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Research Article

The Local Strong and Weak Solutions for a Generalized Pseudoparabolic Equation

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The Cauchy problem for a nonlinear generalized pseudoparabolic equation is investigated. The well-posedness of local strong solutions for the problem is established in the Sobolev space $C([0,T); H^s(R)) \bigcap C^1([0,T); H^{s-1}(R))$ with $s > 3/2$, while the existence of local weak solutions is proved in the space $H^s(R)$ with $1 \leq s \leq 3/2$. Further, under certain assumptions of the nonlinear terms in the equation, it is shown that there exists a unique global strong solution to the problem in the space $C([0,\infty); H^s(R)) \bigcap C^1([0,\infty); H^{s-1}(R))$ with $s \geq 2$.

1. Introduction

Davis [1] investigated the pseudoparabolic equation

$$u_t(t,x) = \frac{\partial}{\partial x}\varphi(u_x) + \alpha u_{txx}, \tag{1.1}$$

where the constant $\alpha \geq 0$, the function $\varphi \in C^2(-\infty,\infty)$, $\varphi(0) = 0$ and $\varphi'(\xi) > 0$, and the subscripts x and t indicate partial derivatives. Equation (1.1) arises from the study of shearing flows of incompressible simple fluids. The quantity $\varphi(u_x) + \alpha u_{tx}$ is viewed as an approximation to the stress functional during such a flow. Much attention has been given to this approximation when the function φ is linear (see [2, 3]). The existence and uniqueness of the global weak solution of the initial value problem for (1.1) were established in [1].

Recently, Chen and Xue [4] investigated the Cauchy problem for the nonlinear generalized pseudoparabolic equation

$$u_t - \alpha u_{txx} - \lambda u_{xx} + \gamma u_x + f(u)_x = \frac{\partial}{\partial x}\varphi(u_x) + g(u) - \alpha g(u)_{xx}, \quad x \in R,\ t > 0, \tag{1.2}$$

where $u(t,x)$ is an unknown function, $\alpha > 0$, $\lambda \geq 0$, γ is a real number, $f(s)$, $\varphi(s)$, and $g(s)$ denote given nonlinear functions. The well-posedness of global strong solution in a Sobolev space, the global classical solution and its asymptotic behavior are studied in [4] in which several key assumptions are imposed on the functions $\varphi(s)$ and $g(s)$. In fact, various dynamic properties for many special cases of (1.2) have been established in [5–7]. For example, when $\varphi(s) = g(s) = 0$, (1.2) becomes the generalized regularized long wave Burger equation.

Motivated by the works in [1, 4], we study the problem

$$\begin{gathered} u_t - \alpha u_{txx} = \frac{\partial}{\partial x}\varphi(u_x) + \beta u^{2m} u_{xx}, \quad x \in R,\ t > 0, \\ u(0,x) = u_0(x), \quad x \in R, \end{gathered} \tag{1.3}$$

where $\alpha > 0$ and $\beta \geq 0$, m is a nature number, $\varphi(s)$ is a given function, and $u_0(x)$ is a given initial value function. Here we should address that (1.2) does not include the first equation of problem (1.3) due to the term $\beta u^{2m} u_{xx}$. Letting $\beta = 0$, the first equation of problem (1.3) reduces to (1.1).

The objectives of this work are threefold. The first objective is to establish the local well-posedness of system (1.3) in the space $C([0,T); H^s(R)) \bigcap C^1([0,T); H^{s-1}(R))$ with $s > 3/2$. We should address that the Sobolev index $s \geq 2$ is required to guarantee the local well-posedness of (1.1) and (1.2) in the works of Davis [1] and Chen and Xue [4]. The second aim is to study the existence of local weak solutions for system (1.3). The third aim is to discuss the well-posedness of the global strong solution for problem (1.3). Under the assumptions of the function $\varphi(s)$ and the initial value $u_0(x)$ similar to those presented in [1, 4], problem (1.3) is shown to have a unique global solution in the space $C([0,\infty); H^s(R)) \bigcap C^1([0,\infty); H^{s-1}(R))$.

The organization of this paper is as follows. The well-posedness of local strong solutions for problem (1.3) is investigated in Section 2, and the existence of local weak solutions is established in Section 3. Section 4 deals with the well-posedness of the global strong solution.

