Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 627813, 12 pages
doi:10.1155/2012/627813

Research Article

Long-Time Decay to the Global Solution of the 2D Dissipative Quasigeostrophic Equation

Jamel Benameur and Mongi Blel

Department of Mathematics, College of Science, King Saud University, Riyadh 11451, Saudi Arabia

Correspondence should be addressed to Mongi Blel, mongi_blel@yahoo.fr

Received 10 January 2012; Accepted 8 March 2012

Academic Editor: Muhammad Aslam Noor

We study the behavior at infinity in time of any global solution $\theta \in \mathcal{C}(\mathbb{R}^+, \dot{H}^{2-2\alpha}(\mathbb{R}^2))$ of the surface quasigeostrophic equation with subcritical exponent $2/3 \leq \alpha \leq 1$. We prove that $\lim_{t\to\infty}\|\theta(t)\|_{\dot{H}^{2-2\alpha}} = 0$. Moreover, we prove also the nonhomogeneous version of the previous result, and we prove that if $\theta \in \mathcal{C}(\mathbb{R}^+, \dot{H}^{2-2\alpha}(\mathbb{R}^2))$ is a global solution, then $\lim_{t\to\infty}\|\theta(t)\|_{H^{2-2\alpha}} = 0$.

1. Introduction

We consider the $2D$ dissipative quasi-geostrophic equation with subcritical exponent $1/2 < \alpha \leq 1$,

$$\begin{aligned} \partial_t\theta + (-\Delta)^{\alpha}\theta + (u\cdot\nabla)\theta &= 0 \quad \text{in } \mathbb{R}^+\times\mathbb{R}^2, \\ \theta(0,x) &= \theta^0(x) \quad \text{in } \mathbb{R}^2, \end{aligned} \tag{$\mathcal{S}_\alpha$}$$

where $x \in \mathbb{R}^2$, $t > 0$, $\theta = \theta(x,t)$ is the unknown potential temperature, and $u = (u_1, u_2)$ is the divergence free velocity which is determined by the Riesz transformation of θ in the following way:

$$\begin{aligned} u_1 &= -\mathcal{R}_2\theta = -\partial_2(-\Delta)^{-1/2}\theta, \\ u_2 &= \mathcal{R}_1\theta = \partial_1(-\Delta)^{-1/2}\theta. \end{aligned} \tag{1.1}$$

This equation is a two-dimensional model of the $3D$ incompressible Euler equations, and if $\alpha = 1$, the equation $(\mathcal{S}_1)$ is the $2D$ Navier-Stokes equation. We refer the reader to [1] where the authors explain the physical origin and the signification of the parameters of this equation.

The critical homogeneous Sobolev space of the system $(\mathcal{S}_\alpha)$ is $\dot{H}^{2-2\alpha}(\mathbb{R}^2)$, and we have

$$\left\| \lambda^{2\alpha-1} f(\lambda .) \right\|_{\dot{H}^{2-2\alpha}} = \| f \|_{\dot{H}^{2-2\alpha}}, \quad \forall \lambda > 0. \tag{1.2}$$

The local well-posedness of $(\mathcal{S}_\alpha)$ with $\dot{H}^{2-2\alpha}(\mathbb{R}^2)$ data is established by [2] and [3] separately if $\alpha \in (0, 1/2]$. In [4], Dong and Du study the critical case $\alpha = 1/2$ in the critical space $\dot{H}^1(\mathbb{R}^2)$. They prove the global existence if the initial condition is in the critical space $H^1(\mathbb{R}^2)$.

The global existence when $\alpha \in (1/2, 1]$ is an open problem. We have only the local existence. In this case [5], Niche and Schonbek prove that if the initial data θ^0 is in $L^2(\mathbb{R}^2)$, then the L^2 norm of the solution tends to zero but with no uniform rate, that is, there are solutions with arbitrary slow decay. If $\theta^0 \in L^p(\mathbb{R}^2)$, with $1 \leq p \leq 2$, they obtain a uniform decay rate in L^2. They consider also the solution in other L^q spaces. For the proof of their results, they use the kernel $P_\alpha(t, x)$ associated to the operator $\partial_t + (-\Delta)^\alpha$, and they use the Littlewood-Paley decomposition. Our main result is the following.

Theorem 1.1. *Assume that* $2/3 \leq \alpha \leq 1$.

(i) *If* $\theta \in C(\mathbb{R}^+, \dot{H}^{2-2\alpha}(\mathbb{R}^2))$ *is a global solution of* $(\mathcal{S}_\alpha)$, *then*

$$\lim_{t \to \infty} \|\theta(t)\|_{\dot{H}^{2-2\alpha}} = 0. \tag{1.3}$$

(ii) *If* $\theta \in C(\mathbb{R}^+, H^{2-2\alpha}(\mathbb{R}^2))$ *is a global solution of* $(\mathcal{S}_\alpha)$, *then*

$$\lim_{t \to \infty} \|\theta(t)\|_{H^{2-2\alpha}} = 0. \tag{1.4}$$

2. Notations and Preliminary Results

2.1. Notations and Technical Lemmas

In this short section, we collect some notations and definitions that will be used later, and we give some technical lemmas.

