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Research Article

Qualitative Analysis of Coating Flows on a Rotating Horizontal Cylinder

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We consider a nonlinear 4th-order degenerate parabolic partial differential equation that arises in modelling the dynamics of an incompressible thin liquid film on the outer surface of a rotating horizontal cylinder in the presence of gravity. The parameters involved determine a rich variety of qualitatively different flows. We obtain sufficient conditions for finite speed of support propagation and for waiting time phenomena by application of a new extension of Stampacchia's lemma for a system of functional equations.

1. Introduction

The time evolution of thickness of a viscous liquid film spreading over a solid surface under the action of the surface tension and gravity can be described by lubrication models [1–5]. These models approximate the full Navier-Stokes system that describes the motion of the liquid flow. Thin films play an increasingly important role in a wide range of applications, for example, packaging, barriers, membranes, sensors, semiconductor devices, and medical implants [6–8].

In this paper we consider the dynamics of a viscous incompressible thin fluid film on the outer surface of a horizontal circular cylinder that is rotating around its axis in the presence of a gravitational field. The motion of the liquid film is governed by four physical effects: viscosity, gravity, surface tension, and centrifugal forces. These are reflected in the parameters: R: the radius of the cylinder, ω: its rate of rotation (assumed constant), g: the acceleration due to gravity, ν: the kinematic viscosity, ρ: the fluid's density, and σ: the surface tension. These parameters yield three independent dimensionless numbers: the Reynolds number $\mathrm{Re} = (R^2\omega)/\nu$, $\gamma = g/(R\omega^2)$, and the Weber number $\mathrm{We} = (\rho R^3\omega^2)/\sigma$. The understanding of coating flow dynamics is important for industrial printing process where rotating cylinder transports the coating material in the form of liquid paint. The rotating thin fluid film can exhibit variety of different behaviour including: interesting pattern formations ("shark teeth" and "duck bill" patterns), fluid curtains, hydroplaning drops, and frontal avalanches [8–10]. As a result, the coating flow has been the subject of continuous study since the pioneering model was derived in 1977 by Moffatt (see [11]):

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial \theta}\left[\omega h - \frac{1}{3}\frac{g}{\nu R}h^3 \cos\theta\right] = 0. \tag{1.1}$$

The surface tension and inertial effects were neglected in (1.1). Here $h(x,t)$ is the thickness of the fluid film, θ is a rotation angle, and t is a time variable. The linear stability of rigidly rotating films on a rotating circular cylinder under three-dimensional disturbances was examined in [12, 13]. It was shown that the most unstable mode for thin film flows on the surface of a cylinder is the purely axial one that leads to so-called "ring instabilities". During the past decade, coating and rimming problems attracted many researchers who analyzed different types of flow regime asymptotically [14–17] and numerically [18–20]. For a detailed review of a growing literature on different types of thin film flows please see [21] and references there in.

The coating flow is generated by viscous forces due to cylinder's surface motion relative to the fluid. There is no temperature gradient, hence the interface does not experience a shear stress. If the cylinder is fully coated there is only one free boundary where the liquid meets the surrounding air. Otherwise, there is also a free boundary (or contact line) where the air and liquid meet the cylinder's surface.

The asymptotic evolution equation for the thickness of the fluid film with the surface tension effect:

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial \theta}\left[\omega h - \frac{1}{3}\frac{g}{\nu R}h^3 \cos\theta + \frac{1}{3}\frac{\sigma}{\rho R^4 \nu}h^3\left(\frac{\partial h}{\partial \theta} + \frac{\partial^3 h}{\partial \theta^3}\right)\right] = 0, \tag{1.2}$$

was derived by Pukhnachev [22] in 1977. It is valid under the assumptions that the fluid film is thin $h \ll R$ and its slope is small $(1/R)(\partial h/\partial\theta) \ll 1$. Later in 2009, taking into account inertial effects, Kelmanson [23] presented a more general model:

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial \theta}\left[\omega h - \frac{1}{3}\frac{g}{\nu R}h^3 \cos\theta + \frac{1}{3}\frac{\sigma}{\rho R^4 \nu}h^3\left(\frac{\partial h}{\partial \theta} + \frac{\partial^3 h}{\partial \theta^3}\right) + \frac{1}{3}\frac{\omega^2 \rho}{\nu R}h^3\frac{\partial h}{\partial \theta}\right] = 0. \tag{1.3}$$

He analyzed, asymptotically and numerically, diverse effects of inertia in both small- and large-surface-tension limits.

We should mention that all three lubrication approximation models described above were based on the assumption of the no-slip boundary condition. It is well known [24] that the combination of constant viscosity and no-slip boundary conditions at the liquid-solid interface yields a logarithmic divergence in the rate of dissipation at moving contact line, that is, an infinite energy is needed to make the droplet expand. The most common way to overcome this difficulty is to introduce effective slip conditions (see (2.1)) that indeed removes the force singularity at advancing contact lines (see [25]).

The main goal of our paper is to study waiting time phenomenon for the coating flows under an assumption of effective slip conditions, that is, we analyze (2.1) that is a modified version of (1.3). Our approach is based on now well-established nonlinear PDE analysis for degenerate higher order parabolic equations.

The sufficient conditions: $h_0(x) \leq A|x|^{4/n}$ for $0 < n < 2$, $|h_{0x}(x)| \leq B|x|^{4/n-1}$ for $2 \leq n < 3$, (where A and B are some positive constants) on nonnegative initial data, h_0 for the occurrence of waiting time phenomena were derived by Dal Passo et al. [26] for the classic thin film equation:

$$h_t + \left(|h|^n h_{xxx}\right)_x = 0. \tag{1.4}$$

These results were based on an energy method developed in [27] for quasilinear parabolic equations. To the best of our knowledge, there is only one publication [28], where the waiting time phenomenon in the classic thin film equation (1.4) was discovered for $h_0(x) \sim |x|^{\alpha}$ for $2 < \alpha < 4/n$. The result was obtained by means of matching asymptotic methods and was supported by numerous numerical simulations. For more general nonlinear degenerate parabolic equations with nonlinear lower order terms the waiting time phenomenon was analyzed in [29–31].

It is well known [32] that the similarity solutions of the second order nonlinear parabolic equation:

$$c_t = (c^m c_x)_x, \quad m > 0, \tag{1.5}$$

subject to prescribing appropriate initial data, demonstrate the existence of a waiting-time phenomena before the free boundary moves. The comparison theorem, that is not applicable in our case, then enabled a number of results to be obtained about the existence and length of waiting times for general initial data. Our approach is completely different and based on local entropy/energy functional estimates.

We also analyze speed of support propagation and obtain an upper bound on it for the modified version of (1.3) (see (2.1)). The first finite speed results for nonnegative generalized solutions of the classic thin film equation (1.4) were obtained in [33, 34] for the case $0 < n < 2$ and $2 \leq n < 3$, respectively. For more general types of thin film equations the finite speed of support propagation phenomenon was studied in [35–39] (see also references there in).

The outline of our paper is as follows. We first prove for $n > 0$ the long-time existence of a generalized weak solution and then prove that it can have an additional regularity in Section 2. In Sections 3 and 4 we show finite speed support propagation in the "slow" convection case ($n > 1$): for $1 < n < 3$ and waiting time phenomena for $1 < n < 2$, accordingly. The general strategy is to use an extension of Stampacchia's lemma for a system of functional equations (see Lemma 3.1 [26], where this extension is proved for a single equation and Lemma A.2 in [37], where this extension is proved for systems in the homogeneous case). This result to our knowledge is new and might be of independent interest. We leave as an open problem the "fast" convection case ($0 < n < 1$): finite speed of support propagation and sufficient conditions for waiting time phenomenon.

2. Existence and Regularity of Solutions

We are interested in the existence of nonnegative generalized weak solutions to the following initial-boundary value problem:

$$(P)\begin{cases} h_t + \left(f(h)(a_0 h_{xxx} + a_1 h_x + w_x)\right)_x = 0 & \text{in } Q_T, \\ \dfrac{\partial^i h}{\partial x^i}(-a,t) = \dfrac{\partial^i h}{\partial x^i}(a,t) & \text{for } t > 0,\ i = \overline{0,3}, \\ h(x,0) = h_0(x) \geqslant 0, \end{cases} \tag{2.1}$$

where $f(h) = |h|^n$, $h = h(x,t)$, $\Omega = (-a,a)$, $Q_T = (0,T) \times \Omega$, $n > 0$, $a_0 > 0$, $a_1 \geq 0$, and $w(x,t)$ such that

$$w(x,\cdot) \in W^1_\infty(0,T) \quad \text{for a.e. } x \in \overline{\Omega}, \qquad w(\cdot,t) \in W^2_\infty(\Omega) \quad \text{for a.e. } t \in [0,T]. \tag{2.2}$$

ote that (1.3) is a particular case of (2.1) that corresponds to $n = 3$ and $w(x,t) = \cos(x - \omega t)$.

We consider a generalized weak solution in the following sense [40, 41].

Definition 2.1. A generalized weak solution of problem (P) is a nonnegative function h satisfying

$$\begin{gathered} h \in C^{1/2,1/8}_{x,t}\left(\overline{Q}_T\right) \cap L^\infty\left(0,T; H^1(\Omega)\right), \qquad h_t \in L^2\left(0,T; \left(H^1(\Omega)\right)'\right), \\ h \in C^{4,1}_{x,t}(\mathcal{P}), \qquad \sqrt{f(h)}\ (a_0 h_{xxx} + a_1 h_x + w_x) \in L^2(\mathcal{P}), \end{gathered} \tag{2.3}$$

where $\mathcal{P} := \{h > 0\}$. The solution h satisfies (2.1) in the following sense:

$$\int_0^T \langle h_t(\cdot,t), \phi \rangle dt - \iint_{\mathcal{P}} f(h)(a_0 h_{xxx} + a_1 h_x + w_x)\phi_x dx dt = 0, \tag{2.4}$$

for all $\phi \in C^1(Q_T) \cap C(\overline{Q}_T)$ with $\phi(-a,\cdot) = \phi(a,\cdot)$;

$$h(\cdot,t) \longrightarrow h(\cdot,0) = h_0 \text{ pointwise \& strongly in } L^2(\Omega) \quad \text{as } t \longrightarrow 0, \tag{2.5}$$

$$h(-a,t) = h(a,t) \quad \forall t \in [0,T], \ \frac{\partial^i h}{\partial x^i}(-a,t) = \frac{\partial^i h}{\partial x^i}(a,t), \tag{2.6}$$

for $i = 1,2,3$ at all points of the lateral boundary where $h \neq 0$.

Because the second term of (2.4) has an integral over $\{h > 0\}$ rather than over Q_T, the generalized weak solution is "weaker" than a standard weak solution. Here, $\{h > 0\}$ is short hand for $\{(x,t) \in \overline{Q}_T : h(x,t) > 0\}$. This short hand is used throughout: the time interval included in $\{h > 0\}$ is to be inferred from the context it appears in.

A key object for proving additional properties of a generalized weak solution is an integral quantity introduced by Bernis and Friedman [42]: the "entropy" $\int G_0(h(x,t))dx$. The function $G_0(z)$ is defined by

$$G_0(z) := \begin{cases} \dfrac{z^{2-n}}{(2-n)(1-n)} + dz + c & \text{if } n \neq 1, 2, \\ z \ln z - z + e & \text{if } n = 1, \\ -\ln z + \dfrac{z}{e} + 1 & \text{if } n = 2, \end{cases} \tag{2.7}$$

where

$$d = \begin{cases} 1 & \text{if } 1 < n < 2, \\ 0 & \text{otherwise}, \end{cases} \qquad c = \begin{cases} \dfrac{(n-1)^{1/(1-n)}}{2-n} & \text{if } 1 < n < 2, \\ 0 & \text{otherwise}. \end{cases} \tag{2.8}$$

By construction, G_0 is a nonnegative convex function on $[0,\infty)$. For $1 \le n \le 2$, the linear part of G_0 is chosen to ensure that G_0 has a positive lower bound on $[0,\infty)$. Also in the statement of Theorem 2.2 we use an "α-entropy", $\int G_0^{(\alpha)}(h(x,t))dx$, where

$$G_0^{(\alpha)}(z) := \begin{cases} z \ln z - z + e & \text{if } \alpha = n-1, \\ -\ln z + \dfrac{z}{e} + 1 & \text{if } \alpha = n-2, \\ \dfrac{z^{2-n+\alpha}}{(2-n+\alpha)(1-n+\alpha)} + dz + c & \text{otherwise}, \end{cases} \tag{2.9}$$

$$d = \begin{cases} 1 & \text{if } \alpha \in (n-2, n-1), \\ 0 & \text{otherwise}, \end{cases} \qquad c = \begin{cases} \dfrac{(n-1-\alpha)^{1/(1+\alpha-n)}}{2+\alpha-n} & \text{if } \alpha \in (n-2, n-1), \\ 0 & \text{otherwise}. \end{cases} \tag{2.10}$$

$G_0^{(\alpha)}$ is a nonnegative convex function on $[0,\infty)$. The linear part of $G_0^{(\alpha)}$ is chosen to ensure that $G_0^{(\alpha)}$ has a positive lower bound on $[0,\infty)$ if $n-2 \le \alpha \le n-1$. If $\alpha = 0$, the α-entropy is the same as the entropy (2.7).