2. Local Well-Posedness

Let $L^p = L^p(R)$ $(1 \leq p < +\infty)$ be the space of all measurable functions h such that $\|h\|_{L^p}^p = \int_R |h(t,x)|^p dx < \infty$. We define $L^\infty = L^\infty(R)$ with the standard norm $\|h\|_{L^\infty} = \inf_{m(e)=0} \sup_{x \in R \setminus e} |h(t,x)|$. For any real number s, $H^s = H^s(R)$ denotes the Sobolev space with the norm defined by

$$\|h\|_{H^s} = \left(\int_R (1 + |\xi|^2)^s |\widehat{h}(t,\xi)|^2 d\xi \right)^{1/2} < \infty, \tag{2.1}$$

where $\widehat{h}(t,\xi) = \int_R e^{-ix\xi} h(t,x) dx$.

For $T > 0$ and nonnegative number s, $C([0,T); H^s(R))$ denotes the Frechet space of all continuous H^s-valued functions on $[0,T)$. We set $\Lambda = (1-\partial_x^2)^{1/2}$. For simplicity, throughout this paper, we let c denote any positive constants.

The local well-posedness theorem is stated as follows.

Theorem 2.1. *Provided that $s \geq 3/2$, $u_0 \in H^s(R)$, φ is a polynomial of order N with $\varphi(0) = 0$. Then problem* (1.3) *admits a unique local solution:*

$$u(t,x) \in C([0,T); H^s(R)) \bigcap C^1\left([0,T); H^{s-1}(R)\right). \tag{2.2}$$

Proof. In fact, the first equation of problem (1.3) is equivalent to the equation

$$u_t = \Lambda^{-2}\left(\frac{\partial}{\partial x}\varphi(u_x) + \beta u^{2m} u_{xx}\right), \tag{2.3}$$

which leads to

$$u = u_0 + \int_0^t \Lambda^{-2}\left(\frac{\partial}{\partial x}\varphi(u_x) + \beta u^{2m} u_{xx}\right) d\tau. \tag{2.4}$$

Suppose that both u and v are in the closed ball $B_{M_0}(0)$ of radius $M_0 > 1$ about the zero function in $C([0,T]; H^s(R))$ and A is the operator in the right-hand side of (2.4), for fixed $t \in [0,T]$, we get

$$\begin{aligned}
&\left\|\int_0^t \Lambda^{-2}\left(\varphi(u_x)_x + \beta u^{2m} u_{xx}\right) dt - \int_0^t \Lambda^{-2}\left(\varphi(v_x)_x + \beta v^{2m} v_{xx}\right) dt\right\|_{H^s} \\
&\quad \leq T\left(\sup_{0\leq t\leq T}\left\|\varphi(u_x) - \varphi(v_x)\right\|_{H^{s-1}} + \sup_{0\leq t\leq T}\left\|u^{2m}u_{xx} - v^{2m}v_{xx}\right\|_{H^{s-2}}\right).
\end{aligned} \tag{2.5}$$

The algebraic property of $H^{s_0}(R)$ with $s_0 > 1/2$ (see [8–10]) and $s > 3/2$ derives that

$$\begin{aligned}
\left\|u_x^j - v_x^j\right\|_{H^{s-1}} &= \left\|(u_x - v_x)\left(u_x^{j-1} + u_x^{j-2}v_x + \cdots + u_x v_x^{j-2} + v_x^{j-1}\right)\right\|_{H^{s-1}} \\
&\leq \left\|u_x - v_x\|_{H^{s-1}}\|\left(u_x^{j-1} + u_x^{j-2}v_x + \cdots + u_x v_x^{j-2} + v_x^{j-1}\right)\right\|_{H^{s-1}} \\
&\leq c\|u_x - v_x\|_{H^{s-1}} \sum_{i=0}^{j-1} \|u_x\|_{H^{s-1}}^{j-1-i} \|v_x\|_{H^{s-1}}^{i} \leq c M_0^{j-1} \|u - v\|_{H^s}, \\
\|\varphi(u_x) - \varphi(v_x)\|_{H^{s-1}} &\leq c\sum_{j=0}^{N} \|(u_x)^j - (v_x)^j\|_{H^{s-1}} \leq c M_0^{N-1} \|u - v\|_{H^s}.
\end{aligned} \tag{2.6}$$

