

Research Article

On Integral Inequalities of Hermite-Hadamard Type for s -Geometrically Convex Functions

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Received 12 May 2012; Accepted 19 May 2012

Academic Editor: Yonghong Yao

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The authors introduce the concept of the s -geometrically convex functions. By the well-known Hölder inequality, they establish some integral inequalities of Hermite-Hadamard type related to the s -geometrically convex functions and apply these inequalities to special means.

1. Introduction

We firstly list several definitions and some known results.

Definition 1.1. A function $f : I \subset \mathbb{R} = (-\infty, +\infty) \rightarrow \mathbb{R}$ is said to be convex if

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y) \quad (1.1)$$

for $x, y \in I$ and $\lambda \in [0, 1]$.

Definition 1.2 (see [1]). A function $f : I \subset \mathbb{R}_0 = [0, +\infty) \rightarrow \mathbb{R}_0$ is said to be s -convex if

$$f(\lambda x + (1 - \lambda)y) \leq \lambda^s f(x) + (1 - \lambda)^s f(y) \quad (1.2)$$

for some $s \in (0, 1]$, where $x, y \in I$, and $\lambda \in [0, 1]$.

If $s = 1$, the s -convex function becomes a convex function on $\mathbb{R}_0$.

Theorem 1.3 ([2], Theorem 2.2). Let $f : I^\circ \subset \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I° , $a, b \in I^\circ$, $a < b$.

(i) If $|f'(x)|$ is a convex function on $[a, b]$, then

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{(b-a)(|f'(a)| + |f'(b)|)}{8}. \quad (1.3)$$

(ii) If $|f'(x)|^{p/(p-1)}$ is a convex function on $[a, b]$, for $p > 1$, then

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{b-a}{2(p+1)^{(1/p)}} \left(\frac{|f'(a)|^{p/(p-1)} + |f'(b)|^{p/(p-1)}}{2} \right)^{(p-1)/p}. \quad (1.4)$$

Theorem 1.4 ([3], Theorems 2.3 and 2.4). Let $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be differentiable on I° , $a, b \in I$, $a < b$. If $|f'(x)|^p$ is convex on $[a, b]$, for $p > 1$, then

$$\begin{aligned} \left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| &\leq \frac{b-a}{4} \left(\frac{4}{p+1}\right)^{1/p} (|f'(a)| + |f'(b)|), \\ \left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| &\leq \frac{b-a}{16} \left(\frac{4}{p+1}\right)^{1/p} \\ &\times \left[\left(|f'(a)|^{p/(p-1)} + 3|f'(b)|^{p/(p-1)} \right)^{(p-1)/p} \right. \\ &\left. + \left(3|f'(a)|^{p/(p-1)} + |f'(b)|^{p/(p-1)} \right)^{(p-1)/p} \right]. \end{aligned} \quad (1.5)$$

Theorem 1.5 ([4], Theorem 1-4). Let $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be differentiable on I° , $a, b \in I$, $a < b$. If $|f'(x)|^q$ is s -convex on $[a, b]$, for $q > 1$, then

$$\begin{aligned} \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ \leq \frac{b-a}{2} \left(\frac{1}{2}\right)^{(q-1)/q} \left(\frac{2 + (1/2)^s}{(s+1)(s+2)}\right)^{1/q} (|f'(a)|^s + |f'(b)|^s)^{1/q}. \end{aligned} \quad (1.6)$$

Theorem 1.6 ([5], Theorem 4). Let $f : I \rightarrow \mathbb{R}_0$ be differentiable on I° , $a, b \in I$, $a < b$, and $f' \in L([a, b])$. If $|f'(x)|^q$ is s -convex on $[a, b]$ for some $s \in (0, 1]$, and $p, q \geq 1$, such that $(1/q) + (1/p) = 1$ then

$$\begin{aligned} \left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| &\leq 2^{-1/p} \frac{(|f'(a)|^q + (s+1)|f'((a+b)/2)|^q)^{1/q}}{\{(s+1)(s+2)\}^{1/q}} \\ &\quad + 2^{-1/p} \frac{(|f'(b)|^q + (s+1)|f'((a+b)/2)|^q)^{1/q}}{\{(s+1)(s+2)\}^{1/q}} \\ &= 2^{-1/p} \left[\left(\beta(s+1, 2)|f'(a)|^q + \beta(s+2, 1) \left| f'\left(\frac{a+b}{2}\right) \right|^q \right)^{1/q} \right. \\ &\quad \left. + \left(\beta(s+1, 2)|f'(b)|^q + \beta(s+2, 1) \left| f'\left(\frac{a+b}{2}\right) \right|^q \right)^{1/q} \right]. \end{aligned} \quad (1.7)$$

Theorem 1.7 ([6], Theorems 2.2–2.4). Let $f : I \rightarrow \mathbb{R}_0$ be differentiable on I° , $a, b \in I$, $a < b$, and $f' \in L([a, b])$.

