

Research Article

Chaos Control and Synchronization in Fractional-Order Lorenz-Like System

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The present paper deals with fractional-order version of a dynamical system introduced by Chongxin et al. (2006). The chaotic behavior of the system is studied using analytic and numerical methods. The minimum effective dimension is identified for chaos to exist. The chaos in the proposed system is controlled using simple linear feedback controller. We design a controller to place the eigenvalues of the system Jacobian in a stable region. The effectiveness of the controller in eliminating the chaotic behavior from the state trajectories is also demonstrated using numerical simulations. Furthermore, we synchronize the system using nonlinear feedback.

1. Introduction

A variety of problems in engineering and natural sciences are modeled using chaotic dynamical systems. A chaotic system is a nonlinear deterministic system possessing complex dynamical behaviors such as being extremely sensitive to tiny variations of initial conditions, unpredictability, and having bounded trajectories in the phase space [1]. Controlling the chaotic behavior in the dynamical systems using some form of control mechanism has recently been the focus of much attention. So many approaches are proposed for chaos control namely, OGY method [2], backstepping design method [3], differential geometric method [4], inverse optimal control [5], sampled-data feedback control [6], adaptive control [7], and so on. One simple approach is the linear feedback control [8]. Linear feedback controllers are easy to implement, they can perform the job automatically, and stabilize the overall control system efficiently [9].

The controllers can also be used to synchronize two identical or distinct chaotic systems [10–13]. Synchronization of chaos refers to a process wherein two chaotic systems

adjust a given property of their motion to a common behavior due to a coupling. Synchronization has many applications in secure communications of analog and digital signals [14] and for developing safe and reliable cryptographic systems [15].

Fractional calculus deals with derivatives and integration of arbitrary order [16–18] and has deep and natural connections with many fields of applied mathematics, engineering, and physics. Fractional calculus has wide range of applications in control theory [19], viscoelasticity [20], diffusion [21–25], turbulence, electromagnetism, signal processing [26, 27], and bioengineering [28]. Study of chaos in fractional order dynamical systems and related phenomena is receiving growing attention [29, 30]. I. Grigorenko and E. Grigorenko investigated fractional ordered Lorenz system and observed that below a threshold order the chaos disappears [31]. Further, many systems such as Li and Peng [32], Lu [33], Li and Chen [34], Daftardar-Gejji and Bhalekar [35], and unified system [36] were investigated in this regard. Effect of delay on chaotic solutions in fractional order dynamical system is investigated by the present author [37]. It is demonstrated that the chaotic systems can be transformed into limit cycles or stable orbits with appropriate choice of delay parameter. Synchronization of fractional order chaotic systems was also studied by many researchers [38–41].

In this paper, we propose fractional version of the Lorenz-like chaotic dynamical system [42]. We investigate minimum effective dimension of the system for chaos to exist. Then we control the chaos using simple linear feedback control. Further, we synchronize the proposed fractional order system using feedback control.

2. Preliminaries

2.1. Fractional Calculus

Few definitions of fractional derivatives are known [16–18]. Probably the best known is the Riemann-Liouville formulation.

The Riemann-Liouville integral of order $\mu, \mu > 0$ is given by

$$I^\mu f(t) = \frac{1}{\Gamma(\mu)} \int_0^t (t - \tau)^{\mu-1} f(\tau) d\tau, \quad t > 0. \quad (2.1)$$

An alternative definition was introduced by Caputo. Caputo's derivative is defined as

$$\begin{aligned} D^\mu f(t) &= \frac{d^m}{dt^m} f(t), \quad \mu = m \\ &= I^{m-\mu} \frac{d^m f(t)}{dt^m}, \quad m - 1 < \mu < m, \end{aligned} \quad (2.2)$$

where $m \in \mathbb{N}$. The main advantage of the Caputo's formulation is that the Caputo derivative of a constant is equal to zero, that is not the case for the Riemann-Liouville derivative. Note that for $m - 1 < \mu \leq m, m \in \mathbb{N}$,

$$I^\mu D^\mu f(t) = f(t) - \sum_{k=0}^{m-1} \frac{d^k f}{dt^k}(0) \frac{t^k}{k!},$$

$$I^\mu t^\nu = \frac{\Gamma(\nu + 1)}{\Gamma(\mu + \nu + 1)} t^{\mu + \nu}.$$
(2.3)