Using $u^{2m}u_{xx} = \partial_x[u^{2m}u_x] - 2mu^{2m-1}(u_x)^2$ and $v^{2m}v_{xx} = \partial_x[v^{2m}v_x] - 2mv^{2m-1}(v_x)^2$, we get

$$\begin{aligned}\left\|u^{2m}u_{xx} - v^{2m}v_{xx}\right\|_{H^{s-2}} &\leq \|\partial_x\left[u^{2m}u_x - v^{2m}v_x\right]\|_{H^{s-2}} + c\|u^{2m-1}u_x^2 - v^{2m-1}v_x^2\|_{H^{s-2}} \\ &\leq c\left\|\left(u^{2m} - v^{2m}\right)v_x + u^{2m}(u_x - v_x)\right\|_{H^{s-1}} \\ &\quad + c\left\|u^{2m-1}\left(u_x^2 - v_x^2\right) + \left(u^{2m-1} - v^{2m-1}\right)v_x^2\right\|_{H^{s-1}} \\ &\leq c\Big(\|\left(u^{2m} - v^{2m}\right)v_x\|_{H^{s-1}} + \left\|u^{2m}(u_x - v_x)\right\|_{H^{s-1}} \\ &\quad + \left\|u^{2m-1}\left(u_x^2 - v_x^2\right)\right\|_{H^{s-1}} + \left\|\left(u^{2m-1} - v^{2m-1}\right)v_x^2\right\|_{H^{s-1}}\Big) \\ &\leq cM_0^{2m}\|u - v\|_{H^s},\end{aligned} \tag{2.7}$$

in which $s > 3/2$ is used.

From (2.5)–(2.7), we obtain

$$\|Au - Av\|_{H^s} \leq \|u - v\|_{H^s}, \tag{2.8}$$

where $\theta = \max(cTM_0^{N-1}, cTM_0^{2m})$ and c is independent of T. Choosing T sufficiently small such that $\theta < 1$, we know that A is a contractive mapping. Applying the above inequality and (2.4) yields

$$\|Au\|_{H^s} \leq \|u_0\|_{H^s} + \theta\|u\|_{H^s}. \tag{2.9}$$

Choosing T sufficiently small such that $\theta M_0 + \|u_0\|_{H^s} < M_0$, we know that A maps $B_{M_0}(0)$ to itself. It follows from the contractive mapping principle that the mapping A has a unique fixed point u in $B_{M_0}(0)$. This completes the proof of Theorem 2.1. □

3. Existence of Local Weak Solutions

In this section, we assume that $\varphi(\eta) = \eta^{2N+1}$ where N is a nature number. In order to establish the existence of local weak solution, we need the following lemmas.

Lemma 3.1 (see Kato and Ponce [8]). *If $r \geq 0$, then $H^r \bigcap L^\infty$ is an algebra. Moreover,*

$$\|uv\|_r \leq c(\|u\|_{L^\infty}\|v\|_r + \|u\|_r\|v\|_{L^\infty}), \tag{3.1}$$

where c is a constant depending only on r.

Lemma 3.2 (see Kato and Ponce [8]). *Let $r > 0$. If $u \in H^r \bigcap W^{1,\infty}$ and $v \in H^{r-1} \bigcap L^\infty$, then*

$$\|[\Lambda^r, u]v\|_{L^2} \leq c\Big(\|\partial_x u\|_{L^\infty}\|\Lambda^{r-1}v\|_{L^2} + \|\Lambda^r u\|_{L^2}\|v\|_{L^\infty}\Big). \tag{3.2}$$

Lemma 3.3. *Let $s \geq 3/2$, $\varphi(u_x) = u_x^{2N+1}$, and the function $u(t,x)$ is a solution of problem* (1.3) *and the initial data $u_0(x) \in H^s$. Then the following results hold.*

For $q \in (0, s-1]$, there is a constant c such that

$$\int_R \left(\Lambda^{q+1}u\right)^2 dx \leq \int_R \left[\left(\Lambda^{q+1}u_0\right)^2\right] dx + c\int_0^t \|u\|_{H^{q+1}}^2 \Big(\|u_x\|_{L^\infty}^{2N} + \|u_x\|_{L^\infty}^2 \|u\|_{L^\infty}^{2m-2} + \|u_x\|_{L^\infty}\|u\|_{L^\infty}^{2m-1} + \|u\|_{L^\infty}^{2m}\Big) d\tau. \tag{3.3}$$

For $q \in [0, s-1]$, there is a constant c such that

$$\|u_t\|_{H^q} \leq c\|u\|_{H^{q+1}}\left(\|u\|_{L^\infty}^{2m-1} + \|u_x\|_{L^\infty}^{2N-1}\right). \tag{3.4}$$