Theorem 2.2. *(a) (**Existence**). Let $n > 0$ and the nonnegative initial data $h_0 \in H^1(\Omega)$, $h_0(-a) = h_0(a)$ satisfy*

$$\int_\Omega G_0(h_0)dx < \infty. \tag{2.11}$$

Then for any time $0 < T < \infty$ there exists a nonnegative generalized weak solution, h, on Q_T in the sense of the Definition 2.1. Furthermore,

$$h \in L^2\Big(0, T; H^2(\Omega)\Big). \tag{2.12}$$

Let

$$\mathcal{E}_0(T) := \frac{1}{2}\int_\Omega \Big(a_0 h_x^2(x,T) - a_1 h^2(x,T) - 2w(x,T)h(x,T)\Big)dx \tag{2.13}$$

then the weak solution satisfies

$$\mathcal{E}_0(T) + \iint_{\{h>0\}} h^n (a_0 h_{xxx} + a_1 h_x + w_x)^2 dx dt \le \mathcal{E}_0(0) - \iint_{Q_T} h w_t dx dt. \tag{2.14}$$

*(b) (**Regularity**). If the initial data also satisfies*

$$\int_\Omega G_0^{(\alpha)}(h_0)dx < \infty, \tag{2.15}$$

for some $-1/2 < \alpha < 1$, $\alpha \ne 0$ then the nonnegative generalized weak solution has the extra regularity $h^{(\alpha+2)/2} \in L^2(0,T;H^2(\Omega))$ and $h^{(\alpha+2)/4} \in L^4(0,T;W_4^1(\Omega))$.

The theorem above was proved earlier in [41] for the case $n = 3$ only. We note that the analogue of Theorem 4.2 in [42] also holds: there exists a nonnegative weak solution with the integral formulation

$$\int_0^T \langle h_t(\cdot,t),\phi\rangle dt + a_0\iint_{Q_T}\Big(nh^{n-1}h_x h_{xx}\phi_x + h^n h_{xx}\phi_{xx}\Big)dx dt - \iint_{Q_T} h^n (a_1 h_x + w_x)\phi_x dx dt = 0. \tag{2.16}$$

If initial data satisfy finite α-entropy condition, that is, $\int G_0^{(\alpha)}(h_0)dx < \infty$ then one can prove existence of nonnegative solutions with some additional regularity properties and use an integral formulation [43] to define them that is similar to that of (2.16) except that the second integral is replaced by the results of one more integration by parts (there are no h_{xxx} terms). It is worth to mention that for the case $0 < n < 2$ the finite entropy assumption in Theorem 2.2 can be omitted because it does not impose any restriction on nonnegative initial data. One needs to impose finite entropy and finite α-entropy conditions on initial data if $n \ge 2$ only.

2.1. Regularized Problem

Given δ, $\varepsilon > 0$, a regularized parabolic problem, similar to one that was studied by Bernis and Friedman [42] can be written as:

$(P_{\delta,\varepsilon})$

$$h_t + \big(f_{\delta\varepsilon}(h)(a_0 h_{xxx} + a_1 h_x + w_x)\big)_x = 0, \tag{2.17}$$

$$\frac{\partial^i h}{\partial x^i}(-a,t) = \frac{\partial^i h}{\partial x^i}(a,t) \quad \text{for } t > 0,\ i = \overline{0,3}, \tag{2.18}$$

$$h(x,0) = h_{0,\varepsilon}(x), \tag{2.19}$$

where

$$f_{\delta\varepsilon}(z) := f_\varepsilon(z) + \delta = \frac{|z|^4}{|z|^{4-n} + \varepsilon} + \delta \quad \forall z \in \mathbb{R}^1,\ \delta > 0,\ \varepsilon > 0. \tag{2.20}$$

The $\delta > 0$ in (2.20) makes the problem (2.17) regular (i.e., uniformly parabolic). The parameter ε is an approximating parameter which has the effect of increasing the degeneracy from $f(h) \sim |h|^n$ to $f_\varepsilon(h) \sim h^4$. The nonnegative initial data, h_0, is approximated via

$$\begin{gathered} h_0 + \varepsilon^\theta \leqslant h_{0,\varepsilon} \in C^{4+\gamma}(\Omega) \quad \text{for some } 0 < \theta < \frac{2}{5}, \\ \frac{\partial^i h_{0,\varepsilon}}{\partial x^i}(-a) = \frac{\partial^i h_{0,\varepsilon}}{\partial x^i}(a) \quad \text{for } i = \overline{0,3}, \\ h_{0,\varepsilon} \longrightarrow h_0 \text{ strongly in } H^1(\Omega) \quad \text{as } \varepsilon \longrightarrow 0. \end{gathered} \tag{2.21}$$

The ε term in (2.21) "lifts" the initial data so that they are smoothing from $H^1(\Omega)$ to $C^{4+\gamma}(\Omega)$. By Eĭdel'man [44, Theorem 6.3, p.302], the regularized problem has a unique classical solution $h_{\delta\varepsilon} \in C^{4+\gamma,1+\gamma/4}_{x,t}(\Omega \times [0, \tau_{\delta\varepsilon}])$ for some time $\tau_{\delta\varepsilon} > 0$. For any fixed value of δ and ε, by Eĭdel'man [44, Theorem 9.3, p.316] if one can prove a uniform in time a priori bound $|h_{\delta\varepsilon}(x,t)| \leq A_{\delta\varepsilon} < \infty$ for some longer time interval $[0, T_{\text{loc},\delta\varepsilon}]$ ($T_{\text{loc},\delta\varepsilon} > \tau_{\delta\varepsilon}$) and for all $x \in \Omega$ then Schauder-type interior estimates [44, Corollary 2, p.213] imply that the solution $h_{\delta\varepsilon}$ can be continued in time to be in $C^{4+\gamma,1+\gamma/4}_{x,t}(\Omega \times [0, T_{\text{loc},\delta\varepsilon}])$.

Although the solution $h_{\delta\varepsilon}$ is initially positive, there is no guarantee that it will remain nonnegative. The goal is to take $\delta \to 0$, $\varepsilon \to 0$ in such a way that (1) $T_{\text{loc},\delta\varepsilon} \to T_{\text{loc}} > 0$, (2) the solutions $h_{\delta\varepsilon}$ converge to a (nonnegative) limit, h, which is a generalized weak solution, and (3) h inherits certain a priori bounds. This is done by proving various a priori estimates for $h_{\delta\varepsilon}$ that are uniform in δ and ε and hold on a time interval $[0, T_{\text{loc}}]$ that is independent of δ and ε. As a result, $\{h_{\delta\varepsilon}\}$ will be a uniformly bounded and equicontinuous (in the $C^{1/2,1/8}_{x,t}$ norm) family of functions in $\overline{\Omega} \times [0, T_{\text{loc}}]$. Taking $\delta \to 0$ will result in a family of functions $\{h_\varepsilon\}$ that are classical, positive, unique solutions to the regularized problem with $\delta = 0$. Taking $\varepsilon \to 0$ will then result in the desired generalized weak solution h. This last step is where the possibility of nonunique weak solutions arise; see [40] for simple examples of how such constructions applied to $h_t = -(|h|^n h_{xxx})_x$ can result in two different solutions arising from the same initial data.

2.2. A Priori Estimates

Our first task is to derive a priori estimates for classical solutions of (2.17)–(2.21). The lemmas given in this section are proved in the Section 4.

We use an integral quantity based on a function $G_{\delta\varepsilon}$ chosen such that

$$G''_{\delta\varepsilon}(z) = \frac{1}{f_{\delta\varepsilon}(z)}, \quad G_{\delta\varepsilon}(z) \geqslant 0. \tag{2.22}$$

This is analogous to the "entropy" function first introduced by Bernis and Friedman [42].

Lemma 2.3. *Let $h_{0\varepsilon}$ satisfy* (2.21) *and be built from a nonnegative function h_0 that satisfies the hypotheses of Theorem 2.2. Then there exist $\delta_0 > 0$, $\varepsilon_0 > 0$ and time $T_{\text{loc}} > 0$ such that if $\delta \in [0, \delta_0)$, $\varepsilon \in [0, \varepsilon_0)$, and $h_{\delta\varepsilon}$ is a solution of the problem* (2.17)–(2.21) *with initial data $h_{0\varepsilon}$, then for any $T \in [0, T_{\text{loc}}]$ the following inequalities:*

$$\int_\Omega \left\{ h_{\delta\varepsilon,x}^2(x,T) + \frac{2c_1}{a_0} G_{\delta\varepsilon}(h_{\delta\varepsilon}(x,T)) \right\} dx + a_0 \iint_{Q_T} f_{\delta\varepsilon}(h_{\delta\varepsilon}) h_{\delta\varepsilon,xxx}^2 \, dx\,dt \le K_1 < \infty, \tag{2.23}$$

$$\int_\Omega G_{\delta\varepsilon}(h_{\delta\varepsilon}(x,T))dx + a_0 \iint_{Q_T} h_{\delta\varepsilon,xx}^2 \, dx\,dt \le K_2 < \infty \tag{2.24}$$

hold. The energy $\mathcal{E}_{\delta\varepsilon}(t)$ (see (2.13)*) satisfies*

$$\mathcal{E}_{\delta\varepsilon}(T) + \iint_{Q_T} f_{\delta\varepsilon}(h_{\delta\varepsilon})\left(a_0 h_{\delta\varepsilon,xxx} + a_1 h_{\delta\varepsilon,x} + w_x\right)^2 dx\,dt = \mathcal{E}_{\delta\varepsilon}(0) - \iint_{Q_T} h_{\delta\varepsilon} w_t \, dx\,dt. \tag{2.25}$$

The time T_{loc} and the constants K_i are independent of δ and ε.

The proof of existence of δ_0, ε_0, T_{loc}, K_1, and K_2 is constructive; how to find them and what quantities determine them are shown with details in Section 4.

Lemma 2.3 yields uniform-in-δ-and-ε bounds for $\int h_{\delta\varepsilon,x}^2$, $\int G_{\delta\varepsilon}(h_{\delta\varepsilon})$, $\iint h_{\delta\varepsilon,xx}^2$, and $\iint f_{\delta\varepsilon}(h_{\delta\varepsilon}) h_{\delta\varepsilon,xxx}^2$. However, these bounds are found in a different manner than in earlier work for the equation $h_t = -(|h|^n h_{xxx})_x$, for example. Although the inequality (2.24) is unchanged, the inequality (2.23) has an extra term involving $G_{\delta\varepsilon}$. In the proof, this term was introduced to control additional, lower-order terms. This idea of a "blended" $\|h_x\|_2$-entropy bound was first introduced by Shishkov and Taranets for long-wave stable thin film equations with convection [30].

The final a priori bounds for positive, classical solutions use the following functions, parameterized by α for $\alpha \notin \{2,3\}$,

$$G_\varepsilon^{(\alpha)}(z) = G_0^{(\alpha)}(z) + \varepsilon \frac{z^{\alpha-2}}{(\alpha-3)(\alpha-2)} \Longrightarrow \left(G_\varepsilon^{(\alpha)}(z)\right)'' = \frac{z^\alpha}{f_\varepsilon(z)}, \tag{2.26}$$

where $G_0^{(\alpha)}$ is given by (2.9). In the following lemma, we restrict ourselves to the case $\alpha \in [-1/2, 1]$; note that $G_\varepsilon^{(\alpha)}(z) \ge 0$ for such α.