2.2. Numerical Method for Solving Fractional Differential Equations

Numerical methods used for solving ODEs have to be modified for solving fractional differential equations (FDEs). A modification of Adams-Bashforth-Moulton algorithm is proposed by Diethelm et al. in [43–45] to solve FDEs.

Consider for $\alpha \in (m - 1, m]$ the initial value problem (IVP)

$$D^\alpha y(t) = f(t, y(t)), \quad 0 \leq t \leq T,$$

$$y^{(k)}(0) = y_0^{(k)}, \quad k = 0, 1, \dots, m - 1.$$
(2.4)

The IVP (2.4) is equivalent to the Volterra integral equation

$$y(t) = \sum_{k=0}^{m-1} y_0^{(k)} \frac{t^k}{k!} + \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} f(\tau, y(\tau)) d\tau.$$
(2.5)

Consider the uniform grid $\{t_n = nh/n = 0, 1, \dots, N\}$ for some integer N and $h := T/N$. Let $y_h(t_n)$ be approximation to $y(t_n)$. Assume that we have already calculated approximations $y_h(t_j), j = 1, 2, \dots, n$, and we want to obtain $y_h(t_{n+1})$ by means of the equation

$$y_h(t_{n+1}) = \sum_{k=0}^{m-1} \frac{t_{n+1}^k}{k!} y_0^{(k)} + \frac{h^\alpha}{\Gamma(\alpha + 2)} f(t_{n+1}, y_h^P(t_{n+1}))$$

$$+ \frac{h^\alpha}{\Gamma(\alpha + 2)} \sum_{j=0}^n a_{j,n+1} f(t_j, y_n(t_j)),$$
(2.6)

where

$$a_{j,n+1} = \begin{cases} n^{\alpha+1} - (n - \alpha)(n + 1)^\alpha & \text{if } j = 0, \\ (n - j + 2)^{\alpha+1} + (n - j)^{\alpha+1} \\ - 2(n - j + 1)^{\alpha+1} & \text{if } 1 \leq j \leq n, \\ 1 & \text{if } j = n + 1. \end{cases}$$
(2.7)

The preliminary approximation $y_h^P(t_{n+1})$ is called predictor and is given by

$$y_h^P(t_{n+1}) = \sum_{k=0}^{m-1} \frac{t^{n+1-k}}{k!} y_0^{(k)} + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^n b_{j,n+1} f(t_j, y_n(t_j)), \quad (2.8)$$

where

$$b_{j,n+1} = \frac{h^\alpha}{\alpha} ((n+1-j)^\alpha - (n-j)^\alpha). \quad (2.9)$$

Error in this method is

$$\max_{j=0,1,\dots,N} |y(t_j) - y_h(t_j)| = O(h^p), \quad (2.10)$$

where $p = \min(2, 1 + \alpha)$.

2.3. Asymptotic Stability of the Fractional-Ordered System

Consider the following fractional-ordered dynamical system:

$$D^\alpha x_i = f_i(x_1, x_2, x_3), \quad 1 \leq i \leq 3. \quad (2.11)$$

Let $p \equiv (x_1^*, x_2^*, x_3^*)$ be an equilibrium point of the system (2.11) that is, $f_i(p) = 0, 1 \leq i \leq 3$ and $\xi_i = x_i - x_i^*$ a small disturbance from a fixed point. Then