Proof. For $q \in (0, s-1]$, applying $(\Lambda^q u)\Lambda^q$ to both sides of the first equation of system (1.3) and integrating with respect to x by parts, we have the identity

$$\frac{1}{2}\frac{d}{dt}\int_R \left[(\Lambda^q u)^2 + \alpha(\Lambda^q u_x)^2\right] dx = \int_R (\Lambda^q u)\Lambda^q(\varphi(u_x)_x) dx + \beta\int_R \Lambda^q u \Lambda^q\left[u^{2m}u_{xx}\right] dx. \tag{3.5}$$

We will estimate the two terms on the right-hand side of (3.5), respectively. For the first term, by using the Cauchy-Schwartz inequality and Lemmas 3.1 and 3.2, we have

$$\begin{aligned}\left|\int_R (\Lambda^q u)\Lambda^q(\varphi(u_x)_x) dx\right| &= \left|\int_R (\Lambda^q u_x)\Lambda^q(\varphi(u_x)) dx\right| \\ &\leq c\|\Lambda^q u_x\|_{L^2}\left\|\Lambda^q (u_x)^{2N+1}\right\|_{L^2} \leq c\|u\|_{H^{q+1}}^2 \|u_x\|_{L^\infty}^{2N}.\end{aligned} \tag{3.6}$$

For the second term, we have

$$\begin{aligned}\int_R \Lambda^q u\Lambda^q\left[u^{2m}u_{xx}\right] dx &= \int_R \Lambda^q u \Lambda^q\left[\left(u^{2m}u_x\right)_x - 2mu^{2m-1}u_x^2\right] dx \\ &= \int_R \Lambda^q u_x \Lambda^q\left(u^{2m}u_x\right) dx - 2m\int_R \Lambda^q u\Lambda^q\left[u^{2m-1}u_x^2\right] dx = K_1 + K_2.\end{aligned} \tag{3.7}$$

For K_1, applying Lemma 3.1 derives

$$|K_1| \leq c\|u\|_{H^{q+1}}^2\left(\|u\|_{L^\infty}^{2m} + \|u_x\|_{L^\infty}\|u\|_{L^\infty}^{2m-1}\right). \tag{3.8}$$

For K_2, we get

$$
\begin{aligned}
|K_2| &\le c\|u\|_{H^q}\|u^{2m-1}u_x^2\|_{H^q} \\
&\le c\|u\|_{H^q}\left(\|u^{2m-1}u_x\|_{L^\infty}\|u_x\|_{H^q} + \|u^{2m-1}u_x\|_{H^q}\|u_x\|_{L^\infty}\right) \\
&\le c\|u\|_{H^{q+1}}^2\left(\|u_x\|_{L^\infty}\|u\|_{L^\infty}^{2m-1} + \|u_x\|_{L^\infty}^2\|u\|_{L^\infty}^{2m-2}\right).
\end{aligned} \tag{3.9}
$$

It follows from (3.5)–(3.9) that there exists a constant c such that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_R\left[(\Lambda^q u)^2 + (\Lambda^q u_x)^2\right]dx \\
&\quad\le c\|u\|_{H^{q+1}}^2\left(\|u_x\|_{L^\infty}^{2N} + \|u_x\|_{L^\infty}^2\|u\|_{L^\infty}^{2m-2} + \|u_x\|_{L^\infty}\|u\|_{L^\infty}^{2m-1} + \|u\|_{L^\infty}^{2m}\right).
\end{aligned} \tag{3.10}
$$

Integrating both sides of the above inequality with respect to t results in inequality (3.3).

To estimate the norm of u_t, we apply the operator $(1-\partial_x^2)^{-1}$ to both sides of the first equation of system (1.3) to obtain the equation

$$
u_t = \Lambda^{-2}\left(\frac{\partial}{\partial x}\varphi(u_x) + \beta u^{2m}u_{xx}\right). \tag{3.11}
$$

Applying $(\Lambda^q u_t)\Lambda^q$ to both sides of (3.11) for $q \in [0, s-1]$ gives rise to

$$
\int_R (\Lambda^q u_t)^2 dx = \int_R (\Lambda^q u_t)\Lambda^{q-2}\left[\partial_x\varphi(u_x) + u^{2m}u_{xx}\right]d\tau. \tag{3.12}
$$