(i) If $|f'(x)|$ is s -convex on $[a, b]$ for some $s \in (0, 1]$, then

$$\begin{aligned} \left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| &\leq \frac{b-a}{4(s+1)(s+2)} \left[|f'(a)| + 2(s+1) \left| f'\left(\frac{a+b}{2}\right) \right| + |f'(b)| \right] \\ &\leq \frac{(2^{2-s} + 1)(b-a)}{4(s+1)(s+2)} [|f'(a)| + |f'(b)|]. \end{aligned} \quad (1.8)$$

(ii) If $|f'(x)|^{p/(p-1)}$ ($p > 1$) is a s -convex function on $[a, b]$ for some $s \in (0, 1]$, then

$$\begin{aligned} \left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| &\leq \frac{b-a}{4} \left(\frac{1}{p+1} \right)^{1/p} \left(\frac{1}{s+1} \right)^{2/q} \left[\left((2^{1-s} + s + 1) |f'(a)|^q + 2^{1-s} |f'(b)|^q \right)^{1/q} \right. \\ &\quad \left. + \left(2^{1-s} |f'(a)|^q |f'(a)|^{p/(p-1)} + (2^{1-s} + s + 1) |f'(b)|^q \right)^{1/q} \right], \end{aligned} \quad (1.9)$$

where $1/p + 1/q = 1$.

(iii) If $|f'(x)|^q$ ($q \geq 1$) is s -convex on $[a, b]$ for some $s \in (0, 1]$, then

$$\begin{aligned} & \left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{b-a}{8} \left(\frac{2}{(s+1)(s+2)} \right)^{1/q} \left[\left((2^{1-s} + 1) |f'(a)|^q + 2^{1-s} |f'(b)|^q \right)^{1/q} \right. \\ & \quad \left. + \left(2^{1-s} |f'(a)|^q |f'(a)|^q + (2^{1-s} + 1) |f'(b)|^q \right)^{1/q} \right]. \end{aligned} \quad (1.10)$$

Now we introduce the definition of the s -geometrically convex function.

Definition 1.8. A function $f : I \subset \mathbb{R}_+ = (0, +\infty) \rightarrow \mathbb{R}_+$ is said to be a geometrically convex function if

$$f(x^\lambda y^{1-\lambda}) \leq [f(x)]^\lambda [f(y)]^{1-\lambda} \quad (1.11)$$

for $x, y \in I$ and $\lambda \in [0, 1]$.

Definition 1.9. A function $f : I \subset \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is said to be a s -geometrically convex function if

$$f(x^\lambda y^{1-\lambda}) \leq [f(x)]^{\lambda^s} [f(y)]^{(1-\lambda)^s} \quad (1.12)$$

for some $s \in (0, 1]$, where $x, y \in I$ and $\lambda \in [0, 1]$.

If $s = 1$, the s -geometrically convex function becomes a geometrically convex function on $\mathbb{R}_+$.

In this paper, we will establish some integral inequalities of Hermite-Hadamard type related to the s -geometrically convex functions and then apply these inequalities to special means.

2. A Lemma

In order to prove our results, we need the following lemma.

Lemma 2.1. Let $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be differentiable on I° , and $a, b \in I$, with $a < b$. If $f' \in L([a, b])$, then

$$\begin{aligned} & f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \\ & = \frac{b-a}{4} \int_0^1 \left[t f'\left((1-t)a + t \frac{a+b}{2}\right) + (t-1) f'\left((1-t) \frac{a+b}{2} + tb\right) \right] dt, \end{aligned}$$

$$\begin{aligned}
& \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \\
&= \frac{b-a}{4} \int_0^1 \left[(t-1)f' \left((1-t)a + t \frac{a+b}{2} \right) + tf' \left((1-t) \frac{a+b}{2} + tb \right) \right] dt.
\end{aligned} \tag{2.1}$$