$$\begin{aligned} D^\alpha \xi_i &= D^\alpha x_i \\ &= f_i(x_1, x_2, x_3) = f_i(\xi_1 + x_1^*, \xi_2 + x_2^*, \xi_3 + x_3^*) \\ &= f_i(x_1^*, x_2^*, x_3^*) + \xi_1 \frac{\partial f_i(p)}{\partial x_1} + \xi_2 \frac{\partial f_i(p)}{\partial x_2} + \xi_3 \frac{\partial f_i(p)}{\partial x_3} \\ &\quad + \text{higher-ordered terms} \\ &\approx \xi_1 \frac{\partial f_i(p)}{\partial x_1} + \xi_2 \frac{\partial f_i(p)}{\partial x_2} + \xi_3 \frac{\partial f_i(p)}{\partial x_3}. \end{aligned} \quad (2.12)$$

System (2.12) can be written as

$$D^\alpha \xi = J \xi, \quad (2.13)$$

where $\xi = (\xi_1, \xi_2, \xi_3)^t$ and

$$J = \begin{pmatrix} \partial_1 f_1(p) & \partial_2 f_1(p) & \partial_3 f_1(p) \\ \partial_1 f_2(p) & \partial_2 f_2(p) & \partial_3 f_2(p) \\ \partial_1 f_3(p) & \partial_2 f_3(p) & \partial_3 f_3(p) \end{pmatrix}. \quad (2.14)$$

Consider the linear autonomous system

$$D^\alpha \xi = J\xi, \quad \xi(0) = \xi_0, \quad (2.15)$$

where J is $n \times n$ matrix and $0 < \alpha < 1$ is asymptotically stable if and only if $|\arg(\lambda)| > \alpha\pi/2$ for all eigenvalues λ of J . In this case, each component of solution $\xi(t)$ decays towards 0 like $t^{-\alpha}$ [29, 46].

This shows that if $|\arg(\lambda)| > \alpha\pi/2$ for all eigenvalues λ of J then the solution $\xi_i(t)$ of the system (2.13) tends to 0 as $t \rightarrow \infty$. Thus, the equilibrium point p of the system is asymptotically stable if $|\arg(\lambda)| > \alpha\pi/2$, for all eigenvalues λ of J , that is, if

$$\min_i |\arg(\lambda_i)| > \frac{\alpha\pi}{2}. \quad (2.16)$$

3. Fractional Lorenz-Like System

In [42], Chongxin et al. proposed novel Lorenz-like chaotic system

$$\begin{aligned} \dot{x} &= a(y - x), \\ \dot{y} &= bx - lxz, \\ \dot{z} &= -cz + hx^2 + my^2, \end{aligned} \quad (3.1)$$

where $a = 10, b = 40, c = 2.5, m = h = 2, l = 1$ and initial conditions ((2.2),(2.3), and (28)). In this paper, we study the corresponding fractional order system

$$\begin{aligned} D^\alpha x &= a(y - x), \\ D^\alpha y &= bx - lxz, \\ D^\alpha z &= -cz + hx^2 + my^2, \end{aligned} \quad (3.2)$$

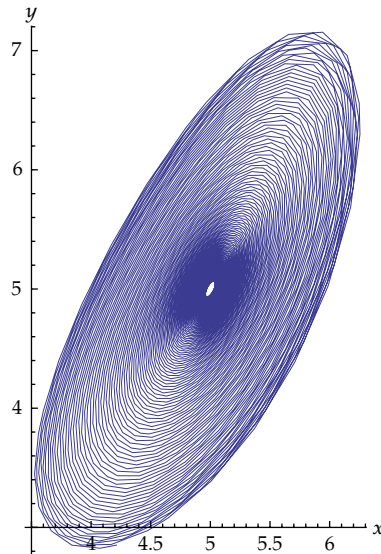
where $\alpha \in (0, 1)$. The equilibrium points of the system (3.1) and the eigenvalues of corresponding Jacobian matrix

$$J(x, y, z) = \begin{pmatrix} -a & a & 0 \\ b - lz & 0 & -lx \\ 2hx & 2my & -c \end{pmatrix} \quad (3.3)$$

are given in Table 1. An equilibrium point p of the system (3.1) is called as saddle point if the Jacobian matrix at p has at least one eigenvalue with negative real part (stable) and one eigenvalue with nonnegative real part (unstable). A saddle point is said to have index one (/two) if there is exactly one (/two) unstable eigenvalue/s. It is established in the literature [47–51] that scrolls are generated only around the saddle points of index two. Saddle points of index one are responsible only for connecting scrolls.