For the right-hand of (3.12), we have

$$
\begin{aligned}
&\left|\int_R(\Lambda^q u_t)\left(1-\partial_x^2\right)^{-1}\Lambda^q\partial_x\varphi(u_x)dx\right| \\
&\quad\le c\|u_t\|_{H^q}\left(\int_R\left(1+\xi^2\right)^{q-1}\left[\int_R\left[\widehat{u_x^{2N}}(\xi-\eta)\widehat{u_x}(\eta)\right]d\eta\right]^2\right)^{1/2} \\
&\quad\le c\|u_t\|_{H^q}\|u\|_{H^1}\|u\|_{H^{q+1}}\|u_x\|_{L^\infty}^{2N-1}, \\
&\left|\int_R(\Lambda^q u_t)\left(1-\partial_x^2\right)^{-1}\Lambda^q\partial_x\left(u^{2m}u_x\right)dx\right| \\
&\quad\le c\|u_t\|_{H^q}\left(\int_R\left(1+\xi^2\right)^{q-1}\left[\int_R\left[\widehat{u^{2m}}(\xi-\eta)\widehat{u_x}(\eta)\right]d\eta\right]^2\right)^{1/2} \\
&\quad\le c\|u_t\|_{H^q}\|u\|_{H^1}\|u\|_{H^{q+1}}\|u\|_{L^\infty}^{2m-1}, \\
&\left|\int_R(\Lambda^q u_t)\left(1-\partial_x^2\right)^{-1}\Lambda^q\left(u^{2m-1}u_x^2\right)dx\right| \\
&\quad\le c\|u_t\|_{H^q}\left(\int_R\left(1+\xi^2\right)^{q-1}\left[\int_R\left[\widehat{u^{2m-1}u_x}(\xi-\eta)\widehat{u_x}(\eta)\right]d\eta\right]^2\right)^{1/2} \\
&\quad\le c\|u_t\|_{H^q}\|u\|_{H^1}\|u\|_{H^{q+1}}\|u\|_{L^\infty}^{2m-1}.
\end{aligned} \tag{3.13}
$$

Applying (3.13) into (3.12) yields the inequality

$$\|u_t\|_{H^q} \leq c\|u\|_{H^1}\|u\|_{H^{q+1}}\left(\|u\|_{L^\infty}^{2m-1} + \|u_x\|_{L^\infty}^{2N-1}\right) \tag{3.14}$$

for a constant $c > 0$. This completes the proof of Lemma 3.3. □

Lemma 3.4. *If $u(t,x)$ is a solution of problem* (1.3)*, $\alpha > 0$, $\varphi(\eta) = \eta^{2N+1}$, then*

$$\|u\|_{L^\infty} \leq c\|u\|_{H^1(R)} \leq c\|u_0\|_{H^1(R)}, \tag{3.15}$$

where c is a constant.

Proof. Multiplying both sides of the first equation of (1.3) by $u(t,x)$ and integrating with respect to x over R, we have

$$\frac{1}{2}\frac{d}{dt}\int_R\left[u(t,x)^2 + \alpha u_x(t,x)^2\right]dx = \int_R \varphi(u_x)_x u(t,x)dx + \beta\int_R u^{2m+1}u_{xx}dx. \tag{3.16}$$

Since

$$\int_R \varphi(u_x)_x u(t,x)dx + \beta\int_R u^{2m+1}u_{xx}dx = -\int_R u_x^{2N+2}dx - \beta(2m+1)\int_R u^{2m}u_x^2dx < 0, \tag{3.17}$$

we derive that

$$\frac{1}{2}\frac{d}{dt}\int_R\left[u(t,x)^2 + \alpha u_x(t,x)^2\right]dx < 0, \tag{3.18}$$

which results in

$$\int_R\left[u(t,x)^2 + \alpha u_x(t,x)^2\right] < \int_R\left[u(0,x)^2 + \alpha u_x(0,x)^2\right] \leq c\|u_0\|_{H^1}^2. \tag{3.19}$$

From (3.19), we know that (3.15) holds. This completes the proof. □

Defining

$$\phi(x) = \begin{cases} e^{1/(x^2-1)}, & |x| < 1, \\ 0, & |x| \geq 1 \end{cases} \tag{3.20}$$

and setting $\phi_\varepsilon(x) = \varepsilon^{-1/4}\phi(\varepsilon^{-1/4}x)$ with $0 < \varepsilon < 1/4$ and $u_{\varepsilon 0} = \phi_\varepsilon \star u_0$, we know that $u_{\varepsilon 0} \in C^\infty$ for any $u_0 \in H^s(R)$ and $s > 0$.