Proof. Integrating by part and changing variables of integration yields

$$\begin{aligned}
& \int_0^1 \left[tf' \left((1-t)a + t \frac{a+b}{2} \right) + (t-1)f' \left((1-t) \frac{a+b}{2} + tb \right) \right] dt \\
&= \frac{2}{b-a} \left[tf \left((1-t)a + t \frac{a+b}{2} \right) \Big|_0^1 - \int_0^1 f \left((1-t)a + t \frac{a+b}{2} \right) dt \right] \\
&\quad + \frac{2}{b-a} \left[(t-1)f \left((1-t) \frac{a+b}{2} + tb \right) \Big|_0^1 - \int_0^1 f \left((1-t) \frac{a+b}{2} + tb \right) dt \right] \\
&= \frac{4}{b-a} \left(f \left(\frac{a+b}{2} \right) - \frac{1}{b-a} \int_a^b f(x) dx \right), \\
& \int_0^1 \left[(t-1)f' \left((1-t)a + t \frac{a+b}{2} \right) + tf' \left((1-t) \frac{a+b}{2} + tb \right) \right] dt \\
&= \frac{2}{b-a} \left[(t-1)f \left((1-t)a + t \frac{a+b}{2} \right) \Big|_0^1 - \int_0^1 f \left((1-t)a + t \frac{a+b}{2} \right) dt \right] \\
&\quad + \frac{2}{b-a} \left[tf \left((1-t) \frac{a+b}{2} + tb \right) \Big|_0^1 - \int_0^1 f \left((1-t) \frac{a+b}{2} + tb \right) dt \right] \\
&= \frac{4}{b-a} \left(\frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right).
\end{aligned} \tag{2.2}$$

This completes the proof of Lemma 2.1. $\square$

3. Main Results

Theorem 3.1. Let $f : I \subset \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be differentiable on I° , $a, b \in I$, with $a < b$, and $f' \in L([a, b])$. If $|f'(x)|^q$ is s -geometrically convex and monotonically decreasing on $[a, b]$ for $q \geq 1$ and $s \in (0, 1]$, then

$$\left| f \left(\frac{a+b}{2} \right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{b-a}{4} \left(\frac{1}{2} \right)^{1-1/q} G_1(s, q; g_1(\alpha), g_2(\alpha)) \tag{3.1}$$

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{b-a}{4} \left(\frac{1}{2} \right)^{1-1/q} G_1(s, q; g_2(\alpha), g_1(\alpha)), \tag{3.2}$$

where

$$g_1(\alpha) = \begin{cases} \frac{1}{2}, & \alpha = 1, \\ \frac{\frac{1}{\alpha} \ln \alpha - \alpha + 1}{[\ln \alpha]^2}, & \alpha \neq 1, \end{cases} \quad g_2(\alpha) = \begin{cases} \frac{1}{2}, & \alpha = 1, \\ \frac{\frac{1}{\alpha} - \ln \alpha - 1}{[\ln \alpha]^2}, & \alpha \neq 1, \end{cases} \quad (3.3)$$

$$\alpha(u, v) = |f'(a)|^{-u} |f'(b)|^v, u, v > 0, \quad (3.4)$$

$$G_1(s, q; g_1(\alpha), g_2(\alpha))$$

$$= \begin{cases} |f'(a)|^s \left[g_1\left(\alpha\left(\frac{sq}{2}, \frac{sq}{2}\right)\right) \right]^{1/q} + |f'(a)f'(b)|^{s/2} \left[g_2\left(\alpha\left(\frac{sq}{2}, \frac{sq}{2}\right)\right) \right]^{1/q}, & |f'(a)| \leq 1, \\ |f'(a)|^{1/s} \left[g_1\left(\alpha\left(\frac{q}{2s}, \frac{sq}{2}\right)\right) \right]^{1/q} + |f'(a)|^{1/2s} |f'(b)|^{s/2} \left[g_2\left(\alpha\left(\frac{q}{2s}, \frac{sq}{2}\right)\right) \right]^{1/q}, & |f'(b)| \leq 1 \leq |f'(a)|, \\ |f'(a)|^{1/s} \left[g_1\left(\alpha\left(\frac{q}{2s}, \frac{q}{2s}\right)\right) \right]^{1/q} + |f'(a)f'(b)|^{1/2s} \left[g_2\left(\alpha\left(\frac{q}{2s}, \frac{q}{2s}\right)\right) \right]^{1/q}, & 1 \leq |f'(b)|. \end{cases} \quad (3.5)$$