Lemma 2.4. *Assume ε_0 and T_{loc} are from Lemma 2.3, $\delta = 0$, and $\varepsilon \in [0, \varepsilon_0)$. Assume $\alpha \in [-1/2, 1]$ and that h_ε is a positive, classical solution of the problem* (2.17)–(2.21) *with initial data $h_{0,\varepsilon}$ satisfying Lemma 2.3. If the initial data $h_{0,\varepsilon}$ is built from h_0 which also satisfies*

$$\int_\Omega G_0^{(\alpha)}(h_0(x))dx < \infty \tag{2.27}$$

then there exists K_4 such that

$$\int_\Omega \left\{ h_{\varepsilon,x}^2(x,T) + G_\varepsilon^{(\alpha)}(h_\varepsilon(x,T)) \right\} dx + \iint_{Q_T} \left[\beta h_\varepsilon^\alpha h_{\varepsilon,xx}^2 + \gamma h_\varepsilon^{\alpha-2} h_{\varepsilon,x}^4\right] dx\,dt \le K_4 < \infty \tag{2.28}$$

holds for all $T \in [0, T_{\text{loc}}]$ *and* K_4 *is independent of* ε *and is determined by* α, ε_0, a_0, a_1, w_x, Ω *and* h_0. *Here*

$$\beta = \begin{cases} a_0 & \text{if } 0 \le \alpha \le 1, \\ a_0 \dfrac{1+2\alpha}{4(1-\alpha)} & \text{if } -\dfrac{1}{2} \le \alpha < 0, \end{cases} \qquad \gamma = \begin{cases} a_0 \dfrac{\alpha(1-\alpha)}{6} & \text{if } 0 \le \alpha \le 1, \\ a_0 \dfrac{(1+2\alpha)(1-\alpha)}{36} & f \ -\dfrac{1}{2} \le \alpha < 0. \end{cases} \tag{2.29}$$

Furthermore, if $\alpha \in (-1/2, 1) \setminus \{0\}$ *then*

$$\left\{ h_\varepsilon^{(\alpha+2)/2} \right\}_{\varepsilon \in (0,\varepsilon_0)} \subset L^2\Big(0, T_{\text{loc}}; H^2(\Omega)\Big), \qquad \left\{ h_\varepsilon^{(\alpha+2)/4} \right\}_{\varepsilon \in (0,\varepsilon_0)} \subset L^4\Big(0, T_{\text{loc}}; W^{1,4}(\Omega)\Big) \tag{2.30}$$

are uniformly bounded.

The α-entropy, $\int G_0^{(\alpha)}(h)dx$, was first introduced for $\alpha = -1/2$ in [45] and an a priori bound like that of Lemma 2.4 and regularity results like those of Theorem 2.2 were found simultaneously and independently in [40, 43].

The proof of existence starts from a construction of a classical solution $h_{\delta\varepsilon}$ on $[0, T_{\text{loc}}]$ that satisfies the hypotheses of Lemma 2.3 if $\delta \in (0, \delta_0)$ and $\varepsilon \in (0, \varepsilon_0)$. Taking the regularizing parameter, δ, to zero, one proves that there is a limit h_ε and that h_ε is a generalized weak solution. After that additional nonlinear estimates are required to analyze properties of the limit h_ε; specifically to show that it is strictly positive, classical, and unique. Hence, the a priori bounds given by Lemmas 2.3 and 2.4 are applicable to h_ε. This allows us to take the approximating parameter, ε, to zero and to construct the desired nonnegative generalized weak solution of Theorems 2.2 (see, e.g., [41]).

2.3. Long-Time Existence of Solutions

Lemma 2.5. *Let* h *be a generalized solution of Theorem 2.2. Then*

$$\frac{a_0}{4} \|h(\cdot, T_{\text{loc}})\|_{H^1(\Omega)}^2 \le \mathcal{E}_0(0) + K_5 + K_6 T_{\text{loc}}, \tag{2.31}$$

where $\mathcal{E}_0(0)$ *is defined in* (2.13), $M = \int h_0$, *and*

$$K_5 = \|w\|_\infty M + \frac{2\sqrt{6}}{3} \frac{(a_0 + a_1)^{3/2}}{a_0} M^2 + \frac{a_0 + a_1}{2} \frac{M^2}{|\Omega|}, \qquad K_6 = \|w_t\|_\infty M. \tag{2.32}$$

Proof of Lemma 2.5. By (2.13),

$$\frac{a_0}{2} \int_\Omega h_x^2(x, T)dx \le \mathcal{E}_0(T) + \frac{a_1}{2} \int_\Omega h^2(x, T)dx + \int_\Omega h(x, T)w(x, T)dx - \iint_{Q_T} h w_t dx dt. \tag{2.33}$$

The linear-in-time bound (2.14) on $\mathcal{E}_0(T_{\mathrm{loc}})$ then implies

$$\frac{a_0}{2}\|h(\cdot,T_{\mathrm{loc}})\|_{H^1}^2 \le \mathcal{E}_0(0) + \frac{a_0+a_1}{2}\int_\Omega h^2 dx + (\|w\|_\infty + \|w_t\|_\infty T)M. \tag{2.34}$$

Using the estimate (see [41, Lemma 4.1, page 1837])

$$\|h\|_{L^2(\Omega)}^2 \le 6^{2/3}M^{4/3}\left(\int_\Omega h_x^2\, dx\right)^{1/3} + \frac{M^2}{|\Omega|}, \tag{2.35}$$

and Young's inequality:

$$\begin{aligned}\frac{a_0+a_1}{2}\int_\Omega h^2 dx &\le \frac{a_0+a_1}{2}\left(6^{2/3}M^{4/3}\left(\int_\Omega h_x^2 dx\right)^{1/3} + \frac{M^2}{|\Omega|}\right)\\ &\le \frac{a_0}{4}\int_\Omega h_x^2(x,T_{\mathrm{loc}})dx + \frac{2\sqrt{6}}{3}\frac{(a_0+a_1)^{3/2}}{\sqrt{a_0}}M^2 + \frac{a_0+a_1}{2}\frac{M^2}{|\Omega|}.\end{aligned} \tag{2.36}$$

Using this in (2.34), the desired bound (2.31) follows immediately. □

This H^1-estimate will be used to extend the short-time existence of a solution to the long-time existence result of Theorem 2.2 (see [41, Proof of Theorem 3, page 1838]).

3. Finite Speed of Support Propagation

Theorem 3.1. *Let $1 < n < 3$. Assume h_0 is nonnegative, $h_0 \in H^1(\Omega)$ and* supp $h_0 \subset (-r_0, r_0) \Subset \Omega$. *Then the solution h of Theorem 2.2 has finite speed of support propagation, that is, there exists a continuous nondecreasing function $\Gamma(T)$, $\Gamma(0) = 0$ such that* supp $h(T,\cdot) \subset (-r_0-\Gamma(T), r_0+\Gamma(T)) \Subset \Omega$ *for all $T \le T_0 := \Gamma^{-1}(a - r_0)$.*

In the following theorem, we find the explicit upper bounds of the $\Gamma(T)$ for a solution of the corresponding Cauchy problem with a compactly supported nonnegative initial data $h_0 \in H^1(\mathbb{R}^1)$. Note that the definition of generalized weak solution of the Cauchy problem is as Definition 2.1 except that Ω is replaced by $\mathbb{R}^1$ and the relation (2.6) is dropped. Using Lemma 2.5, we can show that the upper estimate of $\Gamma(T)$ from Theorem 3.1 is independent on Ω therefore the solution from Theorem 2.2 can be extended to be identically zero for $|x| > r_0 - \Gamma(T)$ and thus is a solution on the line for all $T \le T_0$. Performing a similar procedure in $[T_0, 2T_0], \ldots, [mT_0, (m+1)T_0], \ldots$, we obtain a compactly supported nonnegative solution of the Cauchy problem for all $T \ge 0$ and Theorem 2.2 holds with $\Omega = \mathbb{R}^1$.

Theorem 3.2. *Let $1 < n < 3$. Assume h_0 is nonnegative, $h_0 \in H^1(\mathbb{R}^1)$,* supp $h_0 \subset (-r_0, r_0)$ *and h is a solution of the Cauchy problem. Then the following estimates:*

$\Gamma(T) \le D_1(T^{1/(n+4)} + T^{5/(n+4)})$ *for all $T > 0$ if $1 < n < 2$,*

$\Gamma(T) \le D_2 T^{1/(n+4)}$ *for small enough time if $2 \le n < 3$,*

are valid. Here the constants D_i depend on the parameters problem and initial data only.

3.1. Proof of Theorem 3.1 for the Case $1 < n < 2$

The following lemma contains the local entropy estimate. The proof of Lemma 3.3 is similar to (A.16), (A.29), therefore it is omitted.

Lemma 3.3. *Let* $\zeta \in C^{1,2}_{t,x}(\overline{Q}_T)$ *such that* $\operatorname{supp}\zeta \subset \Omega$, $(\zeta^4)' = 0$ *on* $\partial\Omega$, *and* $\zeta^4(-a,t) = \zeta^4(a,t)$. *Assume that* $-1/2 < \alpha < 1$, *and* $\alpha \neq 0$. *Then there exist constants* C_i $(i = 1,2,3)$ *dependent on* n, m, α, a_0, *and* a_1, *independent of* Ω, *such that for all* $0 < T < \infty$

$$\begin{aligned}
&\int_\Omega \zeta^4(x,T)G_0^{(\alpha)}(h(x,T))dx - \iint_{Q_T}\left(\zeta^4\right)_t G_0^{(\alpha)}(h)dxdt \\
&\quad + C_1\iint_{Q_T}\left(h^{(\alpha+2)/2}\right)^2_{xx}\zeta^4 dxdt \le \int_\Omega \zeta^4(x,0)G_0^{(\alpha)}(h_0)dx \\
&\quad + C_2\iint_{Q_T} h^{\alpha+2}\left(\zeta^4 + \zeta_x^4 + \zeta^2\zeta_{xx}^2 + \zeta^2\zeta_x^2 + \zeta^3|\zeta_{xx}|\right)dxdt \\
&\quad + C_3\iint_{Q_T} h^{\alpha+1}\left(\left|\zeta^3\right||\zeta_x| + \zeta^4\right)dxdt.
\end{aligned} \tag{3.1}$$

Let $0 < n < 2$, and let $\operatorname{supp} h_0 \subseteq (-r_0, r_0) \Subset \Omega$. For an arbitrary $s \in (0, a - r_0)$ and $\delta > 0$ we consider the families of sets

$$\Omega(s) = \Omega \setminus (-r_0 - s, r_0 + s), \qquad Q_T(s) = (0,T) \times \Omega(s). \tag{3.2}$$

We introduce a nonnegative cutoff function $\eta(\tau)$ from the space $C^2(\mathbb{R}^1)$ with the following properties:

$$\eta(\tau) = \begin{cases} 0 & \text{if } \tau \le 0, \\ \tau^2(3 - 2\tau) & \text{if } 0 < \tau < 1, \\ 1 & \text{if } \tau \le 1. \end{cases} \tag{3.3}$$

Next we introduce our main cut-off functions $\eta_{s,\delta}(x) \in C^2(\overline{\Omega})$ such that $0 \le \eta_{s,\delta}(x) \le 1$ for all $x \in \overline{\Omega}$ and possess the following properties:

$$\eta_{s,\delta}(x) = \eta\left(\frac{|x| - (r_0 + s)}{\delta}\right) = \begin{cases} 1, & x \in \Omega(s+\delta), \\ 0, & x \in \Omega \setminus \Omega(s), \end{cases} \quad \left|(\eta_{s,\delta})_x\right| \le \frac{3}{\delta}, \ \left|(\eta_{s,\delta})_{xx}\right| \le \frac{6}{\delta^2}, \tag{3.4}$$

for all $s > 0$, $\delta > 0 : r_0 + s + \delta < a$. Choosing $\zeta^4(x,t) = \eta_{s,\delta}(x)e^{-t/T}$, from (3.1) we arrive at

$$\begin{aligned}
&\int_{\Omega(s+\delta)} h^{\alpha-n+2}(T)dx + \frac{1}{T}\iint_{Q_T(s+\delta)} h^{\alpha-n+2}dxdt + C\iint_{Q_T(s+\delta)}\left(h^{(\alpha+2)/2}\right)^2_{xx}dxdt \\
&\quad \le \frac{C}{\delta^4}\iint_{Q_T(s)} h^{\alpha+2}dxdt + \frac{C}{\delta}\iint_{Q_T(s)} h^{\alpha+1}dxdt =: C\sum_{i=1}^{2}\delta^{-\alpha_i}\iint_{Q_T(s)} h^{\xi_i},
\end{aligned} \tag{3.5}$$

for all $s \in (0, a - r_0)$, where $(n-1)_+ < \alpha < 1$ and $0 < \delta < 1$ is enough small. We apply the Nirenberg-Gagliardo interpolation inequality (see Lemma B.2) in the region $\Omega(s+\delta)$ to a function $v := h^{(\alpha+2)/2}$ with $a = (2\xi_i)/(\alpha+2)$, $b = (2(\alpha - n + 2))/(\alpha+2)$, $d = 2$, $i = 0$, $j = 2$, and $\theta_i = ((\alpha+2)(\xi_i - \alpha + n - 2))/(\xi_i(4\alpha - 3n + 8))$ under the conditions:

$$\alpha - n + 2 < \xi_i \quad \text{for } i = 1, 2. \tag{3.6}$$

Integrating the resulted inequalities with respect to time and taking into account (3.5), we arrive at the following relations:

$$\iint_{Q_T(s+\delta)} h^{\xi_i} \le C T^{1-(\theta_i \xi_i)/(\alpha+2)} \left(\sum_{i=1}^{2} \delta^{-\alpha_i} \iint_{Q_T(s)} h^{\xi_i} \right)^{1+\nu_i} + CT \left(\sum_{i=1}^{2} \delta^{-\alpha_i} \iint_{Q_T(s)} h^{\xi_i} \right)^{\xi_i/(\alpha-n+2)}, \tag{3.7}$$

where $\nu_i = (4(\xi_i - \alpha + n - 2))/(4\alpha - 3n + 8)$. These inequalities are true provided that

$$\frac{\theta_i \xi_i}{\alpha+2} < 1 \Longleftrightarrow \xi_i < 5\alpha - 4n + 10 \quad \text{for } i = 1, 2. \tag{3.8}$$

Simple calculations show that inequalities (3.6) and (3.8) hold with some $(n-1)_+ < \alpha < 1$ if and only if $1 < n < 2$. The finite speed of propagations follows from (3.7) by applying Lemma B.3 with $s_1 = 0$. Hence,

$$\operatorname{supp} h(T, \cdot) \subset (-r_0 - \Gamma(T), r_0 + \Gamma(T)) \Subset \Omega \quad \text{for all } T : T \in [0, T_0], \tag{3.9}$$

where $T_0 := \Gamma^{-1}(a - r_0)$.