It follows from Theorem 2.1 that for each ε the Cauchy problem

$$u_t - u_{txx} = \frac{\partial}{\partial x}\varphi(u_x) + \beta u^{2m} u_{xx}, \qquad u(0,x) = u_{\varepsilon 0}(x), \quad x \in R \tag{3.21}$$

has a unique solution $u_\varepsilon(t,x) \in C^\infty([0,T); H^\infty)$.

Lemma 3.5. *Under the assumptions of problem* (3.21), *the following estimates hold for any* ε *with* $0 < \varepsilon < 1/4$, $u_0 \in H^s(R)$ *and* $s > 0$:

$$\|u_{\varepsilon 0x}\|_{L^\infty} \le c_1 \|u_{0x}\|_{L^\infty}, \qquad \|u_{\varepsilon 0}\|_{H^q} \le c_1, \quad \text{if } q \le s, \tag{3.22}$$

where c_1 *is a constant independent of* ε.

Proof. Using the definition of $u_{\varepsilon 0}$ and $u_{\varepsilon 0x}$ results in the conclusion of the lemma. □

Lemma 3.6. *Suppose that* $u_0(x) \in H^s(R)$ *with* $s \in [1, 3/2]$ *such that* $\|u_{0x}\|_{L^\infty} < \infty$. *Let* $u_{\varepsilon 0}$ *be defined as in system* (3.21) *and let* $\varphi(\eta) = \eta^{2N+1}$. *Then there exist two positive constants* T *and* c, *independent of* ε, *such that the solution* u_ε *of problem* (3.21) *satisfies* $\|u_{\varepsilon x}\|_{L^\infty} \le c$ *for any* $t \in [0,T)$.

Proof. Using notation $u = u_\varepsilon$ and differentiating both sides of the first equation of problem (3.11) with respect to x give rise to

$$u_{tx} = -\varphi(u_x) - \beta u^{2m} u_x + \Lambda^{-2}\Big[\varphi(u_x) + \beta u^{2m} u_x - 2m\beta\Big(u^{2m-1}u_x\Big)_x\Big]. \tag{3.23}$$

Letting $p > 0$ be an integer and multiplying the above equation by $(u_x)^{2p+1}$ and then integrating the resulting equation with respect to x yield the equality

$$\frac{1}{2p+2}\frac{d}{dt}\int_R (u_x)^{2p+2} dx = -\int_R \varphi(u_x) u_x^{2p+1} dx - \beta \int_R u^{2m} u_x^{2p+2} dx + \int_R J u_x^{2p+1} dx, \tag{3.24}$$

where

$$J = \Lambda^{-2}\Big[\varphi(u_x) + \beta u^{2m} u_x - 2m\beta\Big(u^{2m-1}u_x\Big)_x\Big]. \tag{3.25}$$

Applying the Hölder's inequality to (3.24) and noting Lemmas 3.4 and 3.5, we obtain

$$\begin{aligned}\frac{1}{2p+2}\frac{d}{dt}\int_R (u_x)^{2p+2} dx &\le c\|u_x\|_{L^\infty}^{2N} \int_R |u_x|^{2p+2} dx + c\int_R u_x^{2p+2} dx \\ &\quad + \left(\int_R |J|^{2p+2} dx\right)^{1/(2p+2)} \left(u_x^{2p+2} dx\right)^{2(p+1)/2(p+2)}\end{aligned} \tag{3.26}$$

or

$$\frac{d}{dt}\left(\int_R (u_x)^{2(p+2)}dx\right)^{1/(2p+2)} \le c\|u_x\|_{L^\infty}^{2N}\left(\int_R u_x^{2p+2}dx\right)^{1/(2p+2)} + c\left(\int_R u_x^{2p+2}dx\right)^{1/(2p+2)} + \left(\int_R |J|^{2p+2}dx\right)^{1/(2p+2)}. \tag{3.27}$$

Since $\|f\|_{L^p} \to \|f\|_{L^\infty}$ as $p \to \infty$ for any $f \in L^\infty \bigcap L^2$, integrating both sides of the inequality (3.27) with respect to t and taking the limit as $p \to \infty$ result in the estimate

$$\|u_x\|_{L^\infty} \le \|u_{0x}\|_{L^\infty} + \int_0^t c\left(\|u_x\|_{L^\infty} + \|u_x\|_{L^\infty}^{2N+1} + \|J\|_{L^\infty}\right)d\tau. \tag{3.28}$$