Proof. (1) Since $|f'(x)|^q$ is s -geometrically convex and monotonically decreasing on $[a, b]$, from Lemma 2.1 and Hölder inequality, we have

$$\begin{aligned} & \left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{b-a}{4} \int_0^1 \left| t \left| f'\left((1-t)a + t\frac{a+b}{2}\right) \right| + |t-1| \left| f'\left((1-t)\frac{a+b}{2} + tb\right) \right| \right| dt \\ & \leq \frac{b-a}{4} \left\{ \left(\int_0^1 t dt \right)^{1-1/q} \left[\int_0^1 t \left| f'\left((1-t)a + t\frac{a+b}{2}\right) \right|^q dt \right]^{1/q} \right. \\ & \quad \left. + \left(\int_0^1 (1-t) dt \right)^{1-1/q} \left[\int_0^1 (1-t) \left| f'\left((1-t)\frac{a+b}{2} + tb\right) \right|^q dt \right]^{1/q} \right\} \\ & \leq \frac{b-a}{4} \left(\frac{1}{2} \right)^{1-1/q} \left\{ \left[\int_0^1 t \left| f'\left(a^{(2-t)/2} b^{t/2}\right) \right|^q dt \right]^{1/q} \right. \\ & \quad \left. + \left[\int_0^1 (1-t) \left| f'\left(a^{(1-t)/2} b^{(1+t)/2}\right) \right|^q dt \right]^{1/q} \right\} \end{aligned}$$

$$\begin{aligned} &\leq \frac{b-a}{4} \left(\frac{1}{2}\right)^{1-1/q} \left\{ \left[\int_0^1 t |f'(a)|^{q((2-t)/2)^s} |f'(b)|^{q(t/2)^s} dt \right]^{1/q} \right. \\ &\quad \left. + \left[\int_0^1 (1-t) |f'(a)|^{q((1-t)/2)^s} |f'(b)|^{q((1+t)/2)^s} dt \right]^{1/q} \right\}. \end{aligned} \quad (3.6)$$

If $0 < \mu \leq 1 \leq \eta, 0 < \alpha, s \leq 1$, then

$$\mu^{\alpha^s} \leq \mu^{\alpha s}, \quad \eta^{\alpha^s} \leq \eta^{\alpha/s}. \quad (3.7)$$

(i) If $|f'(a)| \leq 1$, by (3.7), we obtain that

$$\begin{aligned} &\int_0^1 t \left(|f'(a)|^{q((2-t)/2)^s} |f'(b)|^{q(t/2)^s} \right) dt \\ &\leq \int_0^1 t \left(|f'(a)|^{(sq/2)(2-t)} |f'(b)|^{(sq/2)t} \right) dt = |f'(a)|^{sq} g_1 \left(\alpha \left(\frac{sq}{2}, \frac{sq}{2} \right) \right), \\ &\int_0^1 (1-t) \left(|f'(a)|^{q((1-t)/2)^s} |f'(b)|^{q((1+t)/2)^s} \right) dt \\ &\leq \int_0^1 (1-t) \left(|f'(a)|^{(sq/2)(1-t)} |f'(b)|^{(sq/2)(1+t)} \right) dt = |f'(a)f'(b)|^{sq/2} g_2 \left(\alpha \left(\frac{sq}{2}, \frac{sq}{2} \right) \right). \end{aligned} \quad (3.8)$$

(ii) If $|f'(b)| \leq 1 \leq |f'(a)|$, by (3.7), we obtain that

$$\begin{aligned} &\int_0^1 t \left(|f'(a)|^{q((2-t)/2)^s} |f'(b)|^{q(t/2)^s} \right) dt \\ &\leq \int_0^1 t \left(|f'(a)|^{(q/2s)(2-t)} |f'(b)|^{(sq/2)t} \right) dt = |f'(a)|^{q/s} g_1 \left(\alpha \left(\frac{q}{2s}, \frac{sq}{2} \right) \right), \\ &\int_0^1 (1-t) \left(|f'(a)|^{q((1-t)/2)^s} |f'(b)|^{q((1+t)/2)^s} \right) dt \\ &\leq \int_0^1 (1-t) \left(|f'(a)|^{(q/2s)(1-t)} |f'(b)|^{(sq/2)(1+t)} \right) dt = |f'(a)|^{q/2s} |f'(b)|^{sq/2} g_2 \left(\alpha \left(\frac{q}{2s}, \frac{sq}{2} \right) \right). \end{aligned} \quad (3.9)$$