(i) The Fourier transformation in $\mathbb{R}^2$ is normalized as

$$\mathcal{F}(f)(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^2} \exp(-ix \cdot \xi) f(x) dx, \quad \xi = (\xi_1, \xi_2) \in \mathbb{R}^2. \tag{2.1}$$

(ii) The inverse Fourier formula is

$$\mathcal{F}^{-1}(g)(x) = (2\pi)^{-2} \int_{\mathbb{R}^2} \exp(i\xi \cdot x) f(\xi) d\xi, \quad x = (x_1, x_2) \in \mathbb{R}^2. \tag{2.2}$$

(iii) For $s \in \mathbb{R}$, $H^s(\mathbb{R}^2)$ denotes the usual nonhomogeneous Sobolev space on $\mathbb{R}^2$ and $\langle \cdot, \cdot \rangle_{H^s(\mathbb{R}^2)}$ its scalar product.

(iv) For $s \in \mathbb{R}$, $\dot{H}^s(\mathbb{R}^2)$ denotes the usual homogeneous Sobolev space on $\mathbb{R}^2$ and $\langle \cdot, \cdot \rangle_{\dot{H}^s(\mathbb{R}^2)}$ its scalar product.

(v) For $s,\ s' \in \mathbb{R}$ and $t \in [0,1]$,

$$\|f\|_{H^{ts+(1-t)s'}} \leq \|f\|_{H^s}^t \|f\|_{H^{s'}}^{1-t}, \tag{2.3}$$

$$\|f\|_{\dot{H}^{ts+(1-t)s'}} \leq \|f\|_{\dot{H}^s}^t \|f\|_{\dot{H}^{s'}}^{1-t}. \tag{2.4}$$

These two inequalities are called the interpolation inequalities, respectively, in the homogeneous and nonhomogeneous Sobolev spaces.

(i) For any Banach space $(B, \|\cdot\|)$, any real number $1 \leq p \leq \infty$, and any time $T > 0$, we denote by $L_T^p(B)$ the space of measurable functions $t \in [0,T] \mapsto f(t) \in B$ such that $(t \mapsto \|f(t)\|) \in L^p([0,T])$.

(ii) If $f = (f_1, f_2)$ and $g = (g_1, g_2)$ are two vector fields, we set

$$\begin{gathered} f \otimes g := (g_1 f, g_2 f), \\ \operatorname{div}(f \otimes g) := (\operatorname{div}(g_1 f), \operatorname{div}(g_2 f)). \end{gathered} \tag{2.5}$$

We recall a fundamental lemma concerning some product laws in homogeneous Sobolev spaces.

Lemma 2.1 (see [6]). *Let s_1, s_2 be two real numbers such that*

$$s_1 < 1, \qquad s_1 + s_2 > 0. \tag{2.6}$$

There exists a constant $C := C(s_1, s_2)$, such that for all $f, g \in \dot{H}^{s_1}(\mathbb{R}^2) \cap \dot{H}^{s_2}\mathbb{R}^2)$,

$$\|fg\|_{\dot{H}^{s_1+s_2-1}(\mathbb{R}^2)} \leq C\Big(\|f\|_{\dot{H}^{s_1}(\mathbb{R}^2)}\|g\|_{\dot{H}^{s_2}} + \|f\|_{\dot{H}^{s_2}}\|g\|_{\dot{H}^{s_1}}\Big). \tag{2.7}$$

If $s_1, s_2 < 1$ and $s_1 + s_2 > 0$, there exists a constant $c = c(s_1, s_2)$ such that for all $f \in \dot{H}^{s_1}(\mathbb{R}^2)$ and $g \in \dot{H}^{s_2}\mathbb{R}^2)$,

$$\|fg\|_{\dot{H}^{s_1+s_2-1}(\mathbb{R}^2)} \leq c\|f\|_{\dot{H}^{s_1}}\|g\|_{\dot{H}^{s_2}}. \tag{2.8}$$

For the proof of the main result, we need the following lemma.

Lemma 2.2. *With the same conditions of Theorem 1.1, for all $\sigma \geq 0$,*

$$\int_{\mathbb{R}^2} |\xi|^{2\sigma} |\mathcal{F}((u\cdot\nabla)\theta)\mathcal{F}(w)| d\xi \leq C\|\theta\|_{\dot{H}^{2-2\alpha}}\|\theta\|_{\dot{H}^{\sigma+\alpha}}\|w\|_{\dot{H}^{\sigma+\alpha}}. \tag{2.9}$$

Remark 2.3. (i) In the case where $\sigma = 0$, the formula (2.9) gives

$$\int_{\mathbb{R}^2} |\mathcal{F}((u\cdot\nabla)\theta)\mathcal{F}(w)|d\xi \le C\|\theta\|_{\dot{H}^{2-2\alpha}}\|\theta\|_{\dot{H}^{\alpha}}\|w\|_{\dot{H}^{\alpha}}. \tag{2.10}$$

In the case where $\sigma = 2-2\alpha$, the formula (2.9) gives

$$\int_{\mathbb{R}^2} |\xi|^{2(2-2\alpha)}|\mathcal{F}((u\cdot\nabla)\theta)\mathcal{F}(w)|d\xi \le C\|\theta\|_{\dot{H}^{2-2\alpha}}\|\theta\|_{\dot{H}^{2-\alpha}}\|w\|_{\dot{H}^{2-\alpha}}. \tag{2.11}$$