3.2. Proof of Theorem 3.2 for the Case $1 < n < 2$

We can repeat the previous procedure from Section 3.1 for $\Omega(s) = \mathbb{R}^1 \setminus (-r_0 - s, r_0 + s)$ and we obtain

$$G_i(s+\delta) := \iint_{Q_T(s+\delta)} h^{\xi_i} \le C T^{1-(\theta_i \xi_i)/(\alpha+2)} \left(\sum_{i=1}^{2} \delta^{-\alpha_i} \iint_{Q_T(s)} h^{\xi_i} \right)^{1+\nu_i}, \tag{3.10}$$

instead of (3.7), and

$$\begin{aligned} \Gamma(T) = {} & C\Big(T^{(1-(\theta_1\xi_1)/(\alpha+2))(1+\nu_2)} T^{(1-(\theta_2\xi_2)/(\alpha+2))\nu_1(1+\nu_1)} (G(0))^{\nu_1} \Big)^{1/(4(1+\nu_1)(1+\nu_2))} \\ & + C\Big(T^{(1-(\theta_2\xi_2)/(\alpha+2))(1+\nu_1)} T^{(1-(\theta_1\xi_1)/(\alpha+2))\nu_2(1+\nu_2)} (G(0))^{\nu_2} \Big)^{1/((1+\nu_1)(1+\nu_2))}, \end{aligned} \tag{3.11}$$

where

$$G(0) = C\Big(T^{(1-(\theta_1\xi_1)/(\alpha+2))(1+\nu_2)} (G_2(0))^{1+\nu_1} + T^{(1-(\theta_2\xi_2)/(\alpha+2))(1+\nu_1)} (G_1(0))^{1+\nu_2} \Big). \tag{3.12}$$

Now we need to estimate $G(0)$. With that end in view, we obtain the following estimates:

$$G_i(0) \le C_1(C_2 + C_3 T)^{(\xi_i - 1)/(\alpha+5)} T^{1-(\xi_i-1)/(\alpha+5)}, \quad i = 1, 2, \tag{3.13}$$

where $1 < \xi_i < \alpha + 6$, and C_i depends on initial data only. Really, applying the Nirenberg-Gagliardo interpolation inequality (see Lemma B.2) in $\Omega = \mathbb{R}^1$ to a function $v := h^{(\alpha+2)/2}$ with $a = (2\xi_i)/(\alpha+2)$, $b = 2/(\alpha+2)$, $d = 2$, $i = 0$, $j = 2$, and $\widetilde{\theta}_i = ((\alpha+2)(\xi_i - 1))/(\xi_i(\alpha+5))$ under the condition $\xi_i > 1$, we deduce that

$$\int_{\mathbb{R}^1} h^{\xi_i} \le c\|h_0\|_1^{(2(3\xi_i+\alpha+2))/((\alpha+2)(\alpha+5))} \left(\int_{\mathbb{R}^1} \left(h^{(\alpha+2)/2}\right)_{xx}^2 dx\right)^{(\xi_i-1)/(\alpha+5)}. \tag{3.14}$$

Integrating (3.14) with respect to time and taking into account the Hölder inequality ($(\xi_i - 1)/(\alpha+5) < 1 \Rightarrow \xi_i < \alpha + 6$), we arrive at the following relations:

$$\iint_{Q_T} h^{\xi_i} \le c\|h_0\|_1^{(2(3\xi_i+\alpha+2))/((\alpha+2)(\alpha+5))} T^{1-(\xi_i-1)/(\alpha+5)} \left(\iint_{Q_T} \left(h^{(\alpha+2)/2}\right)_{xx}^2 dx\right)^{(\xi_i-1)/(\alpha+5)}. \tag{3.15}$$

From (3.15), due to (A.16) (as $\varepsilon \to 0$) and (2.31), we find (3.13).

Inserting (3.13) into (3.12), we obtain after straightforward computations that

$$\Gamma(T) \le C\left(T^{1/(n+4)} + T^{5/(n+4)}\right) \quad \text{for all } T \ge 0. \tag{3.16}$$

3.3. *Proof of Theorem 3.1 for the Case* $4/3 < n < 3$

The following lemma contains the local energy estimate. The proof of Lemma 3.4 is Appendix A.

Lemma 3.4. *Let $n \in (1/2, 3)$ and $\beta > (1-n)/3$. Let $\zeta \in C^2(\overline{\Omega})$ such that* $\operatorname{supp} \zeta$ *in Ω and $(\zeta^6)' = 0$ on $\partial\Omega$, and $\zeta(-a) = \zeta(a)$. Then there exist constants C_i $(i = \overline{1,3})$ dependent on n, m, a_0, and a_1, independent of Ω and ε, such that for any $0 < T < \infty$*

$$\begin{aligned}
&\int_\Omega \zeta^6 h_x^2(x,T)dx + \int_\Omega \zeta^4 h^{\beta+1}(T)dx + C_1 \iint_{Q_T} \zeta^6 \left(h^{(n+2)/2}\right)_{xxx}^2 dxdt \\
&\quad \le \int_\Omega \zeta^6 h_0^2(x)dx + \int_\Omega \zeta^4 h_0^{\beta+1} dx + C_2 \iint_{Q_T} h^{n+2}\left(\zeta^6 + \zeta_x^6 + \zeta^3|\zeta_{xx}|^3\right)dxdt \\
&\quad\quad + C_3 \iint_{Q_T} \left\{\chi_{\{\zeta>0\}} h^{n+3\beta-1} + h^n \zeta^6\right\} dxdt,
\end{aligned} \tag{3.17}$$

$$\begin{aligned}
&\int_\Omega \zeta^6 h_x^2(x,T)dx + C_1 \iint_{Q_T} \zeta^6 \left(h^{(n+2)/2}\right)_{xxx}^2 dxdt \leqslant \int_\Omega \zeta^6 h_0^2(x)dx \\
&\quad + C_2 \iint_{Q_T} h^{n+2}\left(\zeta^6 + \zeta_x^6 + \zeta^3|\zeta_{xx}|^3\right)dxdt + C_3 \iint_{Q_T} h^n \zeta^6 dxdt.
\end{aligned} \tag{3.18}$$

Let $\eta_{s,\delta}(x)$ be denoted by (3.4). Setting $\zeta^6(x) = \eta_{s,\delta}(x)$ into (3.17), after simple transformations, we obtain

$$\int_{\Omega(s+\delta)} h_x^2(x,T)dx + \int_{\Omega(s+\delta)} h^{\beta+1}(T)dx + C\iint_{Q_T(s+\delta)} \left(h^{(n+2)/2}\right)_{xxx}^2 dxdt$$
$$\leq \frac{C}{\delta^6}\iint_{Q_T(s)} h^{n+2}dxdt + C\iint_{Q_T(s)}\left\{h^{n+3\beta-1}+h^n\right\}dxdt =: C\sum_{i=1}^{3}\delta^{-\alpha_i}\iint_{Q_T(s)} h^{\xi_i}, \tag{3.19}$$

for all for all $s \in (0, a - r_0)$, $\delta > 0 : r_0 + s + \delta < a$. We apply the Nirenberg-Gagliardo interpolation inequality (see Lemma B.2) in the region $\Omega(s+\delta)$ to a function $v := h^{(n+2)/2}$ with $a = (2\xi_i)/(n+2)$, $b = (2(\beta+1))/(n+2)$, $d = 2$, $i = 0$, $j = 3$, and $\theta_i = ((n+2)(\xi_i - \beta - 1))/(\xi_i(n+5\beta+7))$ under the conditions:

$$\beta < \xi_i - 1 \quad \text{for } i = \overline{1,3}. \tag{3.20}$$

Integrating the resulted inequalities with respect to time and taking into account (3.19), we arrive at the following relations:

$$\iint_{Q_T(s+\delta)} h^{\xi_i} \leq C\, T^{1-(\theta_i\xi_i)/(n+2)}\left(\sum_{i=1}^{3}\delta^{-\alpha_i}\iint_{Q_T(s)} h^{\xi_i}\right)^{1+\nu_i} + CT\left(\sum_{i=1}^{3}\delta^{-\alpha_i}\iint_{Q_T(s)} h^{\xi_i}\right)^{\xi_i/(\beta+1)}, \tag{3.21}$$

where $\nu_i = (6(\xi_i - \beta - 1))/(n + 5\beta + 7)$. These inequalities are true provided that

$$\frac{\theta_i\xi_i}{n+2} < 1 \Longleftrightarrow \beta > \frac{\xi_i - n - 8}{6} \quad \text{for } i = \overline{1,3}. \tag{3.22}$$

Simple calculations show that inequalities (3.20) and (3.22) hold with some $\beta \in ((2-n)/2, n-1)$ if and only if $4/3 < n < 3$. Since all integrals on the right-hand sides of (3.21) vanish as $T \to 0$, the finite speed of propagations follows from (3.21) by applying Lemma B.3 with $s_1 = 0$ and sufficiently small T. Hence,

$$\operatorname{supp} h(T,\cdot) \subset (-r_0 - \Gamma(T), r_0 + \Gamma(T)) \Subset \Omega \quad \text{for all } T : 0 \leq T \leq T_0. \tag{3.23}$$

3.4. Proof of Theorem 3.2 for the Case $4/3 < n < 3$

Suppose that $\Omega(s) = \mathbb{R}^1 \setminus \{x : |x| < s\}$, $Q_T(s) = (0,T) \times \Omega(s)$ for all $s > r_0$, $\operatorname{supp} h_0 \subseteq (-r_0, r_0)$, and $\Gamma(T) = r(T) - r_0$. Since the time interval is small, we can assume that $r(T) < 2r_0$.