Using the algebra property of $H^{s_0}(R)$ with $s_0 > 1/2$ yields ($\|u_\varepsilon\|_{H^{1/2+}}$ means that there exists a sufficiently small $\delta > 0$ such that $\|u_\varepsilon\|_{1/2+} = \|u_\varepsilon\|_{H^{1/2+\delta}}$)

$$\begin{aligned}\|J\|_{L^\infty} &\le c\|J\|_{H^{1/2+}} \le c\|\Lambda^{-2}\left[\varphi(u_x) + \beta u^{2m}u_x - 2m\beta\left(u^{2m-1}u_x\right)_x\right]\|_{H^{1/2+}}\\ &\le c\left(\|\varphi(u_x)\|_{H^0} + \|u\|_{H^1} + \|u^{2m-1}u_x\|_{H^0}\right)\\ &\le c\left(\|u_x\|^{2N}\|u\|_{H^1} + \|u\|_{H^1} + \|u\|_{L^\infty}^{2m-1}\|u\|_{H^1}\right)\\ &\le c\left(\|u_x\|^{2N} + 1\right),\end{aligned} \tag{3.29}$$

in which Lemmas 3.4 and 3.5 are used. From (3.28) and (3.29), one has

$$\|u_x\|_{L^\infty} \le \|u_{0x}\|_{L^\infty} + c\int_0^t \left[1 + \|u_x\|_{L^\infty} + \|u_x\|_{L^\infty}^{2N} + \|u_x\|_{L^\infty}^{2N+1}\right]d\tau. \tag{3.30}$$

From Lemma 3.5, it follows from the contraction mapping principle that there is a $T > 0$ such that the equation

$$\|W\|_{L^\infty} = \|u_{0x}\|_{L^\infty} + c\int_0^t \left[1 + \|W\|_{L^\infty} + \|W\|_{L^\infty}^{2N} + \|W\|_{L^\infty}^{2N+1}\right]d\tau \tag{3.31}$$

has a unique solution $W \in C[0,T]$. Using the result presented on page 51 in [11] yields that there are constants $T > 0$ and $c > 0$ independent of ε such that $\|u_x\|_{L^\infty} \le \|W(t)\|_{L^\infty} \le c$ for arbitrary $t \in [0,T]$, which leads to the conclusion of Lemma 3.6. □

Using Lemmas 3.3–3.6, notation $u_\varepsilon = u$ and Gronwall's inequality result in the inequalities

$$\begin{aligned}\|u_\varepsilon\|_{H^q} &\le C_T e^{C_T},\\ \|u_{\varepsilon t}\|_{H^r} &\le C_T e^{C_T},\end{aligned} \tag{3.32}$$

where $q \in (0, s]$, $r \in (0, s-1]$ $(1 \leq s \leq 3/2)$ and C_T depends on T. It follows from the Aubin's compactness theorem that there is a subsequence of $\{u_\varepsilon\}$, denoted by $\{u_{\varepsilon_n}\}$, such that $\{u_{\varepsilon_n}\}$ and their temporal derivatives $\{u_{\varepsilon_n t}\}$ are weakly convergent to a function $u(t, x)$ and its derivative u_t in $L^2([0,T], H^s)$ and $L^2([0,T], H^{s-1})$, respectively. Moreover, for any real number $R_1 > 0$, $\{u_{\varepsilon_n}\}$ is convergent to the function u strongly in the space $L^2([0,T], H^q(-R_1, R_1))$ and $\{u_{\varepsilon_n t}\}$ converges to u_t strongly in the space $L^2([0,T], H^r(-R_1, R_1))$ for $r \in [0, s-1]$. Thus, we can prove the existence of a weak solution to (1.3).

Theorem 3.7. *Suppose that $u_0(x) \in H^s$ with $1 \leq s \leq 3/2$, $\|u_{0x}\|_{L^\infty} < \infty$ and $\varphi(\eta) = \eta^{2N+1}$. Then there exists a $T > 0$ such that* (1.3) *subject to initial value $u_0(x)$ has a weak solution $u(t,x) \in L^2([0,T], H^s)$ in the sense of distribution and $u_x \in L^\infty([0,T] \times R)$.*

