that there is a symmetric 1-homoclinic orbit to O if and only if $s = u_1 = v_1 = 0$ in Section 4.

Remark 5.3. In Theorem 4.2 (resp., Theorem 5.1), the geometric meaning of k is that, confined in U_0 , the u - v component of the above 1-periodic orbit bifurcated from Γ makes circle k (resp., $2k$) times around the saddle-center.

Remark 5.4. From $x_1/\sqrt{s} = \delta\sqrt{s}(1 + O(\delta))$, $s = O(\epsilon^2)$ and the constitution of ρ_{11} and ρ_{21} , it is easy to know that the necessary condition for $\rho_{11} = \rho_{21}$ is $O_1(y_1) = O_2(y_1)$.

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