Hence, for all $s \in (r_0, 2r_0)$, we can take (up to regularization) $\zeta = (|x| - s)_+$ in (3.18). As a result, we obtain

$$\frac{1}{2}\int_{\Omega(s)} (|x|-s)_+^6 h_x^2 dx + \delta^6 C_1 \iint_{Q_T(s+\delta)} \left(h^{(n+2)/2}\right)_{xxx}^2 dx dt \leq C_4 \iint_{Q_T(s)} \left\{h^{n+2} + (r(T)-s)_+^6 h^n\right\} dx dt, \tag{3.24}$$

for all $T \leq T_0$, $s \in (r_0, 2r_0)$. Using the Hardy type inequality

$$\int_{\Omega(s)} (|x|-s)_+^{\alpha} f^2 dx \leq C_0 \int_{\Omega(s)} (|x|-s)_+^{\alpha+2} f_x^2 dx, \tag{3.25}$$

where $C_0 = 4/(\alpha+1)^2$ and $\alpha > -1$, we deduce that

$$\int_{\Omega(s+\delta)} h dx \leq \left(\int_{\Omega(s+\delta)} (|x|-s)_+^4 h^2 dx\right)^{1/2} \left(\int_{\Omega(s+\delta)} (|x|-s)_+^{-4} dx\right)^{1/2} \leq \left(\frac{C_0}{3\delta^3}\right)^{1/2} \left(\int_{\Omega(s)} (|x|-s)_+^6 h_x^2 dx\right)^{1/2}, \tag{3.26}$$

whence

$$\left(\int_{\Omega(s+\delta)} h dx\right)^2 \leq \frac{C_0}{3}\delta^{-3} \int_{\Omega(s)} (|x|-s)_+^6 h_x^2 dx, \tag{3.27}$$

for all $\delta > 0$, $s \in (r_0, 2r_0)$. Substituting (3.27) in (3.24), we get

$$\frac{3}{2C_0}\delta^{-3} \sup_t \left(\int_{\Omega(s+\delta)} h dx\right)^2 + C_1 \iint_{Q_T(s+\delta)} \left(h^{(n+2)/2}\right)_{xxx}^2 dx dt \leq \frac{C_4}{\delta^6} \iint_{Q_T(s)} \left\{h^{n+2} + \Gamma^6(T) h^n\right\} dx dt, \tag{3.28}$$

for all $T \le T_0$, $s \in (r_0, 2r_0)$. By the Nirenberg-Gagliardo, Hölder and Young inequalities, after simple transformations, for $\epsilon_i > 0$, we have

$$\frac{C_4}{\delta^6}\iint_{Q_T(s)} h^{n+2}dxdt \le \epsilon_1 \iint_{Q_T(s)} \left(h^{(n+2)/2}\right)^2_{xxx} dxdt + \frac{C(\epsilon_1)}{\delta^{n+7}}\int_0^T \left(\int_{\Omega(s)} hdx\right)^{n+2} dt,$$
$$\frac{C_4\Gamma^6(T)}{\delta^6}\iint_{Q_T(s)} h^{n}dxdt \le \epsilon_2 \iint_{Q_T(s)} \left(h^{(n+2)/2}\right)^2_{xxx} dxdt + C(\epsilon_2)\left(\frac{\Gamma(T)}{\delta}\right)^{3(n+7)/4}\int_0^T \left(\int_{\Omega(s)} hdx\right)^{(3n+1)/4} dt. \tag{3.29}$$

Substituting the estimates (3.29) in (3.28) and making the standard iterative procedure for small enough $0 < \epsilon_i < 1$, we arrive at the inequality

$$\frac{3}{2C_0}\sup_t \left(\int_{\Omega(s+\delta)} hdx\right)^2 + C_5\delta^3 \iint_{Q_T(s+\delta)} \left(h^{(n+2)/2}\right)^2_{xxx} dxdt \le C_6 \sum_{i=1}^{2} \frac{G_T^{(i)}(s)}{\delta^{\alpha_i}}, \tag{3.30}$$

where $\alpha_1 = n+4$, $\alpha_2 = (3(n+3))/4$, $G_T^{(1)}(s) := \int_0^T (\int_{\Omega(s)} h\, dx)^{n+2}dt$, $G_T^{(2)}(s) := \Gamma^{(3(n+7))/4}(T)\int_0^T (\int_{\Omega(s)} h\, dx)^{(3n+1)/4}dt$. Thus, (3.30) yields

$$G_T^{(i)}(s+\delta) \le C_7 T\ \Gamma^{\mu_i}(T)\left(\sum_{i=1}^{2} \frac{G_T^{(i)}(s)}{\delta^{\alpha_i}}\right)^{\beta_i}, \tag{3.31}$$

for all $s \in (r_0, 2r_0)$ and $0 < \delta < s$, where $\mu_1 = 0$, $\mu_2 = (3(n+7))/4$, $\beta_1 = (n+2)/2$, $\beta_2 = (3n+1)/8$. By Lemma B.3, from (3.31) we find that $G_T^{(i)}(s_0) = 0$, where $\Gamma(T) \le s_0(T) = C_8(T^{1/\alpha_1} + T^{1/\alpha_2}\Gamma^{\mu_2/\alpha_2}(T))$. As $\mu_2/\alpha_2 = (n+7)/(n+3) > 1$ for any $T \le T_0$, we have $\Gamma(T) \le C_9 T^{1/(n+4)}$.

4. Waiting Time Phenomenon

Let $\Omega(s) = \{x : x \ge s\}$ for all $s \in \mathbb{R}^1$, and

$$\mathbf{h}_0(s) := \int_{\Omega(s)} h_0^{\alpha-n+2}(x)dx = 0 \quad \forall s \ge 0, \tag{4.1}$$

where $(n-1)_+ < \alpha < 1$. Let us assume that the function $\mathbf{h}_0(s)$ satisfies the *flatness* conditions. Namely, for every $s:\ s_0 < s < 0$ the following estimate:

$$\mathbf{h}_0(s) \le \chi \max\left\{(-s)^{1+(4(\alpha-n+2))/n}, (-s)^{1+(4-3n+4(\alpha-n+2))/(4(n-1))}\right\}$$
$$= \begin{cases} \chi(-s)^{1+(4(\alpha-n+2))/n} & \text{for } \dfrac{4}{3} \le n < 2, \\ \chi(-s)^{1+(4-3n+4(\alpha-n+2))/(4(n-1))} & \text{for } 1 < n < \dfrac{4}{3}, \end{cases} \tag{4.2}$$

is valid.

Theorem 4.1. *Let $1 < n < 2$. Assume h_0 is nonnegative, $h_0 \in H^1(\mathbb{R}^1)$ and* $\operatorname{meas}\{\Omega(s) \cap \operatorname{supp} h_0\} = \emptyset$ *for all $s \geq 0$, that is, the condition (4.1) is valid, and the flatness condition (4.2) also holds.*

Then for the solution h of Theorem 2.2 (with $\Omega = \mathbb{R}^1$) there exists the time $T^* = T^*(\chi) > 0$ depending on the known parameters only such that

$$\operatorname{supp} h(t,\cdot) \cap \Omega(0) = \emptyset \quad \forall 0 < t \leq T^*, \tag{4.3}$$

where χ is the constant from the flatness condition. Note, that $T^* \to +\infty$ as $\chi \to 0$.

Remark 4.2. Let the initial data $h_0 \in C(\mathbb{R}^1)$ satisfy the following properties:

(1) if $1 < n < 4/3$ then we suppose that

$$\sup_{x \in \Omega(s)} h_0(x) \leq \chi(-s)^{(4-3n+4(\alpha-n+2))/(4(n-1)(\alpha-n+2))} \quad \text{for some } \alpha \in ((n-1)_+, 1); \tag{4.4}$$

(2) if $4/3 \leq n < 2$ then we suppose that $\sup_{x\in\Omega(s)} h_0(x) \leq \chi\, (-s)^{4/n}$.

These assumptions on the initial data are sufficient for the validity of flatness condition (4.2) and guarantee the appearance of the WTP, that is, the validation of property (4.3).

Remark 4.3. Note that due to Lemma 2.5 we have the estimate $h^{\alpha+2}(x,t) \leq C(1+t)h^{\alpha+1}(x,t)$. Therefore, using this inequality in (3.1) with $\Omega = \mathbb{R}^1$, we could also obtain the waiting time phenomenon by the application of Theorem 2.1 from [46] with $w = h^{(\alpha+2)/2}$, $l = k = p = 2$, $q = (2(\alpha - n + 2))/(\alpha + 2)$, and $s = (2(\alpha+1))/(\alpha+2)$.

Proof of Theorem 4.1. Similar to (3.10) for $\Omega(s) = \{x : x \geq s\}$ and we obtain

$$G_i(s+\delta) := \iint_{Q_T(s+\delta)} h^{\xi_i} \leq K\ T^{1-(\theta_i\xi_i)/(\alpha+2)} \left(\sum_{k=1}^{2} \delta^{-\alpha_k} G_k(s) + \mathbf{h}_0(s) \right)^{1+\nu_i}. \tag{4.5}$$

Let us check that all conditions of Lemma B.4 are satisfied. We denote by

$$\begin{gathered} G_{\max}(s) := \max_{i=1,2} \left\{ c_0 2^{\beta+1} \left(\sum_{k=1}^{2} (G_k(s))^{\overline{\beta}_k} \right)^{\beta_i - 1} (s) \right\}^{1/(\alpha_i\beta)}, \\ g_{\max}(s) := \max_{i=1,2} \left\{ 2^{(\beta+1)/(\alpha_i\beta)} \left(2^{\beta-1} \sum_{k=1}^{2} \left(K\ T^{1-(\theta_k\xi_k)/(\alpha+2)} \right)^{\overline{\beta}_k} \right)^{\overline{\beta}_i/\alpha_i} (\mathbf{h}_0(s))^{(\beta_i-1)/\alpha_i} \right\}, \\ c_0 = 2^{\beta-1} \sum_{k=1}^{2} \left(K\ T^{1-(\theta_k\xi_k)/(\alpha+2)} \right)^{\overline{\beta}_k}, \qquad \beta_i = 1 + \nu_i, \qquad \beta = \beta_1\beta_2. \end{gathered} \tag{4.6}$$

Taking $s = -2\delta$ in (4.5) and passing to the limit $\delta \to \infty$, due to the boundedness of functions $G_k(s)$ and $\mathbf{h}_0(s)$, we deduce

$$G_k(-\infty) \leq K\ T^{1-(\theta_k\xi_k)/(\alpha+2)} \mathbf{h}_0^{\beta_k}(-\infty). \tag{4.7}$$

This implies that the condition (i) of Lemma B.4 is fulfilled. Because of the assumption (4.2) on the function $\mathbf{h}_0(s)$, we can find T^* such that the condition (ii) of Lemma B.4 is valid for all $T \in [0, T^*]$. Here $T^* = T^*(\chi)$ goes to infinity as $\chi \to 0$. Hence, the application of Lemma B.4 ends the proof. □

Appendices

A. Proofs of a Priori Estimates

The first observation is that the periodic boundary conditions imply that classical solutions of (2.17) conserve mass:

$$\int_\Omega h_{\delta\varepsilon}(x,t)dx = \int_\Omega h_{0,\varepsilon}(x)dx = M_\varepsilon < \infty \quad \text{for all } t > 0. \tag{A.1}$$

Further, (2.21) implies $M_\varepsilon \to M = \int h_0$ as $\varepsilon \to 0$. Also, we will relate the L^p norm of h to the L^p norm of its zero-mean part as follows:

$$|h(x)| \le \left| h(x) - \frac{M_\varepsilon}{\Omega} \right| + \frac{M_\varepsilon}{\Omega} \Longrightarrow \|h\|_p^p \le 2^{p-1}\|v\|_p^p + \left(\frac{2}{|\Omega|}\right)^{p-1} M_\varepsilon^p, \tag{A.2}$$

where $v := h - M_\varepsilon/\Omega$. We will use the Poincaré inequality which holds for any zero-mean function in $H^1(\Omega)$

$$\|v\|_p^p \le b_1 \|v_x\|_p^p \quad 1 \le p < \infty, \ b_1 = \frac{|\Omega|^p}{p}. \tag{A.3}$$

Also used will be an interpolation inequality [47, Theorem 2.2, page 62] for functions of zero mean in $H^1(\Omega)$:

$$\|v\|_p^p \le b_2 \|v_x\|_2^{ap} \|v\|_r^{(1-a)p}, \tag{A.4}$$

where $r \ge 1$, $p \ge r$, $a = (1/r - 1/p)/(1/r + 1/2)$, $b_2 = (1 + r/2)^{ap}$. It follows that for any zero-mean function v in $H^1(\Omega)$

$$\|v\|_p^p \le b_3 \|v_x\|_2^p, \Longrightarrow \|h\|_p^p \le b_4 \|h_x\|_2^p + b_5 M_\varepsilon^p, \tag{A.5}$$

where

$$b_3 = \begin{cases} b_1 |\Omega|^{(2-p)/p} & \text{if } 1 \le p \le 2 \\ b_1^{(p+2)/2} b_2 & \text{if } 2 < p < \infty, \end{cases} \qquad b_4 = 2^{p-1}\, b_3, \qquad b_5 = \left(\frac{2}{|\Omega|}\right)^{p-1}. \tag{A.6}$$

To see that (A.5) holds, consider two cases. If $1 \le p < 2$, then by (A.3), $\|v\|_p$ is controlled by $\|v_x\|_p$. By the Hölder inequality, $\|v_x\|_p$ is then controlled by $\|v_x\|_2$. If $p > 2$ then by (A.4),

$\|v\|_p$ is controlled by $\|v_x\|_2^a\|v\|_2^{1-a}$ where $a = 1/2 - 1/p$. By the Poincaré inequality, $\|v\|_2^{1-a}$ is controlled by $\|v_x\|_2^{1-a}$.

Proof of Lemma 2.3. In the following, we denote the solution $h_{\delta\varepsilon}$ by h whenever there is no chance of confusion.