Proof. From Lemma 3.6, we know that $\{u_{\varepsilon_n x}\}$ $(\varepsilon_n \to 0)$ is bounded in the space L^∞. Thus, the sequences $\{u_{\varepsilon_n}\}$, $\{u_{\varepsilon_n x}\}$, $\{u^2_{\varepsilon_n x}\}$, and $\{u^{2N+1}_{\varepsilon_n x}\}$ are weakly convergent to u, u_x, u_x^2, and u_x^{2N+1} in $L^2[0,T], H^r(-R, R)$ for any $r \in [0, s-1)$, separately. Therefore, u satisfies the equation

$$\int_0^T \int_R u(g_t - g_{xxt}) dx\, dt = \int_0^T \int_R \left[u_x^{2N+1} g_x + \beta u^{2m} u_x g_x - 2m\beta u^{2m-1} u_x^2 g\right] dx\, dt, \tag{3.33}$$

with $u(0,x) = u_0(x)$ and $g \in C_0^\infty$. Since $X = L^1([0,T] \times R)$ is a separable Banach space and $\{u_{\varepsilon_n x}\}$ is a bounded sequence in the dual space $X^* = L^\infty([0,T] \times R)$ of X, there exists a subsequence of $\{u_{\varepsilon_n x}\}$, still denoted by $\{u_{\varepsilon_n x}\}$, weakly star convergent to a function v in $L^\infty([0,T] \times R)$. It derives from the weakly convergence of $\{u_{\varepsilon_n x}\}$ to u_x in $L^2([0,T] \times R)$ that $u_x = v$ almost everywhere. Thus, we obtain $u_x \in L^\infty([0,T] \times R)$. □

4. Well-Posedness of Global Solutions

Lemma 4.1. *If $u(t,x)$ is a solution of problem* (1.3), *$\alpha > 0$, $\varphi(\eta) = \eta^{2N+1}$, then*

$$\|u_x\|_{L^\infty} \leq A^{1/2}, \tag{4.1}$$

where

$$A = \int_R \left[\frac{1+\alpha}{\alpha} \left(u_0'(x)\right)^2 + (1+\alpha)\left(u_0''(x)\right)^2\right] dx. \tag{4.2}$$

Proof. Multiplying each side of the first equation of problem (1.3) by u_{xx} and integrating over $[0,t] \times R$ yields

$$\int_0^t \int_R \left(u_{xx}^2 \varphi'(u_x) + u^{2m} u_{xx}^2 + \frac{\alpha}{2} \frac{\partial}{\partial t}\left(u_{xx}^2\right)\right) dx\, dt = \int_0^t \int_R u_t u_{xx} dx\, dt. \tag{4.3}$$

Integrating the right-hand side of the above identity by parts and using $u_x(\pm\infty) = 0$, we get

$$2\int_0^t \int_R u_t u_{xx} dx dt = \int_R \left[u_0'(x)\right]^2 dx - \int_R u_x^2(t,x) dx. \tag{4.4}$$

From (4.3), (4.4) and the assumption of this lemma, we have

$$\alpha\|u_{xx}\|_{L^2}^2 + \|u_x\|_{L^2} \le \int_R \left[\left(u_0'(x)\right)^2 + \alpha\left(u_0''(x)\right)^2\right] dx, \tag{4.5}$$

from which we obtain (4.1). □

Theorem 4.2. *Suppose that* $s \ge 2$, $u_0 \in H^s(R)$, $\varphi(u_x) = u_x^{2N+1}$ *with positive integer* N. *Then problem* (1.3) *has a unique global solution:*

$$u(t,x) \in C([0,\infty); H^s(R)) \bigcap C^1\left([0,\infty); H^{s-1}(R)\right). \tag{4.6}$$

Proof. Using the Gronwall inequality and Lemma 3.3 and choosing $s = q + 1$, we have

$$\|u\|_{H^s} \le c\|u_0\|_{H^s} e^{\int_0^t (\|u_x\|_{L^\infty}^{2N} + \|u_x\|_{L^\infty}^2 \|u\|_{L^\infty}^{2m-2} + \|u_x\|_{L^\infty} \|u\|_{L^\infty}^{2m-1} + \|u\|_{L^\infty}^{2m}) d\tau}. \tag{4.7}$$

From Lemma 4.1, we have

$$\|u_x\| \le A^{1/2} = \left(\int_R \left[\frac{1+\alpha}{\alpha}\left(u_0'(x)\right)^2 + (1+\alpha)\left(u_0''(x)\right)^2\right] dx\right)^{1/2} \le c\|u_0\|_{H^2(R)}. \tag{4.8}$$

Using (4.7) and (4.8) derives

$$\|u\|_{H^s} \le c\|u_0\|_{H^s} e^{ct}, \tag{4.9}$$

which completes the proof of Theorem 4.2. □

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