Proof of Lemma 2.2. From the Cauchy-Schwarz inequality, we have

$$\begin{aligned}\int_{\mathbb{R}^2} |\xi|^{2\sigma}|\mathcal{F}((u\cdot\nabla)\theta)\mathcal{F}(w)|d\xi &\le \int_{\mathbb{R}^2} |\xi|^{\sigma-\alpha}|\mathcal{F}((u\cdot\nabla)\theta)||\xi|^{\sigma+\alpha}|\mathcal{F}(w)(\xi)|d\xi \\ &\le \left(\int_{\mathbb{R}^2} |\xi|^{2(\sigma-\alpha)}|\mathcal{F}((u\cdot\nabla)\theta)|^2 d\xi\right)^{1/2}\|w\|_{\dot{H}^{\sigma+\alpha}}.\end{aligned} \tag{2.12}$$

Using the weak derivatives properties, the product laws (Lemma 2.1), with $s_1+s_2 = \sigma-\alpha+2 > 0$, $s_1 = 2-2\alpha < 1$, and $s_2 = \sigma+\alpha$, we can dominate the nonlinear part as follows:

$$\begin{aligned}\int_{\mathbb{R}^2} |\xi|^{2(\sigma-\alpha)}|\mathcal{F}((u\cdot\nabla)\theta)|^2 d\xi &\le \int_{\mathbb{R}^2} |\xi|^{2(\sigma-\alpha+1)}(|\mathcal{F}(\theta)|*|\mathcal{F}(\theta)|)^2 d\xi \\ &\le C\|\theta\|^2_{\dot{H}^{2-2\alpha}}\|\theta\|^2_{\dot{H}^{\sigma+\alpha}}.\end{aligned} \tag{2.13}$$

□

2.2. Existence Theorem

In [7], Wu proves an existence and uniqueness theorem of $(\mathcal{S}_\alpha)$ in the well-known Besov spaces $\dot{B}^r_{p,q}$. We recall this theorem in the special case, where $p = q = 2$.

Theorem 2.4. *Assume that $\alpha \in (0,1]$ and $\theta^0 \in \dot{H}^{2-2\alpha}(\mathbb{R}^2)$, then there exists a constant $c_\alpha > 0$ such that if*

$$\left\|\theta^0\right\|_{\dot{H}^{2-2\alpha}} < c_\alpha, \tag{2.14}$$

then the initial value problem $(\mathcal{S}_\alpha)$ has a unique solution in $C_b(\mathbb{R}^+, \dot{H}^{2-2\alpha}(\mathbb{R}^2)) \cap L^2(\mathbb{R}^+, \dot{H}^{2-\alpha}(\mathbb{R}^2))$. Moreover,

$$\|\theta(t)\|^2_{\dot{H}^{2-2\alpha}} + \int_0^t \|\theta(\tau)\|^2_{\dot{H}^{2-\alpha}}d\tau \le c'_\alpha, \qquad \forall t \ge 0, \tag{2.15}$$

where $C_b(\mathbb{R}^+, \dot{H}^{2-2\alpha}(\mathbb{R}^2))$ is the space of continuous and bounded functions from $\mathbb{R}^+$ to $\dot{H}^{2-2\alpha}(\mathbb{R}^2)$.

In use of the fact that $\dot{H}^{2-2\alpha}(\mathbb{R}^2)$ is a Hilbert space, one deduces the following.

Corollary 2.5. *Assume that* $\alpha \in (1/2, 1]$ *and* $\theta^0 \in \dot{H}^{2-2\alpha}(\mathbb{R}^2)$*, then there exists a constant* $c_\alpha > 0$ *such that if*

$$\left\| \theta^0 \right\|_{\dot{H}^{2-2\alpha}} < c_\alpha, \tag{2.16}$$

then the initial value problem $(\mathcal{S}_\alpha)$ *has a unique solution in* $C_b(\mathbb{R}^+, \dot{H}^{2-2\alpha}(\mathbb{R}^2)) \cap L^2(\mathbb{R}^+, \dot{H}^{2-\alpha}(\mathbb{R}^2))$*. Moreover,*

$$\|\theta(t)\|^2_{\dot{H}^{2-2\alpha}} + \int_0^t \|\theta(\tau)\|^2_{\dot{H}^{2-\alpha}} d\tau \leq \left\| \theta^0 \right\|^2_{\dot{H}^{2-2\alpha}}, \quad \forall t \geq 0. \tag{2.17}$$

Proof. Taking the scalar product in $\dot{H}^{2-2\alpha}(\mathbb{R}^2)$, we get

$$\begin{aligned} \frac{1}{2}\partial_t \|\theta\|^2_{\dot{H}^{2-2\alpha}} + \|\theta\|^2_{\dot{H}^{2-\alpha}} &\leq |\langle (u\cdot\nabla)\theta, \theta\rangle_{\dot{H}^{2-2\alpha}}| \\ &\leq |\langle \operatorname{div}(\theta u), \theta\rangle_{\dot{H}^{2-2\alpha}}| \\ &\leq \|\operatorname{div}(\theta u)\|_{\dot{H}^{2-3\alpha}} \|\theta\|_{\dot{H}^{2-\alpha}} \\ &\leq \|\theta u\|_{\dot{H}^{3-3\alpha}} \|\theta\|_{\dot{H}^{2-\alpha}}. \end{aligned} \tag{2.18}$$