To prove the bound (2.23) one starts by multiplying (2.17) by $-h_{xx}$, integrating over Q_T, and using the periodic boundary conditions (2.18) yields

$$\begin{aligned}\frac{1}{2}\int_\Omega h_x^2(x,T)dx + a_0\iint_{Q_T} f_{\delta\varepsilon}(h)h_{xxx}^2 dxdt &= \frac{1}{2}\int_\Omega h_{0\varepsilon,x}^2(x)dx \\ - a_1\iint_{Q_T} f_{\delta\varepsilon}(h)h_x h_{xxx}dxdt &- \iint_{Q_T} f_{\delta\varepsilon}(h)w_x h_{xxx}dxdt.\end{aligned} \tag{A.7}$$

By Cauchy and Young inequalities, due to (A.3)–(A.5), it follows from (A.7) that

$$\begin{aligned}\frac{1}{2}\int_\Omega h_x^2(x,T)dx + \frac{a_0}{2}\iint_{Q_T} f_{\delta\varepsilon}(h)h_{xxx}^2 dxdt &\le \frac{1}{2}\int_\Omega h_{0\varepsilon,x}^2 dx \\ + c_1\iint_{Q_T} h_{xx}^2 dxdt + c_2\int_0^T &\max\left\{1, \left(\int_\Omega h_x^2 dx\right)^{\kappa_1}\right\}dt,\end{aligned} \tag{A.8}$$

where $\kappa_1 = \max\{n,3\}$, $c_1 = (a_1^2/4a_0)b_2$, $c_2 = (a_1^2/2a_0)b_4 + (a_1^2/2a_0)b_5M_\varepsilon^{2n} + (a_1^2/4a_0)b_2 + (a_1^2/a_0)\delta + \sup_{t\le T}((\|w_x\|_2^2/a_0)\delta + (\|w_x\|_\infty^2/a_0)b_4 + (\|w_x\|_\infty^2/a_0)b_5M_\varepsilon^n)$. Multiplying (2.17) by $G'_{\delta\varepsilon}(h)$, integrating over Q_T, and using the periodic boundary conditions (2.18), we obtain

$$\begin{aligned}\int_\Omega G_{\delta\varepsilon}(h(x,T))dx + a_0\iint_{Q_T} h_{xx}^2 dxdt &\le \int_\Omega G_{\delta\varepsilon}(h_{0\varepsilon})dx \\ + c_3\int_0^T \max&\left\{1, \int_\Omega h_x^2(x,t)dx\right\}dt,\end{aligned} \tag{A.9}$$

where $c_3 = a_1 + \sup_{t\le T}\|w_x\|_2$. Further, from (A.8) and (A.9) we find

$$\begin{aligned}&\int_\Omega h_x^2 dx + \frac{2c_1}{a_0}\int_\Omega G_{\delta\varepsilon}(h(x,T))dx + a_0\iint_{Q_T} f_{\delta\varepsilon}(h)h_{xxx}^2 dxdt \\ &\quad\le \int_\Omega h_{0\varepsilon,x}^2 dx + \frac{2c_1}{a_0}\int_\Omega G_{\delta\varepsilon}(h_{0\varepsilon})dx + c_4\int_0^T \max\left\{1, \left(\int_\Omega h_x^2(x,t)dx\right)^{\kappa_1}\right\}dt,\end{aligned} \tag{A.10}$$

where $c_4 = 2c_1c_3/a_0 + 2c_2$. Applying the nonlinear Grönwall lemma [48] to $v(T) \le v(0) + c_4\int_0^T \max\{1, v^{\kappa_1}(t)\}dt$ with $v(t) = \int(h_x^2(x,t) + 2c_1/a_0\ G_{\delta\varepsilon}(h(x,t)))\,dx$ yields

$$\begin{aligned}&\int_\Omega h_x^2(x,t) + 2\frac{c_1}{a_0}G_{\delta\varepsilon}(h(x,t))dx \\ &\quad\le 2^{1/(\kappa_1-1)}\max\left\{1, \int_\Omega\left(h_{0\varepsilon,x}^2(x) + \frac{2c_1}{a_0}G_{\delta\varepsilon}(h_{0\varepsilon}(x))\right)dx\right\} = K_{\delta\varepsilon} < \infty\end{aligned} \tag{A.11}$$

for all $t \in [0, T_{\delta\varepsilon,\mathrm{loc}}]$, where

$$T_{\delta\varepsilon,\mathrm{loc}} := \frac{1}{2c_4(\kappa_1 - 1)} \min\left\{1, \left(\int_\Omega \left(h_{0\varepsilon,x}^2(x) + \frac{2c_1}{a_0} G_{\delta\varepsilon}(h_{0\varepsilon}(x))\right) dx\right)^{-(\kappa_1 - 1)}\right\}. \tag{A.12}$$

Using the $\delta \to 0$, $\varepsilon \to 0$ convergence of the initial data and the choice of $\theta \in (0, 2/5)$ (see (2.21)) as well as the assumption that the initial data h_0 has finite entropy (2.11), the times $T_{\delta\varepsilon,\mathrm{loc}}$ converge to a positive limit and the upper bound K in (A.11) can be taken finite and independent of δ and ε for δ and ε sufficiently small. Therefore there exists $\delta_0 > 0$ and $\varepsilon_0 > 0$ and K such that the bound (A.11) holds for all $0 \le \delta < \delta_0$ and $0 \le \varepsilon < \varepsilon_0$ with K replacing $K_{\delta\varepsilon}$ and for all

$$0 \le t \le T_{\mathrm{loc}} := \frac{9}{10} \lim_{\varepsilon \to 0, \delta \to 0} T_{\delta\varepsilon,\mathrm{loc}}. \tag{A.13}$$

Using the uniform bound on $\int h_x^2$ that (A.11) provides, one can find a uniform-in-δ-and-ε bound for the right-hand-side of (A.10) yielding the desired a priori bound (2.23). Similarly, one can find a uniform-in-δ-and-ε bound for the right-hand-side of (A.9) yielding the desired a priori bound (2.24). The time T_{loc} and the constant K are determined by δ_0, ε_0, a_0, a_1, $\sup_{t \le T} \|w_x\|_2$, $\sup_{t \le T} \|w_x\|_\infty$, $\int h_0$, $\|h_{0x}\|_2$, and $\int G_0(h_0)$.

To prove the bound (2.25), multiply (2.17) by $-a_0 h_{xx} - a_1 h - w$, integrate over Q_T, integrate by parts, use the periodic boundary conditions (2.18) to find (2.25). □

Proof of Lemma 2.4. In the following, we denote the positive, classical solution h_ε by h whenever there is no chance of confusion.

Multiplying (2.17) by $(G_\varepsilon^{(\alpha)}(h))'$, integrating over Q_T, taking $\delta \to 0$, and using the periodic boundary conditions (2.18) yield

$$\begin{aligned} &\int_\Omega G_\varepsilon^{(\alpha)}(h(x,T))dx + a_0 \iint_{Q_T} h^\alpha h_{xx}^2 dx dt + a_0 \frac{\alpha(1-\alpha)}{3} \iint_{Q_T} h^{\alpha-2} h_x^4 dx dt \\ &\quad = \int_\Omega G_\varepsilon^{(\alpha)}(h_{0\varepsilon})dx + a_1 \iint_{Q_T} h^\alpha h_x^2 dx dt - \frac{1}{\alpha+1} \iint_{Q_T} h^{\alpha+1} w_{xx} dx dt. \end{aligned} \tag{A.14}$$

Case 1 $(0 < \alpha < 1)$. The coefficient multiplying $\iint h^{\alpha-2} h_x^4$ in (A.14) is positive and can therefore be used to control the term $\iint h^\alpha h_x^2$ on the right-hand side of (A.14). Specifically, using the Cauchy-Schwartz inequality and the Cauchy inequality,

$$a_1 \iint_{Q_T} h^\alpha h_x^2 dx dt \le \frac{a_0 \alpha(1-\alpha)}{6} \iint_{Q_T} h^{\alpha-2} h_x^4 dx dt + \frac{3a_1^2}{2a_0 \alpha(1-\alpha)} \iint_{Q_T} h^{\alpha+2} dx dt. \tag{A.15}$$

Using the bound (A.15) in (A.14), due to (A.5), yields

$$\begin{aligned}
&\int_\Omega G_\varepsilon^{(\alpha)}(h(x,T))dx + a_0\iint_{Q_T} h^\alpha h_{xx}^2 dxdt + a_0\frac{\alpha(1-\alpha)}{6}\iint_{Q_T} h^{\alpha-2}h_x^4 dxdt \\
&\quad\le \int_\Omega G_\varepsilon^{(\alpha)}(h_{0\varepsilon})dx + \frac{3a_1^2}{2a_0\alpha(1-\alpha)}\iint_{Q_T} h^{\alpha+2}dxdt + \frac{\sup_{t\le T}\|w_{xx}\|_\infty}{\alpha+1}\iint_{Q_T} h^{\alpha+1}dxdt. \\
&\quad\le \int_\Omega G_\varepsilon^{(\alpha)}(h_{0\varepsilon})dx + d_1\int_0^T \max\left\{1,\left(\int_\Omega h_x^2 dx\right)^{\alpha/2+1}\right\}dt,
\end{aligned}\tag{A.16}$$

where $d_1 = b_4((3a_1^2)/(2a_0\alpha(1-\alpha))) + b_4((\sup_{t\le T}\|w_{xx}\|_\infty)/(1+\alpha)) + b_5(((3a_1^2)/(2a_0\alpha(1-\alpha)))M_\varepsilon^{\alpha+2} + ((\sup_{t\le T}\|w_{xx}\|_\infty)/(1+\alpha))M_\varepsilon^{\alpha+1})$. Using the Cauchy inequality in (A.7) and taking $\delta \to 0$, after applying the Cauchy-Schwartz inequality and (A.5), yields

$$\begin{aligned}
&\int_\Omega h_x^2 dx + a_0\iint_{Q_T} f_\varepsilon(h)h_{xxx}^2 dxdt \le \int_\Omega h_{0\varepsilon,x}^2 dx \\
&\quad + \frac{2a_1^2}{a_0}\iint_{Q_T} h^n h_x^2 dxdt + \frac{2\sup_{t\le T}\|w_x\|_\infty^2}{a_0}\iint_{Q_T} h^n dxdt \le \int_\Omega h_{0\varepsilon,x}^2 dx \\
&\quad + \frac{a_0\alpha(1-\alpha)}{6}\iint_{Q_T} h^{\alpha-2}h_x^4 dxdt + d_2\int_0^T \max\left\{1,\left(\int_\Omega h_x^2 dx\right)^{n+1-\alpha/2}\right\}dt,
\end{aligned}\tag{A.17}$$

where $d_2 = ((6a_1^4)/(a_0^3\alpha(1-\alpha)))b_4 + ((2\sup_{t\le T}\|w_x\|_\infty^2)/a_0)b_4 + b_5(((6a_1^4)/(a_0^3\alpha(1-\alpha)))M_\varepsilon^{2(n+1)-\alpha} + ((2\sup_{t\le T}\|w_x\|_\infty^2)/a_0)M_\varepsilon^n)$. Using (A.16) yields

$$\begin{aligned}
&\int_\Omega h_x^2(x,T)dx + \int_\Omega G_\varepsilon^{(\alpha)}(h(x,T))dx + a_0\iint_{Q_T} f_\varepsilon(h)h_{xxx}^2 dxdt \\
&\quad\le \int_\Omega h_{0\varepsilon,x}^2 dx + \int_\Omega G_\varepsilon^{(\alpha)}(h_{0\varepsilon})dx + d_3\int_0^T \max\left\{1,\left(\int_\Omega h_x^2 dx\right)^{n+1-\alpha/2}\right\},
\end{aligned}\tag{A.18}$$

where $d_3 = d_1 + d_2$. Applying the nonlinear Grönwall lemma [48] to $v(T) \le v(0) + d_3\int_0^T \max\{1, v^{n+1-\alpha/2}(t)\}dt$ with $v(T) = \int(h_x^2(x,T) + G_\varepsilon^{(\alpha)}(h(x,T)))dx$ yields

$$\begin{aligned}
&\int_\Omega \left(h_x^2(x,T) + G_\varepsilon^{(\alpha)}(h(x,T))\right)dx \\
&\quad\le 4^{1/(2n-\alpha)}\max\left\{1,\int_\Omega\left(h_{0\varepsilon,x}^2(x) + G_\varepsilon^{(\alpha)}(h_{0,\varepsilon}(x))\right)dx\right\} = K_\varepsilon < \infty,
\end{aligned}\tag{A.19}$$

for all T:

$$0 \le T \le T_{\varepsilon,\mathrm{loc}}^{(\alpha)} := \frac{1}{d_3(2n-\alpha)}\min\left\{1,\left(\int_\Omega\left(h_{0\varepsilon,x}^2 + G_\varepsilon^{(\alpha)}(h_{0,\varepsilon})\right)dx\right)^{-(2n-\alpha)/2}\right\}.\tag{A.20}$$

The bound (A.19) holds for all $0 \leq \varepsilon < \varepsilon_0$ where ε_0 is from Lemma 2.3 and for all $t \leq \min\{T_{\rm loc}, T^{(\alpha)}_{\varepsilon,\rm loc}\}$ where $T_{\rm loc}$ is from Lemma 2.3.