(iii) If $1 \leq |f'(b)|$, by (3.7), we obtain that

$$\begin{aligned} &\int_0^1 t \left(|f'(a)|^{q((2-t)/2)^s} |f'(b)|^{q(t/2)^s} \right) dt \\ &\leq \int_0^1 t \left(|f'(a)|^{(q/2s)(2-t)} |f'(b)|^{(q/2s)t} \right) dt = |f'(a)|^{q/s} g_1 \left(\alpha \left(\frac{q}{2s}, \frac{q}{2s} \right) \right), \end{aligned}$$

$$\begin{aligned}
& \int_0^1 (1-t) \left(|f'(a)|^{q((1-t)/2)^s} |f'(b)|^{q((1+t)/2)^s} \right) dt \\
& \leq \int_0^1 (1-t) \left(|f'(a)|^{(q/2s)(1-t)} |f'(b)|^{(q/2s)(1+t)} \right) dt = |f'(a)f'(b)|^{q/2s} g_2 \left(\alpha \left(\frac{q}{2s}, \frac{q}{2s} \right) \right).
\end{aligned} \tag{3.10}$$

From (3.6) to (3.10), (3.1) holds.

(2) Since $|f'(x)|^q$ is s -geometrically convex and monotonically decreasing on $[a, b]$, from Lemma 2.1 and Hölder inequality, we have

$$\begin{aligned}
& \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \\
& \leq \frac{b-a}{4} \left[\left(\int_0^1 (1-t) dt \right)^{1-1/q} \left(\int_0^1 (1-t) \left| f' \left((1-t)a + t \frac{a+b}{2} \right) \right|^q dt \right)^{1/q} \right. \\
& \quad \left. + \left(\int_0^1 t dt \right)^{1-1/q} \left(\int_0^1 t \left| f' \left((1-t) \frac{a+b}{2} + tb \right) \right|^q dt \right)^{1/q} \right] \\
& \leq \frac{b-a}{4} \left(\frac{1}{2} \right)^{1-1/q} \left\{ \left[\int_0^1 (1-t) |f'(a)|^{q((2-t)/2)^s} |f'(b)|^{q(t/2)^s} dt \right]^{1/q} \right. \\
& \quad \left. + \left[\int_0^1 t |f'(a)|^{q((1-t)/2)^s} |f'(b)|^{q((1+t)/2)^s} dt \right]^{1/q} \right\}.
\end{aligned} \tag{3.11}$$

(i) If $|f'(a)| \leq 1$, by (3.7), we have

$$\begin{aligned}
& \int_0^1 (1-t) \left(|f'(a)|^{q((2-t)/2)^s} |f'(b)|^{q(t/2)^s} \right) dt \leq |f'(a)|^{sq} g_2 \left(\alpha \left(\frac{sq}{2}, \frac{sq}{2} \right) \right), \\
& \int_0^1 t \left(|f'(a)|^{q((1-t)/2)^s} |f'(b)|^{q((1+t)/2)^s} \right) dt \leq |f'(a)f'(b)|^{sq/2} g_1 \left(\alpha \left(\frac{sq}{2}, \frac{sq}{2} \right) \right).
\end{aligned} \tag{3.12}$$

(ii) If $|f'(b)| \leq 1 \leq |f'(a)|$, by (3.7), we have

$$\begin{aligned}
& \int_0^1 (1-t) \left(|f'(a)|^{q((2-t)/2)^s} |f'(b)|^{q(t/2)^s} \right) dt \leq |f'(a)|^{q/s} g_2 \left(\alpha \left(\frac{q}{2s}, \frac{sq}{2} \right) \right), \\
& \int_0^1 t \left(|f'(a)|^{q((1-t)/2)^s} |f'(b)|^{q((1+t)/2)^s} \right) dt \leq |f'(a)|^{q/2s} |f'(b)|^{sq/2} g_1 \left(\alpha \left(\frac{q}{2s}, \frac{sq}{2} \right) \right).
\end{aligned} \tag{3.13}$$

(iii) If $1 \leq |f'(b)|$, by (3.7), we have

$$\begin{aligned} \int_0^1 (1-t) \left(|f'(a)|^{q((2-t)/2)^s} |f'(b)|^{q(t/2)^s} \right) dt &\leq |f'(a)|^{q/s} g_2 \left(\alpha \left(\frac{q}{2s}, \frac{q}{2s} \right) \right), \\ \int_0^1 t \left(|f'(a)|^{q((1-t)/2)^s} |f'(b)|^{q((1+t)/2)^s} \right) dt &\leq |f'(a)f'(b)|^{q/2s} g_1 \left(\alpha \left(\frac{q}{2s}, \frac{q}{2s} \right) \right). \end{aligned} \quad (3.14)$$

From (3.11) to (3.14), (3.2) holds. This completes the required proof. $\square$

Applying Theorem 3.1 to $q = 1, s = 1$, respectively, results in the following corollary.