Using Lemma 2.1 with $s_1 = 2 - 2\alpha < 1$ and $s_2 = 2 - \alpha$, we obtain

$$\frac{1}{2}\partial_t \|\theta\|^2_{\dot{H}^{2-2\alpha}} + \|\theta\|^2_{\dot{H}^{2-\alpha}} \leq C_\alpha \|\theta\|_{\dot{H}^{2-2\alpha}} \|\theta\|^2_{\dot{H}^{2-\alpha}}, \quad \left(C_\alpha = \frac{1}{2c_\alpha} \right). \tag{2.19}$$

Then the quadratic term can be absorbed,

$$\frac{1}{2}\partial_t \|\theta\|^2_{\dot{H}^{2-2\alpha}} + \|\theta\|^2_{\dot{H}^{2-\alpha}} \leq 0. \tag{2.20}$$

Taking the integral on the interval $[0, t]$, we obtain

$$\|\theta(t)\|^2_{\dot{H}^{2-2\alpha}} + \int_0^t \|\theta(\tau)\|^2_{\dot{H}^{2-\alpha}} d\tau \leq \left\| \theta^0 \right\|^2_{\dot{H}^{2-2\alpha}}, \quad \forall t \geq 0. \tag{2.21}$$

□

3. Proof of the Main Theorem

The proof of the first part will be in two steps.

First Step (Small Initial Data)

In this case, we suppose that

$$\left\| \theta^0 \right\|_{\dot{H}^{2-2\alpha}} < c_\alpha, \tag{3.1}$$

with c_α a sufficient small number. Then from Corollary 2.5,

$$\theta \in \mathcal{C}_b\Big(\mathbb{R}^+, \dot{H}^{2-2\alpha}\Big(\mathbb{R}^2\Big)\Big) \cap L^2\Big(\mathbb{R}^+, \dot{H}^{2-\alpha}\Big(\mathbb{R}^2\Big)\Big), \tag{3.2}$$

$$\|\theta\|^2_{\dot{H}^{2-2\alpha}} + \int_0^t \|\theta\|^2_{\dot{H}^{2-\alpha}} \leq \left\|\theta^0\right\|^2_{\dot{H}^{2-2\alpha}}, \quad \forall t \geq 0. \tag{3.3}$$

For a strictly positive real number δ and a given distribution f, we define the operators $A_\delta(D)$ and $B_\delta(D)$, respectively, by the following:

$$\begin{aligned} A_\delta(D)f &:= \chi_{B(0,\delta)}(|D|)f = \mathcal{F}^{-1}\left(\chi_{B(0,\delta)}\mathcal{F}(f)\right), \\ B_\delta(D)f &:= (1 - A_\delta(D))f = \mathcal{F}^{-1}\left(\left(1 - \chi_{B(0,\delta)}\right)\mathcal{F}(f)\right). \end{aligned} \tag{3.4}$$

We define $w_\delta = A_\delta(D)\theta$ and $v_\delta = B_\delta(D)\theta$; $\mathcal{F}(\theta) = \mathcal{F}(w_\delta) + \mathcal{F}(v_\delta)$. Then,

$$\begin{aligned} &\partial_t w_\delta + (-\Delta)^\alpha w_\delta + A_\delta(D)(u \cdot \nabla\theta) = 0, \\ &\partial_t \|w_\delta\|^2_{\dot{H}^{2-2\alpha}} + 2\|w_\delta\|^2_{\dot{H}^{2-\alpha}} \leq C\|\theta\|_{\dot{H}^{2-2\alpha}} \cdot \|\theta\|_{\dot{H}^{2-\alpha}} \cdot \|w_\delta\|_{\dot{H}^{2-\alpha}}. \end{aligned} \tag{3.5}$$

We deduce that

$$\|w_\delta\|^2_{\dot{H}^{2-2\alpha}} \leq \|w_\delta(0)\|^2_{\dot{H}^{2-2\alpha}} + C\|\theta(0)\|_{\dot{H}^{2-2\alpha}} \int_0^\infty \|\theta\|_{\dot{H}^{2-\alpha}} \|w_\delta\|_{\dot{H}^{2-\alpha}} d\tau. \tag{3.6}$$

Since $\|w_\delta\|_{\dot{H}^{2-\alpha}} \leq \|\theta\|_{\dot{H}^{2-\alpha}}$, then from the dominate convergence theorem and (3.3), we have

$$\lim_{\delta \to 0} \operatorname{Sup}_{t \geq 0} \|w_\delta\|_{\dot{H}^{2-2\alpha}} = 0. \tag{3.7}$$

The function v_δ satisfies

$$\begin{aligned} &\partial_t v_\delta + (-\Delta)^\alpha v_\delta + B_\delta(D)(u \cdot \nabla\theta) = 0, \\ &\partial_t |\mathcal{F}(v_\delta)|^2 + 2|\xi|^{2\alpha} |\mathcal{F}(v_\delta)|^2 \leq |\mathcal{F}(u \cdot \nabla\theta)\mathcal{F}(v_\delta)|. \end{aligned} \tag{3.8}$$