Table 1: Equilibrium points and corresponding eigenvalues.

Equilibrium point	Eigenvalues
$O(0, 0, 0)$	$-25.6155, 15.6155, -2.5$
$E_1(5, 5, 40)$	$-13.8776, 0.688787 \pm 11.9851i$
$E_2(-5, -5, 40)$	$-13.8776, 0.688787 \pm 11.9851i$

**Figure 1:** xy -phase portrait for $\alpha = 0.96$.

It is clear from Table 1 that the equilibrium points E_1 and E_2 are saddle points of index two; hence, there exists a two-scroll attractor [47] in the system (3.2).

The system (3.2) shows regular behavior if it satisfies (2.16), that is, the system is stable if

$$\alpha < \frac{2}{\pi} \min_i |\arg(\lambda_i)| \approx 0.96345. \quad (3.4)$$

Thus, the system does not show chaotic behavior for $\alpha < 0.96345$. This result is supported by numerical experiments. Figure 1 shows phase portrait in xy -plane for $\alpha = 0.96$. It is observed that the system shows chaotic behavior for $\alpha \geq 0.97$. For $\alpha = 0.97$ xz -phase portrait is shown in Figure 2. Figures 3 and 4 show xy - and yz -phase portraits, respectively, for $\alpha = 0.98$. The phase portraits in xy - and xz -plane are drawn for $\alpha = 0.99$ in Figures 5 and 6 respectively. Thus, the minimum effective dimension of the system is $0.97 \times 3 = 2.91$.

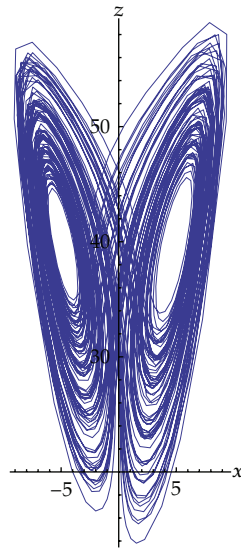


Figure 2: xz -phase portrait for $\alpha = 0.97$.

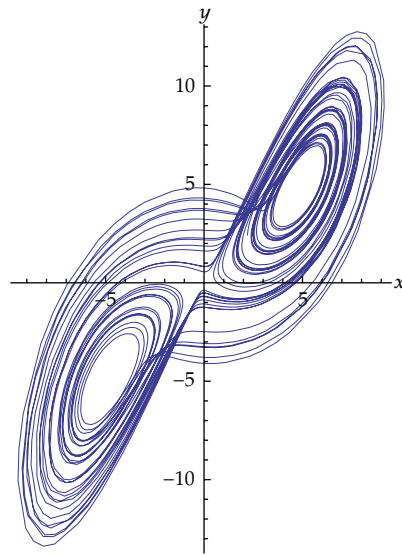


Figure 3: xy -phase portrait for $\alpha = 0.98$.

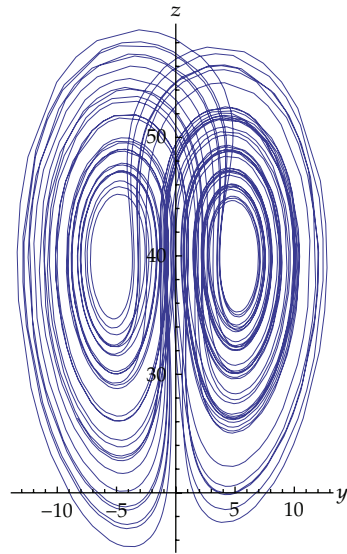


Figure 4: yz -phase portrait for $\alpha = 0.98$.