Corollary 3.2. Let $f : I \subset \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be differentiable on I° , $a, b \in I$ with $a < b$, and $f' \in L([a, b])$. If $|f'(x)|^q$ is s -geometrically convex and monotonically decreasing on $[a, b]$ for $s \in (0, 1]$, then

(i) when $q = 1$, one has

$$\begin{aligned} \left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| &\leq \frac{b-a}{4} G_1(s, 1; g_1(\alpha), g_2(\alpha)), \\ \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| &\leq \frac{b-a}{4} G_1(s, 1; g_2(\alpha), g_1(\alpha)). \end{aligned} \quad (3.15)$$

(ii) when $s = 1$, one has

$$\begin{aligned} \left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| &\leq \frac{b-a}{4} \left(\frac{1}{2}\right)^{1-1/q} G_1(1, q; g_1(\alpha), g_2(\alpha)), \\ \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| &\leq \frac{b-a}{4} \left(\frac{1}{2}\right)^{1-1/q} G_1(1, q; g_2(\alpha), g_1(\alpha)), \end{aligned} \quad (3.16)$$

where $g_1(\alpha), g_2(\alpha), \alpha(u, v), G_1(s, q; g_2(\alpha), g_1(\alpha))$ are same with (3.3)–(3.5).

Theorem 3.3. Let $f : I \subset \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be differentiable on I° , $a, b \in I$, with $a < b$, and $f' \in L([a, b])$. If $|f'(x)|^q$ is s -geometrically convex and monotonically decreasing on $[a, b]$, for $q > 1$ and $s \in (0, 1]$, then

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{b-a}{4} \left(\frac{q-1}{2q-1}\right)^{1-1/q} G_2(s, q; g_3(\alpha)), \quad (3.17)$$

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{b-a}{4} \left(\frac{q-1}{2q-1}\right)^{1-1/q} G_2(s, q; g_3(\alpha)), \quad (3.18)$$

where

$$G_2(s, q; g_3(\alpha)) = \begin{cases} \left(|f'(a)|^s + |f'(a)f'(b)|^{s/2} \right) \left[g_3\left(\alpha\left(\frac{sq}{2}, \frac{sq}{2}\right)\right) \right]^{1/q}, & |f'(a)| \leq 1, \\ \left(|f'(a)|^{1/s} + |f'(a)|^{1/2s} |f'(b)|^{s/2} \right) \left[g_3\left(\alpha\left(\frac{q}{2s}, \frac{sq}{2}\right)\right) \right]^{1/q}, & |f'(b)| \leq 1 \leq |f'(a)|, \\ \left(|f'(a)|^{1/s} + |f'(a)f'(b)|^{1/2s} \right) \left[g_3\left(\alpha\left(\frac{q}{2s}, \frac{q}{2s}\right)\right) \right]^{1/q}, & 1 \leq |f'(b)|, \end{cases} \quad (3.19)$$

$$g_3(\alpha) = \begin{cases} 1, & \alpha = 1, \\ \frac{\alpha - 1}{\ln \alpha}, & \alpha \neq 1, \end{cases}$$

and $\alpha(u, v)$ is the same as in (3.4).

Proof. (1) Since $|f'(x)|^q$ is s -geometrically convex and monotonically decreasing on $[a, b]$, from Lemma 2.1 and Hölder inequality, we have