Multiplying this equation by $|\xi|^{2(2-2\alpha)} e^{2t|\xi|^{2\alpha}}$, we deduce that

$$\begin{aligned} \|v_\delta\|^2_{\dot{H}^{2-2\alpha}} &\leq \int_{|\xi|>\delta} |\xi|^{2(2-2\alpha)} e^{-2t|\xi|^{2\alpha}} \left|\mathcal{F}\left(v^0_\delta\right)\right|^2 \\ &\quad + \int_0^t \int_{|\xi|>\delta} |\xi|^{2(2-2\alpha)} e^{-2(t-\tau)|\xi|^{2\alpha}} |\mathcal{F}(u \cdot \nabla\theta)\mathcal{F}(v_\delta)| d\xi \, d\tau \\ &\leq e^{-2t\delta^{2\alpha}} \left\|v^0_\delta\right\|^2_{\dot{H}^{2-2\alpha}} + C\int_0^t e^{-2(t-\tau)\delta^{2\alpha}} \int_\xi |\xi|^{2(2-2\alpha)} |\mathcal{F}(u \cdot \nabla\theta)\mathcal{F}(v_\delta)| d\xi \, d\tau. \end{aligned} \tag{3.9}$$

Using Remark 2.3 and (3.3), we get

$$\|v_\delta\|^2_{\dot{H}^{2-2\alpha}} \leq e^{-2t\delta^{2\alpha}} \left\|v^0_\delta\right\|^2_{\dot{H}^{2-2\alpha}} + C\left\|\theta^0\right\|_{\dot{H}^{2-2\alpha}} \int_0^t e^{-2(t-\tau)\delta^{2\alpha}} \|\theta\|^2_{\dot{H}^{2-\alpha}} d\tau. \tag{3.10}$$

We set

$$\begin{aligned}
F_\delta(t) &= e^{-2t\delta^{2\alpha}} \left\|v^0_\delta\right\|^2_{\dot{H}^{2-2\alpha}} + C\left\|\theta^0\right\|_{\dot{H}^{2-2\alpha}} \int_0^t e^{-2(t-\tau)\delta^{2\alpha}} \|\theta\|^2_{\dot{H}^{2-\alpha}} d\tau, \\
\int_0^{+\infty} e^{-2t\delta^{2\alpha}} \left\|v^0_\delta\right\|^2_{\dot{H}^{2-2\alpha}} dt &= \frac{\left\|v^0_\delta\right\|^2_{\dot{H}^{2-2\alpha}}}{2\delta^{2\alpha}} \leq \frac{\left\|\theta^0\right\|^2_{\dot{H}^{2-2\alpha}}}{2\delta^{2\alpha}}, \\
\int_0^{+\infty}\int_0^t e^{-2(t-\tau)\delta^{2\alpha}} \|\theta\|^2_{\dot{H}^{2-\alpha}} d\tau\, dt &= \int_0^{+\infty}\left(\int_\tau^{+\infty} e^{-2(t-\tau)\delta^{2\alpha}} dt\right) \|\theta\|^2_{\dot{H}^{2-\alpha}} d\tau \\
&= \frac{1}{2\delta^{2\alpha}} \int_0^{+\infty} \|\theta\|^2_{\dot{H}^{2-\alpha}} dt \leq \frac{\left\|\theta^0\right\|^2_{\dot{H}^{2-2\alpha}}}{4\delta^{2\alpha}}.
\end{aligned} \tag{3.11}$$

Then,

$$\int_0^{+\infty} F_\delta(t)dt \leq \frac{\left\|\theta^0\right\|^2_{\dot{H}^{2-2\alpha}}}{\delta^{2\alpha}}. \tag{3.12}$$

Let $\varepsilon > 0$, from (3.7), there exists $\delta_0 > 0$ such that

$$\|w_{\delta_0}\|_{\dot{H}^{2-2\alpha}} \leq \frac{\varepsilon}{2}, \quad \forall t \geq 0. \tag{3.13}$$

Let $E_{\delta_0} = \{t \geq 0;\ \|v_{\delta_0}\|_{\dot{H}^{2-2\alpha}} > \varepsilon/2\}$, then

$$\int_0^{+\infty} \|v_{\delta_0}\|^2_{\dot{H}^{2-2\alpha}} dt \geq \int_{E_{\delta_0}} \|v_{\delta_0}\|^2_{\dot{H}^{2-2\alpha}} dt \geq \left(\frac{\varepsilon}{2}\right)^2 \lambda_1(E_{\delta_0}), \tag{3.14}$$

where $\lambda_1(E_{\delta_0})$ is the Lebesgue measure of E_{δ_0}. If

$$T_\varepsilon = \left(\frac{2}{\varepsilon}\right)^2 \int_0^{+\infty} \|v_{\delta_0}\|^2_{\dot{H}^{2-2\alpha}} dt, \tag{3.15}$$

then $\lambda_1(E_{\delta_0}) \leq T_\varepsilon$. For $\eta > 0$, there exists $t_0 \in [0, T_\varepsilon + \eta]$ such that $t_0 \notin E_{\delta_0}$, and it results that

$$\|v_{\delta_0}(t_0)\|_{\dot{H}^{2-2\alpha}} \leq \frac{\varepsilon}{2}. \tag{3.16}$$

Equation (3.13) and (3.16) give that

$$\|\theta(t_0)\|_{\dot{H}^{2-2\alpha}} \leq \varepsilon. \tag{3.17}$$

Thus, $\lim_{t\to+\infty}\|\theta(t)\|_{\dot{H}^{2-2\alpha}} = 0$, and this finishes the proof in this case.