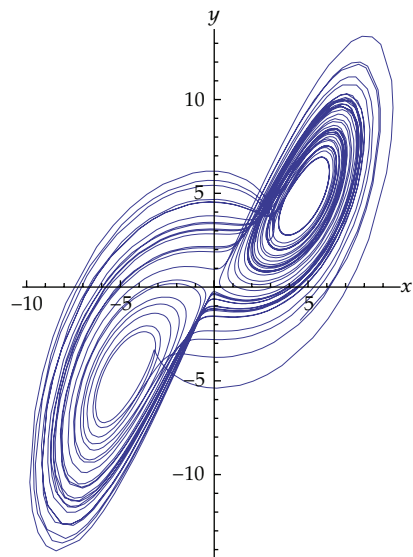


Figure 5: xy -phase portrait for $\alpha = 0.99$.

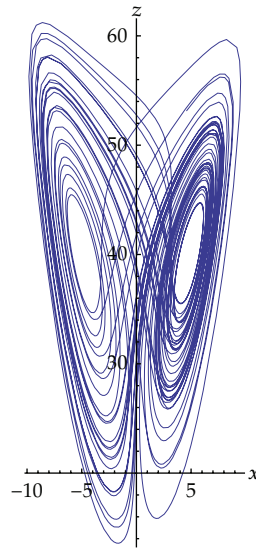


Figure 6: xz -phase portrait for $\alpha = 0.99$.

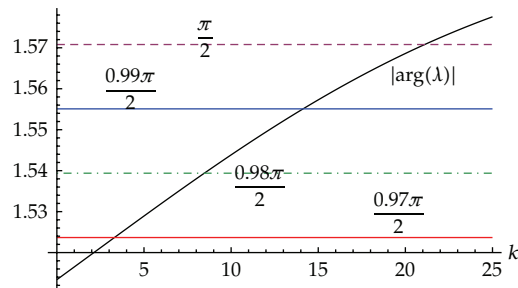


Figure 7: Estimate for the control k .

4. Control of Chaos

In this section, we control the chaos in system (3.2). Consider

$$\begin{aligned}
 D^\alpha x &= 10(y - x), \\
 D^\alpha y &= 40x - xz, \\
 D^\alpha z &= -2.5z + 2x^2 + 2y^2 + u,
 \end{aligned}
 \tag{4.1}$$

where u is the linear feedback control term. We set $u = kx$, where k is a parameter to be determined so that the system (4.1) is stable. Equilibrium points of system (4.1) are

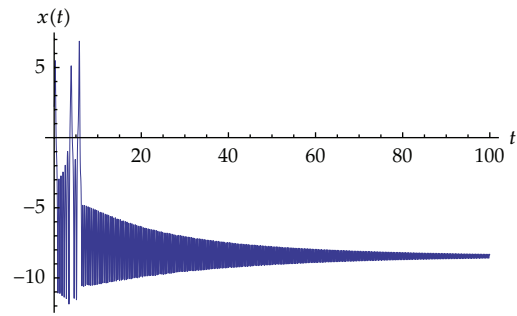


Figure 8: Controlled signal $\alpha = 1$ ($k = 22$).

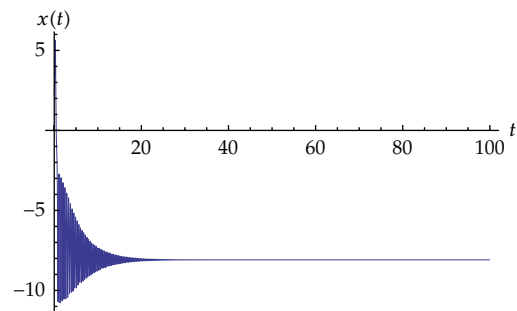


Figure 9: Controlled signal $\alpha = 0.99$ ($k = 20$).