$$\begin{aligned} & \left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{b-a}{4} \int_0^1 \left[t \left| f'\left((1-t)a + t\frac{a+b}{2}\right) \right| + |t-1| \left| f'\left((1-t)\frac{a+b}{2} + tb\right) \right| \right] dt \\ & \leq \frac{b-a}{4} \left[\left(\int_0^1 t^{q/(q-1)} dt \right)^{1-1/q} \left(\int_0^1 \left| f'\left((1-t)a + t\frac{a+b}{2}\right) \right|^q dt \right)^{1/q} \right. \\ & \quad \left. + \left(\int_0^1 (1-t)^{q/(q-1)} dt \right)^{1-1/q} \left(\int_0^1 \left| f'\left((1-t)\frac{a+b}{2} + tb\right) \right|^q dt \right)^{1/q} \right] \\ & \leq \frac{b-a}{4} \left(\frac{q-1}{2q-1} \right)^{1-1/q} \left[\left(\int_0^1 |f'(a)|^{q((2-t)/2)^s} |f'(b)|^{q(t/2)^s} dt \right)^{1/q} \right. \\ & \quad \left. + \left(\int_0^1 |f'(a)|^{q((1-t)/2)^s} |f'(b)|^{q((1+t)/2)^s} dt \right)^{1/q} \right]. \end{aligned} \quad (3.20)$$

(i) If $|f'(a)| \leq 1$, we have

$$\begin{aligned} & \int_0^1 |f'(a)|^{q((2-t)/2)^s} |f'(b)|^{q(t/2)^s} dt \leq |f'(a)|^{sq} g_3\left(\alpha\left(\frac{sq}{2}, \frac{sq}{2}\right)\right), \\ & \int_0^1 |f'(a)|^{q((1-t)/2)^s} |f'(b)|^{q((1+t)/2)^s} dt \leq |f'(a)f'(b)|^{sq/2} g_3\left(\alpha\left(\frac{sq}{2}, \frac{sq}{2}\right)\right), \end{aligned} \quad (3.21)$$

(ii) If $|f'(b)| \leq 1 \leq |f'(a)|$, we have

$$\begin{aligned} \int_0^1 |f'(a)|^{q((2-t)/2)^s} |f'(b)|^{q(t/2)^s} dt &\leq |f'(a)|^{q/s} g_3\left(\alpha\left(\frac{q}{2s}, \frac{sq}{2}\right)\right), \\ \int_0^1 |f'(a)|^{q((1-t)/2)^s} |f'(b)|^{q((1+t)/2)^s} dt &\leq |f'(a)|^{q/2s} |f'(b)|^{sq/2} g_3\left(\alpha\left(\frac{q}{2s}, \frac{sq}{2}\right)\right). \end{aligned} \quad (3.22)$$

(iii) If $1 \leq |f'(b)|$, we have

$$\begin{aligned} \int_0^1 |f'(a)|^{q((2-t)/2)^s} |f'(b)|^{q(t/2)^s} dt &\leq |f'(a)|^{q/s} g_3\left(\alpha\left(\frac{q}{2s}, \frac{q}{2s}\right)\right), \\ \int_0^1 |f'(a)|^{q((1-t)/2)^s} |f'(b)|^{q((1+t)/2)^s} dt &\leq |f'(a)f'(b)|^{q/2s} g_3\left(\alpha\left(\frac{q}{2s}, \frac{q}{2s}\right)\right). \end{aligned} \quad (3.23)$$

From (3.20) to (3.23), (3.17) holds.

(2) Since $|f'(x)|^q$ is s -geometrically convex and monotonically decreasing on $[a, b]$, from Lemma 2.1 and Hölder inequality, we have

$$\begin{aligned} \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| &\leq \frac{b-a}{4} \left(\frac{q-1}{2q-1} \right)^{1-1/q} \left[\left(\int_0^1 |f'(a)|^{q((2-t)/2)^s} |f'(b)|^{q(t/2)^s} dt \right)^{1/q} \right. \\ &\quad \left. + \left(\int_0^1 |f'(a)|^{q((1-t)/2)^s} |f'(b)|^{q((1+t)/2)^s} dt \right)^{1/q} \right]. \end{aligned} \quad (3.24)$$

From (3.24) and (3.21) to (3.23), (3.18) holds. This completes the proof. $\square$

If taking $s = 1$ in Theorem 3.3, we can derive the following corollary.

Corollary 3.4. Let $f : I \subset \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be differentiable on I° , $a, b \in I$, with $a < b$, and $f' \in L([a, b])$. If $|f'(x)|^q$ is geometrically convex and monotonically decreasing on $[a, b]$ for $q > 1$, then

$$\begin{aligned} \left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| &\leq \frac{b-a}{4} \left(\frac{q-1}{2q-1} \right)^{1-1/q} G_2(1, q; g_3(\alpha)), \\ \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| &\leq \frac{b-a}{4} \left(\frac{q-1}{2q-1} \right)^{1-1/q} G_2(1, q; g_3(\alpha)), \end{aligned} \quad (3.25)$$

where $\alpha(u, v)$, $G_2(s, q; g_3(\alpha))$, and $g_3(\alpha)$ are the same as in Theorem 3.3.