Second Step (Large Initial Data)

To prove the result for any initial data, it suffices to prove the existence of some $t_0 \geq 0$ such that

$$\|\theta(t_0)\|_{\dot{H}^{2-2\alpha}} < c_\alpha. \tag{3.18}$$

Let $\theta^0 = a^0 + r^0$, with

$$\begin{gathered} a^0 := \mathcal{F}^{-1}\left(\mathbf{1}_{\{1/N<|\xi|<N\}}\mathcal{F}\left(\theta^0\right)\right), \\ r^0 := \theta^0 - a^0, \\ \left\|r^0\right\|_{\dot{H}^{2-2\alpha}} < c_\alpha. \end{gathered} \tag{3.19}$$

Now, consider the following system:

$$\begin{gathered} \partial_t r + (-\Delta)^\alpha r + (R\cdot\nabla)r = 0 \quad \text{in } \mathbb{R}^+\times\mathbb{R}^2, \\ r(0) = r^0 \quad \text{in } \mathbb{R}^2, \\ R = \nabla^\perp\Delta^{-1/2}r. \end{gathered} \tag{3.20}$$

By Corollary 2.5, there is a unique solution $r \in \mathcal{C}_b(\mathbb{R}^+, \dot{H}^{2-2\alpha}(\mathbb{R}^2)) \cap L^2(\mathbb{R}^+, \dot{H}^{2-\alpha}(\mathbb{R}^2))$ such that

$$\|r(t)\|_{\dot{H}^{2-2\alpha}}^2 + \int_0^t \|r(\tau)\|_{\dot{H}^{2-\alpha}}^2 d\tau \leq \left\|r^0\right\|_{\dot{H}^{2-2\alpha}}^2. \tag{3.21}$$

Let $a := \theta - r \in \mathcal{C}(\mathbb{R}^+, \dot{H}^{2-2\alpha}(\mathbb{R}^2))$, then a is a solution of the following system:

$$\begin{gathered} \partial_t a + (-\Delta)^\alpha a + (A\cdot\nabla)a + (A\cdot\nabla)r + (R\cdot\nabla)a = 0 \quad \text{in } \mathbb{R}^+\times\mathbb{R}^2, \\ a(0) = a^0 \quad \text{in } \mathbb{R}^2, \\ A = \nabla^\perp\Delta^{-1/2}a. \end{gathered} \tag{S^1}$$

Taking a scalar product in $L^2(\mathbb{R}^2)$, we obtain

$$\begin{aligned}\partial_t \|a(t)\|_{L^2}^2 + 2\|a(t)\|_{\dot{H}^\alpha}^2 &\le 2\left|\int_{\mathbb{R}^2} (A\cdot\nabla) r a\right| \\ &\le 2\left|\int_{\mathbb{R}^2} \operatorname{div}\,(rA) a\right| \\ &\le 2\|rA\|_{\dot{H}^{1-\alpha}}\|a\|_{\dot{H}^\alpha}.\end{aligned} \tag{3.22}$$

Using the product law in Lemma 2.1, with $s_1 = 2-2\alpha < 1$ and $s_2 = \alpha < 1$,

$$\begin{aligned}\left|\langle (A\cdot\nabla) r, a\rangle_{L^2(\mathbb{R}^2)}\right| &\le C(\alpha)\|r\|_{\dot{H}^{2-2\alpha}}\|A\|_{\dot{H}^\alpha}\|a\|_{\dot{H}^\alpha} \\ &\le C(\alpha)\|r\|_{\dot{H}^{2-2\alpha}}\|a\|_{\dot{H}^\alpha}^2 \\ &\le \|a\|_{\dot{H}^\alpha}^2,\end{aligned} \tag{3.23}$$

then, for all $t \ge 0$,

$$\begin{aligned}\partial_t \|a(t)\|_{L^2}^2 + \|a(t)\|_{\dot{H}^\alpha}^2 \le 0, \\ \|a(t)\|_{L^2}^2 + \int_0^t \|a(\tau)\|_{\dot{H}^\alpha}^2 d\tau \le \left\|a^0\right\|_{L^2}^2,\end{aligned} \tag{3.24}$$

then $2-2\alpha = \lambda \times 0 + (1-\lambda)\alpha$, with $\lambda := 3 - (2/\alpha) \in [0,1]$,

$$\begin{aligned}\|a(t)\|_{\dot{H}^{2-2\alpha}} &\le \|a(t)\|_{L^2}^{3-2/\alpha}\|a(t)\|_{\dot{H}^\alpha}^{2/\alpha-2} \\ &\le \left\|a^0\right\|_{L^2}^{3-2/\alpha}\|a(t)\|_{\dot{H}^\alpha}^{2/\alpha-2}.\end{aligned} \tag{3.25}$$

Then,

$$\int_0^\infty \|a(t)\|_{\dot{H}^{2-2\alpha}}^{\alpha/(1-\alpha)} dt \le \left\|a^0\right\|_{L^2}^{1/(1-\alpha)}. \tag{3.26}$$