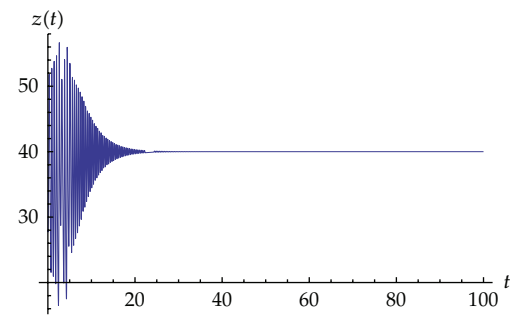


Figure 10: Controlled signal $\alpha = 0.98$ ($k = 15$).

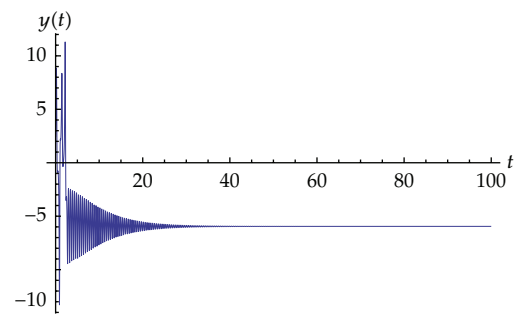


Figure 11: Controlled signal $\alpha = 0.97$ ($k = 7$).

$O = (0, 0, 0)$, $P_1 = (-0.125(k + \sqrt{1600 + k^2}), -0.125(k + \sqrt{1600 + k^2}), 40)$ and $P_2 = (-0.125(k - \sqrt{1600 + k^2}), -0.125(k - \sqrt{1600 + k^2}), 40)$. The points P_1 and P_2 decide. The stability of the system. The Jacobian matrix is given by

$$J_1(x, y, z) = \begin{pmatrix} -10 & 10 & 0 \\ 40 - z & 0 & -x \\ k + 4x & 4y & -2.5 \end{pmatrix}. \quad (4.2)$$

One eigenvalue e_0 of the matrix J_1 at point P_1 is

$$\begin{aligned} & -4.1667 - \left(0.419974 \left(218.75 + 0.375k^2 + 0.375k\sqrt{1600 + k^2}\right)\right) \\ & / \left(-43843.8 - 19.6875k^2 - 19.6875k\sqrt{1600 + k^2}\right. \\ & \quad \left. + \sqrt{\left(\left(-43843.8 - 19.6875k^2 - 19.6875k\sqrt{1600 + k^2}\right)^2\right.}\right. \\ & \quad \left. \left.+ 4\left(218.75 + 0.375k^2 + 0.375k\sqrt{1600 + k^2}\right)^3\right)\right)^{1/3} \\ & - (0.132283 \pm 0.229122i) \left(-43843.8 - 19.6875k^2 - 19.6875k\sqrt{1600 + k^2}\right. \\ & \quad \left. + \sqrt{\left(\left(-43843.8 - 19.6875k^2 - 19.6875k\sqrt{1600 + k^2}\right)^2\right.}\right. \\ & \quad \left. \left.+ 4\left(218.75 + 0.375k^2 + 0.375k\sqrt{1600 + k^2}\right)^3\right)\right)^{1/3}. \end{aligned} \quad (4.3)$$