Using the $\varepsilon \to 0$ convergence of the initial data and the choice of $\theta \in (0, 2/5)$ (see (2.21)) as well as the assumption that the initial data h_0 has finite α-entropy (2.27), the times $T^{(\alpha)}_{\varepsilon,\rm loc}$ converge to a positive limit and the upper bound K_ε in (A.19) can be taken finite and independent of ε. Therefore there exists ε_0 and K such that the bound (A.19) holds for all $0 \leq \varepsilon < \varepsilon_0$ with K replacing K_ε and for all

$$0 \leq t \leq T^{(\alpha)}_{\rm loc} := \min\left\{T_{\rm loc}, \frac{9}{10}\lim_{\varepsilon\to 0} T^{(\alpha)}_{\varepsilon,\rm loc}\right\}, \tag{A.21}$$

where $T_{\rm loc}$ is the time from Lemma 2.3.

Using the uniform bound on $\int h_x^2$ that (A.19) provides, one can find a uniform-in-ε bound for the right-hand-side of (A.16) yielding the desired bound

$$\int_\Omega G^{(\alpha)}_\varepsilon(h(x,T))dx + a_0 \iint_{Q_T} h^\alpha h_{xx}^2 dx dt + a_0 \frac{\alpha(1-\alpha)}{6} \iint_{Q_T} h^{\alpha-2} h_x^4 dx dt \leq K_1, \tag{A.22}$$

which holds for all $0 < \varepsilon < \varepsilon_0$ and all $0 \leq T \leq T^{(\alpha)}_{\rm loc}$. Note, (A.22) implies that for all $0 < \varepsilon < \varepsilon_0$ that $h_\varepsilon^{\alpha/2+1}$ and $h_\varepsilon^{\alpha/4+1/2}$ are contained in balls in $L^2(0,T;H^2(\Omega))$ and $L^4(0,T;W^1_4(\Omega))$ respectively, that is,

$$\iint_{Q_T} \left(h_\varepsilon^{\alpha/2+1}\right)_{xx}^2 dx dt \leq K, \qquad \iint_{Q_T} \left(h_\varepsilon^{\alpha/4+1/2}\right)_x^4 dx dt \leq K. \tag{A.23}$$

From these estimates follows immediately (2.30).

Case 2 ($-1/2 < \alpha < 0$). For $\alpha < 0$ the coefficient multiplying $\iint h^{\alpha-2} h_x^4$ in (A.14) is negative. However, we will show that if $\alpha > -1/2$ then one can replace this coefficient with a positive coefficient while also controlling the term $\iint h^\alpha h_x^2$ on the right-hand side of (A.14). Using the Cauchy-Schwartz inequality, it is easy to show that

$$\iint_{Q_T} h^{\alpha-2} h_x^4 dx dt \leq \frac{9}{(1-\alpha)^2} \iint_{Q_T} h^\alpha h_{xx}^2 dx dt \quad \forall \alpha < 1. \tag{A.24}$$

Using (A.24) in (A.14) yields

$$\begin{aligned} &\int_\Omega G^{(\alpha)}_\varepsilon(h(x,T))dx + a_0 \frac{1+2\alpha}{1-\alpha} \iint_{Q_T} h^\alpha h_{xx}^2 dx dt \\ &\quad\leq \int_\Omega G^{(\alpha)}_\varepsilon(h_{0\varepsilon})dx + a_1 \iint_{Q_T} h^\alpha h_x^2 dx dt + \frac{\sup_{t\leq T}\|w_{xx}\|_\infty}{\alpha+1} \iint_{Q_T} h^{\alpha+1} dx dt. \end{aligned} \tag{A.25}$$

Note that if $\alpha > -1/2$ then all the terms on the left-hand side of (A.25) are positive. We now control the term $\iint h^{\alpha} h_x^2$ on the right-hand side of (A.25). By integration by parts and the periodic boundary conditions

$$\iint_{Q_T} h^{\alpha} h_x^2 dxdt = -\frac{1}{1+\alpha} \iint_{Q_T} h^{\alpha+1} h_{xx} dxdt. \tag{A.26}$$

Applying the Cauchy inequality to (A.26) yields

$$a_1 \iint_{Q_T} h^{\alpha} h_x^2 dxdt \leq \frac{a_0(1+2\alpha)}{2(1-\alpha)} \iint_{Q_T} h^{\alpha} h_{xx}^2 dxdt + \frac{a_1^2(1-\alpha)}{2a_0(1+2\alpha)(1+\alpha)^2} \iint_{Q_T} h^{\alpha+2} dxdt. \tag{A.27}$$

Using inequality (A.27) in (A.25) yields

$$\begin{aligned} &\int_{\Omega} G_{\varepsilon}^{(\alpha)}(h(x,T))dx + a_0 \frac{1+2\alpha}{2(1-\alpha)} \iint_{Q_T} h^{\alpha} h_{xx}^2 dxdt \leq \int_{\Omega} G_{\varepsilon}^{(\alpha)}(h_{0\varepsilon})dx \\ &\quad + \frac{a_1^2(1-\alpha)}{2a_0(1+2\alpha)(1+\alpha)^2} \iint_{Q_T} h^{\alpha+2} dxdt + \frac{\sup_{t\leqslant T}\|w_{xx}\|_{\infty}}{\alpha+1} \iint_{Q_T} h^{\alpha+1} dxdt. \end{aligned} \tag{A.28}$$

Adding $((a_0(1+2\alpha)(1-\alpha))/36)\iint_{Q_T} h^{\alpha-2} h_x^4 dxdt$ to both sides of (A.28) and using the inequality (A.24) yields

$$\begin{aligned} &\int_{\Omega} G_{\varepsilon}^{(\alpha)}(h(x,T))dx + a_0 \frac{(1+2\alpha)}{4(1-\alpha)} \iint_{Q_T} h^{\alpha} h_{xx}^2 dxdt \\ &\quad + \frac{a_0(1+2\alpha)(1-\alpha)}{36} \iint_{Q_T} h^{\alpha-2} h_x^4 dxdt \leq \int_{\Omega} G_{\varepsilon}^{(\alpha)}(h_{0\varepsilon})dx \\ &\quad + e_1 \int_0^T \max\left\{1, \left(\int_{\Omega} h_x^2 dx\right)^{\alpha/2+1}\right\} dt, \end{aligned} \tag{A.29}$$

where $e_1 = ((a_1^2(1-\alpha))/(2a_0(1+2\alpha)(1+\alpha)^2))b_4 + ((\sup_{t\leq T}\|w_{xx}\|_{\infty})/(\alpha+1))b_4 + b_5(((a_1^2(1-\alpha))/(2a_0(1+2\alpha)(1+\alpha)^2))M_{\varepsilon}^{\alpha+2} + ((\sup_{t\leq T}\|w_{xx}\|_{\infty})/(\alpha+1))M_{\varepsilon}^{\alpha+1})$. Recall the bound (A.17). As before, by the Cauchy inequality,

$$\begin{aligned} \frac{2a_1^2}{a_0} \iint_{Q_T} h^n h_x^2 dxdt &\leq \frac{a_0(1+2\alpha)(1-\alpha)}{36} \iint_{Q_T} h^{\alpha-2} h_x^4 dxdt \\ &\quad + \frac{36a_1^4}{a_0^3(1+2\alpha)(1-\alpha)} \iint_{Q_T} h^{2(n+1)-\alpha} dxdt. \end{aligned} \tag{A.30}$$

Using (A.30) in (A.17) yields

$$\begin{aligned}\int_\Omega h_x^2 dx + a_0 \iint_{Q_T} f_\varepsilon(h) h_{xxx}^2 dx dt \le \int_\Omega h_{0\varepsilon,x}^2 dx \\ + \frac{a_0(1+2\alpha)(1-\alpha)}{36} \iint_{Q_T} h^{\alpha-2} h_x^4 dx dt + e_2 \int_0^T \max\left\{1, \left(\int_\Omega h_x^2 dx\right)^{n+1-\alpha/2}\right\} dt,\end{aligned} \tag{A.31}$$

where $e_2 = ((36a_1^4)/(a_0^3(1+2\alpha)(1-\alpha)))b_4 + ((2\sup_{t\le T}\|w_x\|_\infty^2)/a_0)b_4 + b_5(((36a_1^4)/(a_0^3(1+2\alpha)(1-\alpha)))M_\varepsilon^{2(n+1)-\alpha} + ((2\sup_{t\le T}\|w_x\|_\infty^2)/a_0)M_\varepsilon^n)$. Using (A.29) yields

$$\begin{aligned}\int_\Omega h_x^2(x,T) dx + \int_\Omega G_\varepsilon^{(\alpha)}(h(x,T)) dx + a_0 \iint_{Q_T} f_\varepsilon(h) h_{xxx}^2 dx dt \\ \le \int_\Omega h_{0\varepsilon,x}^2 dx + \int_\Omega G_\varepsilon^{(\alpha)}(h_{0\varepsilon}) dx + e_3 \int_0^T \max\left\{1, \left(\int_\Omega h_x^2 dx\right)^{n+1-\alpha/2}\right\},\end{aligned} \tag{A.32}$$

where $e_3 = e_1 + e_2$. The rest of the proof now continues as in the $0 < \alpha < 1$ case. Specifically, one finds a bound

$$\begin{aligned}\int_\Omega \left(h_x^2(x,T) + G_\varepsilon^{(\alpha)}(h(x,T))\right) dx \\ \le 4^{1/(2n-\alpha)} \max\left\{1, \int_\Omega \left(h_{0\varepsilon,x}^2(x) + G_\varepsilon^{(\alpha)}(h_{0\varepsilon}(x))\right) dx\right\} = K_\varepsilon < \infty\end{aligned} \tag{A.33}$$

for all T:

$$0 \le T \le T_{\varepsilon,\mathrm{loc}}^{(\alpha)} := \frac{1}{e_3(2n-\alpha)} \min\left\{1, \left(\int_\Omega \left(h_{0\varepsilon,x}^2(x) + G_\varepsilon^{(\alpha)}(h_{0,\varepsilon}(x))\right) dx\right)^{-(2n-\alpha)/2}\right\}. \tag{A.34}$$

The time $T_{\mathrm{loc}}^{(\alpha)}$ is defined as in (A.21) and the uniform bound (A.33) used to bound the right-hand side of (A.29) yields the desired bound

$$\int_\Omega G_\varepsilon^{(\alpha)}(h(x,T)) dx + \frac{a_0(1+2\alpha)}{4(1-\alpha)} \iint_{Q_T} h^\alpha h_{xx}^2 dx dt + \frac{a_0(1+2\alpha)(1-\alpha)}{36} \iint_{Q_T} h^{\alpha-2} h_x^4 dx dt \le K_2. \tag{A.35}$$

□

Proof of Lemma 3.4. Let $\phi(x) = \zeta^6(x)$. Multiplying (2.17) by $-(\phi(x)h_x)_x$, and integrating on Q_T, yields

$$\frac{1}{2}\int_\Omega (x)h_x^2(x,T)dx\phi - \frac{1}{2}\int_\Omega \phi(x)h_{0\varepsilon,x}^2(x)dx$$
$$= -\iint_{Q_T} f_\varepsilon(h)(a_0h_{xxx} + a_1h_x + w_x)(\phi_{xx}h_x + 2\phi_x h_{xx} + \phi h_{xxx})dxdt$$
$$= -\iint_{Q_T} f_\varepsilon(h)(a_0h_{xxx} + a_1h_x)\phi_{xx}h_x dxdt - 2\iint_{Q_T} f_\varepsilon(h)(a_0h_{xxx} + a_1h_x)\phi_x h_{xx} dxdt$$
$$- \iint_{Q_T} f_\varepsilon(h)(a_0h_{xxx} + a_1h_x)\phi h_{xxx} dxdt - \iint_{Q_T} f_\varepsilon(h)w_x(\phi_{xx}h_x + 2\phi_x h_{xx} + \phi h_{xxx})dxdt$$
$$- a_0\iint_{Q_T} f_\varepsilon(h)h_{xxx}^2\phi dxdt =: I_1 + I_2 + I_3 + I_4 + I_5. \tag{A.36}$$