Now define the set

$$S_\varepsilon := \{t \ge 0; \|a(t)\|_{\dot{H}^{2-2\alpha}} > \varepsilon\} \tag{3.27}$$

as a measurable with respect to the Lebesgue measure. We have

$$\varepsilon^{\alpha/(1-\alpha)}\lambda_1(S_\varepsilon) \le \int_{S_\varepsilon} \|a(t)\|_{\dot{H}^{2-2\alpha}}^{\alpha/(1-\alpha)} dt \le \left\|a^0\right\|_{L^2}^{1/(1-\alpha)}. \tag{3.28}$$

So $\lambda_1(S_\varepsilon) < \infty$ and $\lambda_1(S_\varepsilon) \leq \varepsilon^{\alpha/(1-\alpha)}\|a^0\|_{L^2}^{1/(1-\alpha)}$, then there is

$$t_0 \in [0, \lambda_1(S_\varepsilon) + 1] \setminus S_\varepsilon. \tag{3.29}$$

Then,

$$\|a(t_0)\|_{\dot{H}^{2-2\alpha}} < \varepsilon, \tag{3.30}$$

and then

$$\begin{aligned}\|\theta(t_0)\|_{\dot{H}^{2-2\alpha}} &\leq \|r(t_0)\|_{\dot{H}^{2-2\alpha}} + \|a(t_0)\|_{\dot{H}^{2-2\alpha}} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned} \tag{3.31}$$

Applying the conclusion of Theorem 1.1 for $(\mathcal{S}_\alpha)$ system starting at $\theta(t_0)$, we can deduce the desired result.

In the nonhomogeneous case, we suppose that $\theta \in C(\mathbb{R}^+, H^{2-2\alpha})$, then

$$\lim_{t\to\infty} \|\theta(t)\|_{\dot{H}^{2-2\alpha}} = 0. \tag{3.32}$$

We can suppose that $\|\theta\|_{\dot{H}^{2-2\alpha}} < c_\alpha$, and for all $t \geq 0$,

$$\|\theta(t)\|_{\dot{H}^{2-2\alpha}}^2 + \int_0^t \|\theta(\tau)\|_{\dot{H}^{2-\alpha}}^2 d\tau \leq \left\|\theta^0\right\|_{\dot{H}^{2-2\alpha}}^2. \tag{3.33}$$

Thus, it suffices to prove that

$$\lim_{t\to\infty} \|\theta(t)\|_{L^2} = 0. \tag{3.34}$$

Let $\delta > 0$, then we recall the operators

$$\begin{aligned}A_\delta(D)\theta &= \mathcal{F}^{-1}\left(\chi_{B(0,\delta)}\mathcal{F}(\theta)\right), \\ B_\delta(D)\theta &= \mathcal{F}^{-1}\left(\left(1-\chi_{B(0,\delta)}\right)\mathcal{F}(\theta)\right).\end{aligned} \tag{3.35}$$

We define $w_\delta = A_\delta(D)(\theta)$ and $v_\delta = B_\delta(D)(\theta)$. Then,

$$\partial_t w_\delta + (-\Delta)^\alpha w_\delta + A_\delta(D)(u_\theta \cdot \nabla\theta) = 0, \tag{3.36}$$

and from Lemma 2.2,

$$\partial_t \|w_\delta\|_{L^2}^2 + 2\|w_\delta\|_{\dot{H}^\alpha}^2 \leq C\|\theta\|_{\dot{H}^{2-2\alpha}}.\|\theta\|_{\dot{H}^\alpha} \cdot \|w_\delta\|_{\dot{H}^\alpha}. \tag{3.37}$$

We deduce that

$$\|w_\delta\|_{L^2}^2 \leq \|w_\delta(0)\|_{L^2}^2 + C\left\|\theta^0\right\|_{\dot{H}^{2-2\alpha}} \int_0^{+\infty} \|\theta\|_{\dot{H}^\alpha} \|w_\delta\|_{\dot{H}^\alpha} d\tau. \tag{3.38}$$

Then from the dominate convergence theorem and the following L^2 energy estimate

$$\|\theta\|_{L^2}^2 + 2\int_0^t \|\theta\|_{\dot{H}^\alpha}^2 d\tau \leq \left\|\theta^0\right\|_{L^2}^2, \tag{3.39}$$

we deduce that

$$\begin{gathered} \lim_{\delta\to 0} \operatorname{Sup}_{t\geq 0} \|w_\delta\|_{L^2} = 0, \\ \partial_t v_\delta + (-\Delta)^\alpha v_\delta + B_\delta(D)(u_\theta \cdot \nabla\theta) = 0, \end{gathered} \tag{3.40}$$

$$\partial_t |\mathcal{F}(v_\delta)|^2 + 2|\xi|^{2\alpha} |\mathcal{F}(v_\delta)|^2 \leq |\mathcal{F}(u\cdot\nabla\theta)\mathcal{F}(v_\delta)|. \tag{3.41}$$

Multiplying this equation by $e^{2t|\xi|^{2\alpha}}$, we have

$$\begin{aligned} \|v_\delta\|_{L^2}^2 &\leq e^{-2t\delta^{2\alpha}} \left\|v_\delta^0\right\|_{L^2}^2 + C\int_0^t e^{-2(t-\tau)\delta^{2\alpha}} \left|\left\langle u\cdot\frac{\nabla\theta}{v_\delta}\right\rangle\right|_{L^2(\mathbb{R}^2)}^2 d\tau \\ &\leq e^{-2t\delta^{2\alpha}} \left\|v_\delta^0\right\|_{L^2}^2 + C\left\|\theta^0\right\|_{\dot{H}^{2-2\alpha}} \int_0^t e^{-2(t-\tau)\delta^{2\alpha}} \|\theta\|_{\dot{H}^\alpha}^2 d\tau. \end{aligned} \tag{3.42}$$