The eigenvalue e_0 is having negative real part for all $k \geq 0$ and hence stable for all $0 \leq \alpha \leq 1$. Other two complex-conjugate eigenvalues $\lambda_{+,-}$ are given by

$$\begin{aligned} & -4.1667 - \left((0.209987 \pm 0.363708i) \left(218.75 + 0.375k^2 + 0.375k\sqrt{1600 + k^2}\right)\right) \\ & / \left(-43843.8 - 19.6875k^2 - 19.6875k\sqrt{1600 + k^2}\right) \\ & \quad + \sqrt{\left(\left(-43843.8 - 19.6875k^2 - 19.6875k\sqrt{1600 + k^2}\right)^2\right.}\right. \\ & \quad \left. \left.+ 4\left(218.75 + 0.375k^2 + 0.375k\sqrt{1600 + k^2}\right)^3\right)\right)^{1/3} \\ & - (0.132283 \pm 0.229122i) \left(-43843.8 - 19.6875k^2 - 19.6875k\sqrt{1600 + k^2}\right) \\ & \quad + \sqrt{\left(\left(-43843.8 - 19.6875k^2 - 19.6875k\sqrt{1600 + k^2}\right)^2\right.}\right. \\ & \quad \left. \left.+ 4\left(218.75 + 0.375k^2 + 0.375k\sqrt{1600 + k^2}\right)^3\right)\right)^{1/3}. \end{aligned} \quad (4.4)$$

The stability of eigenvalues $\lambda_{+,-}$ depends on k . We have plotted the curves $|\arg(\lambda)|$ and $\alpha\pi/2$ for $\alpha = 0.97, 0.98, 0.99, 1$ in Figure 7. The intersection points of the curve $|\arg(\lambda)|$ with the curves $0.97\pi/2, 0.98\pi/2, 0.99\pi/2$ and $\pi/2$ are $k \approx 3.5, k \approx 8.5, k \approx 14$ and $k \approx 21$, respectively. Following stability condition (2.16), it is clear that the chaos in the system can be controlled if we take the value of k greater than the corresponding intersection point. Figure 8 shows controlled time series for $\alpha = 1$ and $k = 22$. Similarly, Figures 9, 10, and 11 show controlled time series for $\alpha = 0.99, 0.98, 0.97$, and $k = 20, 15, 7$, respectively.

5. Synchronization

Present section deals with synchronization of proposed fractional-order system. Consider the master system

$$\begin{aligned} D^\alpha x_1 &= a(y_1 - x_1), \\ D^\alpha y_1 &= bx_1 - lx_1z_1, \\ D^\alpha z_1 &= -cz_1 + hx_1^2 + my_1^2, \end{aligned} \quad (5.1)$$

and the slave system

$$\begin{aligned} D^\alpha x_2 &= a(y_2 - x_2) + u_1, \\ D^\alpha y_2 &= bx_2 - lx_2z_2 + u_2, \\ D^\alpha z_2 &= -cz_2 + hx_2^2 + my_2^2 + u_3. \end{aligned} \quad (5.2)$$

The unknown terms u_1, u_2, u_3 in (5.2) are control functions to be determined. Define the error functions as

$$e_1 = x_1 - x_2, \quad e_2 = y_1 - y_2, \quad e_3 = z_1 - z_2. \quad (5.3)$$

Equation (5.3) together with (5.1) and (5.2) yields the error system

$$\begin{aligned} D^\alpha e_1 &= a(e_2 - e_1) - u_1, \\ D^\alpha e_2 &= be_1 + lx_2z_2 - lx_1z_1 - u_2, \\ D^\alpha e_3 &= -ce_3 + h(x_1^2 - x_2^2) + m(y_1^2 - y_2^2) - u_3. \end{aligned} \quad (5.4)$$

The control terms u_i are chosen so that the system (5.4) becomes stable. There is not a unique choice for such functions. We choose

$$\begin{aligned} u_1 &= ae_2, \\ u_2 &= lx_2z_2 - lx_1z_1 + e_2, \\ u_3 &= h(x_1^2 - x_2^2) + m(y_1^2 - y_2^2). \end{aligned} \quad (5.5)$$

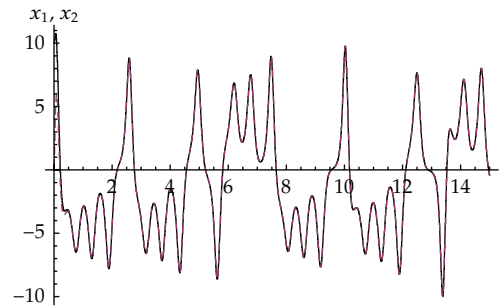


Figure 12: Synchronized signals $x_1; x_2$ ($\alpha = 0.99$).