We now bound the terms I_1, I_2, I_3, and I_4. First,

$$I_1 = -a_0\iint_{Q_T} \phi_{xx}f_\varepsilon(h)h_{xxx}h_x dxdt - a_1\iint_{Q_T} \phi_{xx}f_\varepsilon(h)h_x^2 dxdt$$
$$\leq \epsilon_1\iint_{Q_T} \zeta^6\left\{f_\varepsilon(h)h_{xxx}^2 + h^{n-4}h_x^6\right\}dxdt + C(\epsilon_1)\iint_{Q_T} h^{n+2}\left(\zeta^6 + \zeta_x^6 + \zeta^3|\zeta_{xx}|^3\right)dxdt, \tag{A.37}$$

$$I_2 = -2a_0\iint_{Q_T} \phi_x f_\varepsilon(h)h_{xxx}h_{xx} dxdt - 2a_1\iint_{Q_T} \phi_x f_\varepsilon(h)h_{xx}h_x dxdt$$
$$\leq \epsilon_2\iint_{Q_T} \zeta^6\left\{f_\varepsilon(h)h_{xxx}^2 + h^{n-2}h_x^2h_{xx}^2 + h^{n-1}|h_{xx}|^3\right\}dxdt \tag{A.38}$$
$$+ C(\epsilon_2)\iint_{Q_T} h^{n+2}\left(\zeta^6 + \zeta_x^6\right)dxdt,$$

$$I_3 = -a_0\iint_{Q_T} \phi f_\varepsilon(h)h_{xxx}^2 dxdt - a_1\iint_{Q_T} \phi f_\varepsilon(h)h_{xxx}h_x dxdt$$
$$\leq -a_0\iint_{Q_T} \zeta^6 f_\varepsilon(h)h_{xxx}^2 dxdt + \epsilon_3\iint_{Q_T} \zeta^6\left(f_\varepsilon(h)h_{xxx}^2 + h^{n-4}h_x^6\right)dxdt \tag{A.39}$$
$$+ C(\epsilon_3)\iint_{Q_T} h^{n+2}\zeta^6 dxdt,$$

$$\begin{aligned}
I_4 = &-6\iint_{Q_T} f_\varepsilon(h) h_x w_x \zeta^4\left(5\zeta_x^2+\zeta\zeta_{xx}\right)dx\,dt \\
&-12\iint_{Q_T} f_\varepsilon(h) h_{xx} w_x \zeta^5\zeta_x\,dx\,dt - \iint_{Q_T} f_\varepsilon(h) h_{xxx} w_x \zeta^6\,dx\,dt \\
\le{}& \epsilon_4\iint_{Q_T}\zeta^6\left\{f_\varepsilon(h)h_{xxx}^2 + h^{n-4}h_x^6 + h^{n-1}|h_{xx}|^3\right\}dx\,dt \\
&+ C(\epsilon_4)\iint_{Q_T} h^{n+2}\left(\zeta_x^6+\zeta^3\zeta_{xx}^3\right)dx\,dt + C(\epsilon_4)\iint_{Q_T} h^n\zeta^6\,dx\,dt.
\end{aligned} \tag{A.40}$$

Now, multiplying (2.17) by $\zeta^4(h+\gamma)^\beta$, $\beta > (1-n)/3$, $\gamma > 0$ and integrating on Q_T, using the Young's inequality, letting $\gamma \to 0$, we obtain the following estimate:

$$\begin{aligned}
\int_\Omega \zeta^4 h^{\beta+1}(T)\,dx \le{}& \int_\Omega \zeta^4 h_{0\varepsilon}^{\beta+1}\,dx + \epsilon_4\iint_{Q_T}\zeta^6\left\{f_\varepsilon(h)h_{xxx}^2 + h^{n-4}h_x^6\right\}dx\,dt \\
&+ C(\epsilon_4)\iint_{Q_T}\left\{\chi_{\{\zeta>0\}}h^{n+3\beta-1} + h^{n+2}\left(\zeta^6+\zeta_x^6\right) + h^n\zeta^6\right\}dx\,dt,
\end{aligned} \tag{A.41}$$

where $\beta > (1-n)/3$. If we now add inequalities (A.36) and (A.41), in view of (A.37)–(A.39), then, applying Lemma B.1, choosing $\epsilon_i > 0$, and letting $\varepsilon \to 0$, we obtain (3.17). □

B. Auxiliary Lemmas

Lemma B.1 (see [34, 38]). *Let $\Omega \subset \mathbb{R}^N$, $N < 6$, be a bounded convex domain with smooth boundary, and let $n \in (2-\sqrt{1-N/(N+8)}, 3)$ for $N > 1$, and $1/2 < n < 3$ for $N = 1$. Then the following estimates hold for any strictly positive functions $v \in H^2(\Omega)$ such that $\nabla v\cdot\vec{n} = 0$ on $\partial\Omega$ and $\int_\Omega v^n|\nabla\Delta v|^2 < \infty$:*

$$\begin{gathered}
\int_\Omega \varphi^6\left\{v^{n-4}|\nabla v|^6 + v^{n-2}\left|D^2 v\right|^2|\nabla v|^2\right\} \le c\left\{\int_\Omega \varphi^6 v^n|\nabla\Delta v|^2 + \int_{\{\varphi>0\}} v^{n+2}|\nabla\varphi|^6\right\}, \\
\int_\Omega \varphi^6\left|\nabla\Delta v^{(n+2)/2}\right|^2 \le c\left\{\int_\Omega \varphi^6 v^n|\nabla\Delta v|^2 + \int_{\{\varphi>0\}} v^{n+2}\left\{|\nabla\varphi|^6 + \varphi^2\left|D^2\varphi\right|^2|\nabla\varphi|^2 + \varphi^3|\Delta\varphi|^3\right\}\right\},
\end{gathered} \tag{B.1}$$

where $\varphi \in C^2(\Omega)$ is an arbitrary nonnegative function such that the tangential component of $\nabla\varphi$ is equal to zero on $\partial\Omega$, and the constant $c > 0$ is independent of v.

Lemma B.2 (see [49]). *If $\Omega \subset \mathbb{R}^N$ is a bounded domain with piecewise-smooth boundary, $a > 1$, $b \in (0, a)$, $d > 1$, and $0 \le i < j$, $i, j \in \mathbb{N}$, then there exist positive constants d_1 and d_2 ($d_2 = 0$ if Ω is* *unbounded) depending only on Ω, d, j, b, and N such that the following inequality is valid for every $v(x) \in W^{j,d}(\Omega) \cap L^b(\Omega)$:*

$$\left\| D^i v \right\|_{L^a(\Omega)} \leq d_1 \left\| D^j v \right\|_{L^d(\Omega)}^{\theta} \|v\|_{L^b(\Omega)}^{1-\theta} + d_2 \|v\|_{L^b(\Omega)}, \quad \theta = \frac{1/b + i/N - 1/a}{1/b + j/N - 1/d} \in \left[\frac{i}{j}, 1\right). \tag{B.2}$$

Lemma B.3 (see [37]). *Let $(\beta_1, \ldots, \beta_m) \in \mathbb{R}^m, m \geq 1$ and let $\beta = \prod_{j=1}^m \beta_j$, $\overline{\beta_i} = \beta/\beta_i = \prod_{j=1, j\neq i}^m \beta_j$. Assume that $G_i(s)$ are nonnegative nonincreasing functions satisfying the conditions:*

$$G_i(s+\delta) \leq c_i \left(\sum_{i=1}^m \frac{G_i(s)}{\delta^{\alpha_i}} \right)^{\beta_i} \quad \forall s > 0, \ \delta > 0, \ i = \overline{1,m} \tag{B.3}$$

with real constants $c_i > 0$, $\beta_i > 1$, and $\alpha_i \geq 0$ for $i = \overline{1,m}$, and $\alpha_i > 0$ for $i = \overline{1,\ell}$. Let $G(s) = \sum_{i=1}^m (\overline{c_i^{\overline{\beta}_i}})(G_i(s))^{\overline{\beta}_i}$, and let the function $H(s) = m^{\beta} \sum_{i=\ell+1}^m c_i^{\overline{\beta}_i} \overline{(c_i^{\overline{\beta}_i})}^{\,1-\beta_i} (G_i(s))^{\beta_i - 1}$ be such that $H(s_1) < 1$ at a some $s_1 \geq 0$. Then there exists a positive constant $c > 1$ depending on m, α_i, β_i, ℓ, and $H(s_1)$ such that $G_i(s_0) \equiv 0$ for all $i = \overline{1,\ell}$, where $s_0 = s_1 + c \sum_{i=1}^{\ell} (c_i^{\overline{\beta}_i} \overline{(c_i^{\overline{\beta}_i})}^{\,1-\beta_i} (G(s_1))^{\beta_i - 1})^{1/(\alpha_i \beta)}$. Note, if $\ell = m$ then $s_1 = 0$.

Lemma B.4. *Let $(\beta_1, \ldots, \beta_m) \in \mathbb{R}^m, m \geq 1$, and let $\beta = \prod_{j=1}^m \beta_j$, $\overline{\beta_i} = \beta/\beta_i = \prod_{j=1, j\neq i}^m \beta_j$. Assume that $G_i(s)$, $g(s)$ are nonnegative nonincreasing functions satisfying the conditions:*

$$G_i(s+\delta) \leq c_i \left(\sum_{i=1}^m \frac{G_i(s)}{\delta^{\alpha_i}} + g(s) \right)^{\beta_i} \quad \forall s \in \mathbb{R}^1, \ \delta > 0, \ i = \overline{1,m} \tag{B.4}$$

with real constants $c_i > 0$, $\beta_i > 1$, and $\alpha_i > 0$. Let the functions

$$G_{\max}(s) := \max_{i=\overline{1,m}} \left\{ m c_0 2^{\beta} \left(\sum_{k=1}^m (G_k(s))^{\overline{\beta}_k} \right)^{\beta_i - 1} (s) \right\}^{1/\alpha_i \beta}, \qquad c_0 = 2^{\beta - 1} \sum_{k=1}^m (c_k)^{\overline{\beta}_k} \tag{B.5}$$

and $g_{\max}(s) := \max_{i=\overline{1,m}} (m 2^{\beta})^{1/\alpha_i \beta} (2^{\beta-1} \sum_{k=1}^m (c_k)^{\overline{\beta}_k})^{\overline{\beta}_i/\alpha_i} (g(s))^{(\beta_i - 1)/\alpha_i}$ be such that

(i) *for some $s_1 \in (-\infty, s_0)$ the inequality $G_{\max}(s) \leq k_1 g_{\max}(s)$ holds for all $s < s_1$,*

(ii) *$g_{\max}(s) \leq k_2(s_0 - s)$ for all $s \leq s_0$,*

where $k_1 > (1 - \max_{i=\overline{1,m}}\{2^{-(\beta_i - 1)/(\alpha_i \beta)}\})^{-1}$ and $0 < k_2 < k_1^{-1}(1 - k_1^{-1} - \max_{i=\overline{1,m}}\{2^{-(\beta_i - 1)/(\alpha_i \beta)}\})$. Then $G_i(s) \equiv 0$ for all $s \geq s_0$.

Proof. Let us denote by $G(s) := \sum_{k=1}^{m}(G_k(s))^{\overline{\beta}_k}$. Raising both side of (B.4) to the power $\overline{\beta}_i$ and summing with respect to i, we deduce

$$G(s+\delta) \leq \sum_{k=1}^{m}(c_k)^{\overline{\beta}_k}\left(\sum_{i=1}^{m}\frac{G_i(s)}{\delta^{\alpha_i}} + g(s)\right)^{\beta} \\ \leq c_0 2^{\beta-1}\sum_{i=1}^{m}\frac{G_i^{\beta}(s)}{\delta^{\alpha_i\beta}} + c_0 g^{\beta}(s) \leq c_0 2^{\beta-1}\sum_{i=1}^{m}\frac{G^{\beta_i}(s)}{\delta^{\alpha_i\beta}} + c_0 g^{\beta}(s). \tag{B.6}$$

Choosing $\delta = \delta(s) = \sum_{i=1}^{m}(mc_0 2^{\beta}G^{\beta_i-1}(s))^{1/(\alpha_i\beta)}$, we arrive at

$$G(s+\delta(s)) \leq \frac{1}{2}G(s) + c_0 g^{\beta}(s), \tag{B.7}$$

whence we find that

$$\delta(s+\delta(s)) \leq \epsilon\delta(s) + \widetilde{g}(s), \tag{B.8}$$

where $\epsilon = \max_{i=\overline{1,m}}\{2^{-(\beta_i-1)/(\alpha_i\beta)}\}$, $\widetilde{g}(s) := \sum_{i=1}^{m}(mc_0^{\beta_i}2^{\beta})^{1/(\alpha_i\beta)}(g(s))^{(\beta_i-1)/\alpha_i}$. Applying [27, Lemma 4] to $\delta(s)$, taking into account the conditions (i) and (ii), we obtain $\delta(s) = 0$ for all $s \geq s_0$. □

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