We set

$$\begin{aligned} F_\delta(t) &= e^{-2t\delta^{2\alpha}} \left\|v_\delta^0\right\|_{L^2}^2 + C\left\|\theta^0\right\|_{\dot{H}^{2-2\alpha}} \int_0^t e^{-2(t-\tau)\delta^{2\alpha}} \|\theta\|_{\dot{H}^\alpha}^2 d\tau, \\ \int_0^{+\infty} e^{-2t\delta^{2\alpha}} \left\|v_\delta^0\right\|_{L^2}^2 dt &= \frac{\|\theta^0\|_{L^2}^2}{2\delta^{2\alpha}}, \\ \int_0^{+\infty}\int_0^t e^{-2(t-\tau)\delta^{2\alpha}} \|\theta\|_{\dot{H}^\alpha}^2 d\tau\, dt &= \int_0^{+\infty} \left(\int_\tau^{+\infty} e^{-2(t-\tau)\delta^{2\alpha}} dt\right) \|\theta\|_{\dot{H}^\alpha}^2 d\tau \\ &= \frac{1}{2\delta^{2\alpha}} \int_0^{+\infty} \|\theta\|_{\dot{H}^\alpha}^2 dt \leq \frac{\|\theta^0\|_{L^2}^2}{2\delta^{2\alpha}}. \end{aligned} \tag{3.43}$$

Then,

$$\int_0^{+\infty} F_\delta(t) dt \leq \frac{\|\theta^0\|_{L^2}^2}{\delta^{2\alpha}}. \tag{3.44}$$

Let $\varepsilon > 0$, from (3.40), then there exists $\delta_0 > 0$ such that

$$\|w_{\delta_0}\|_{L^2} \leq \frac{\varepsilon}{2}, \quad \forall t \geq 0. \tag{3.45}$$

Let $E_{\delta_0} = \{t \geq 0;\ \|v_{\delta_0}\|_{L^2} > \varepsilon/2\}$, then

$$\int_0^{+\infty} \|v_{\delta_0}\|_{L^2}^2 dt \geq \int_{E_{\delta_0}} \|v_{\delta_0}\|_{L^2}^2 dt \geq \left(\frac{\varepsilon}{2}\right)^2 \lambda_1(E_{\delta_0}), \tag{3.46}$$

where $\lambda_1(E_{\delta_0})$ is the Lebesgue measure of E_{δ_0}. If

$$T_\varepsilon = \left(\frac{2}{\varepsilon}\right)^2 \int_0^{+\infty} \|v_{\delta_0}\|_{L^2}^2 dt, \tag{3.47}$$

then $\lambda_1(E_{\delta_0}) \leq T_\varepsilon$. For $\eta > 0$, there exists $t_0 \in [0, T_\varepsilon + \eta]$ such that $t_0 \notin E_{\delta_0}$, then

$$\|v_{\delta_0}(t_0)\|_{L^2} \leq \frac{\varepsilon}{2}. \tag{3.48}$$

The equations (3.45) and (3.48) give that

$$\|\theta(t_0)\|_{L^2} < \varepsilon. \tag{3.49}$$

Thus, $\lim_{t\to+\infty}\|\theta(t)\|_{L^2} = 0$, and this finishes the proof.

Acknowledgments

This research is supported by NPST Program of King Saud University, project number 10-MAT1293-02. The authors thank the referee for his/her careful reading of the paper and corrections.

References

[1] P. Constantin, A. J. Majda, and E. Tabak, "Formation of strong fronts in the 2-D quasigeostrophic thermal active scalar," *Nonlinearity*, vol. 7, no. 6, pp. 1495–1533, 1994.
[2] N. Ju, "Dissipative 2D quasi-geostrophic equation: local well-posedness, global regularity and similarity solutions," *Indiana University Mathematics Journal*, vol. 56, no. 1, pp. 187–206, 2007.
[3] H. Miura, "Dissipative quasi-geostrophic equation for large initial data in the critical Sobolev space," *Communications in Mathematical Physics*, vol. 267, no. 1, pp. 141–157, 2006.
[4] H. Dong and D. Du, "Global well-posedness and a decay estimate for the critical dissipative quasi-geostrophic equation in the whole space," *Discrete and Continuous Dynamical Systems-Series A*, vol. 21, no. 4, pp. 1095–1101, 2008.
[5] C. J. Niche and M. E. Schonbek, "Decay of weak solutions to the 2D dissipative quasi-geostrophic equation," *Communications in Mathematical Physics*, vol. 276, no. 1, pp. 93–115, 2007.
[6] J.-Y. Chemin, "About Navier-Stokes equations," Publications of Jaques-Louis Lions Laboratoiry, Paris VI University, R96023, 1996.
[7] J. Wu, "Existence and uniqueness results for the 2-D dissipative quasi-geostrophic equation," *Nonlinear Analysis: Theory, Methods & Applications A*, vol. 67, no. 11, pp. 3013–3036, 2007.