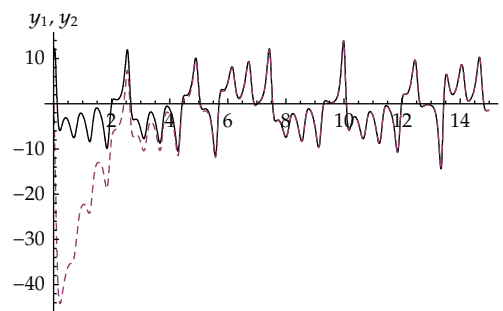


Figure 13: Synchronized signals $y_1; y_2$ ($\alpha = 0.99$).

With the choice of u_i given by (5.5), the error system (5.4) becomes

$$\begin{aligned} D^\alpha e_1 &= -ae_1 = -10e_1, \\ D^\alpha e_2 &= be_1 - e_2 = 40e_1 - e_2, \\ D^\alpha e_3 &= -ce_3 = -2.5e_3. \end{aligned} \quad (5.6)$$

The eigenvalues of the coefficient matrix of linear system (5.6) are $-10, -1$, and -2.5 . Hence, the stability condition (2.16) is satisfied for $0 \leq \alpha \leq 1$ and the errors $e_i(t)$ tend to zero as $t \rightarrow \infty$. Thus, we achieve the required synchronization. The simulation results in case $\alpha = 0.99$ are summarized in Figures 12–15. Synchronization is shown in Figure 12 (signals x_1, x_2), Figure 13 (signals y_1, y_2), and Figure 14 (signals z_1, z_2). Note that the master systems are shown by solid lines whereas slave systems are shown by dashed lines. The errors $e_1(t)$ (solid line), $e_2(t)$ (dashed line) and $e_3(t)$ (dot-dashed line) in the synchronization are shown in Figure 15. We have studied other cases of α namely, 0.97, and 0.98 but the results are omitted.

6. Conclusion

In the present work, we demonstrate the fractional order Lorenz-like system. We have observed that the system is chaotic for the fractional order $\alpha \geq 0.97$, that is, the minimum effective dimension of the system is 2.91. We have used simple linear feedback controller

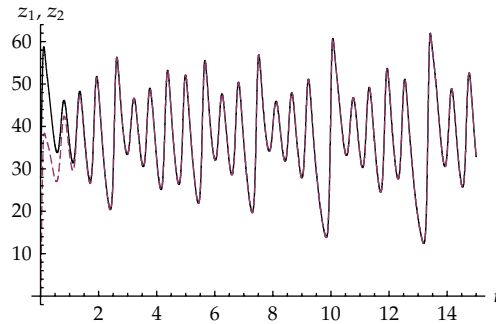


Figure 14: Synchronized signals $z_1; z_2$ ($\alpha = 0.99$).

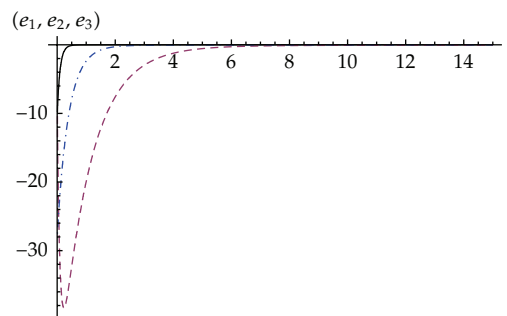


Figure 15: Synchronization errors.

$u = kx$, ($k > 0$) and given sufficient condition on k to control the chaos in the proposed system. Further, we have synchronized the system using feedback control.

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