4. Application to Special Means

Let

$$\begin{aligned} A(a, b) &= \frac{a+b}{2}, & L(a, b) &= \frac{b-a}{\ln b - \ln a} \quad (a \neq b), \\ L_p(a, b) &= \left(\frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)} \right)^{1/p}, & a \neq b, \quad p \in \mathbb{R}, \quad p \neq 0, -1 \end{aligned} \quad (4.1)$$

be the arithmetic, logarithmic, generalized logarithmic means for $a, b > 0$ respectively.

Let $f(x) = x^s/s$, $x \in (0, 1]$, $0 < s < 1$, $q \geq 1$, and then the function

$$|f'(x)|^q = x^{(s-1)q} \quad (4.2)$$

is monotonically decreasing on $(0, 1]$. For $\lambda \in [0, 1]$, we have

$$(s-1)q(\lambda^s - \lambda) \leq 0, \quad (s-1)q((1-\lambda)^s - (1-\lambda)) \leq 0. \quad (4.3)$$

Hence, $|f'(x)|^q$ is s -geometrically convex on $(0, 1]$ for $0 < s < 1$.

Theorem 4.1. Let $0 < a < b \leq 1$, $0 < s < 1$, and $q \geq 1$. Then

$$\begin{aligned} & |[A(a, b)]^s - [L_s(a, b)]^s| \leq \frac{(b-a)^{1-1/q}s}{8} \left(\frac{4s}{(1-s)q} L(a, b) \right)^{1/q} \\ & \quad \times \left[a^{(s-1)/(2s)} \left\{ [L_{(s-1)q/(2s)-1}(a, b)]^{(s-1)q/(2s)-1} L(a, b) - b^{(s-1)q/(2s)} \right\}^{1/q} \right. \\ & \quad \left. + b^{(s-1)/(2s)} \left\{ a^{(s-1)q/(2s)} - [L_{(s-1)q/(2s)-1}(a, b)]^{(s-1)q/(2s)-1} L(a, b) \right\}^{1/q} \right] \\ & |A(a^s, b^s) - [L_s(a, b)]^s| \\ & \leq \frac{(b-a)^{1-1/q}s}{8} \left(\frac{4s}{(1-s)q} L(a, b) \right)^{1/q} \\ & \quad \times \left[b^{(s-1)/(2s)} \left\{ [L_{(s-1)q/(2s)-1}(a, b)]^{(s-1)q/(2s)-1} L(a, b) - b^{(s-1)q/(2s)} \right\}^{1/q} \right. \\ & \quad \left. + a^{(s-1)/(2s)} \left\{ a^{(s-1)q/(2s)} - [L_{(s-1)q/(2s)-1}(a, b)]^{(s-1)q/(2s)-1} L(a, b) \right\}^{1/q} \right]. \end{aligned} \quad (4.4)$$

In particular, if $q = 1$, one has

$$\begin{aligned}
 & |[A(a, b)]^s - [L_s(a, b)]^s| \leq \frac{(b-a)s}{4} [L_{(s-1)/(2s)-1}(a, b)]^{(s-1)/s-2} [L(a, b)]^2 \\
 & |A(a^s, b^s) - [L_s(a, b)]^s| \\
 & \leq \frac{(b-a)s}{4} L(a, b) \left\{ 2[L_{(s-1)/s-1}(a, b)]^{(s-1)/s-1} - [L_{(s-1)/(2s)-1}(a, b)]^{(s-1)/s-2} L(a, b) \right\}.
 \end{aligned} \tag{4.5}$$

Proof. Let $f(x) = x^s/s$, $x \in (0, 1]$, $0 < s < 1$. Then $|f'(a)| = a^{s-1} > b^{s-1} = |f'(b)| \geq 1$ and

$$\begin{aligned}
 & |f'(a)|^{1/s} \left[g_1 \left(\alpha \left(\frac{q}{2s}, \frac{q}{2s} \right) \right) \right]^{1/q} \\
 & = (b-a)^{-1/q} a^{(s-1)/2s} \left(\frac{2sL(a, b)}{(1-s)q} \right)^{1/q} \left\{ [L_{(s-1)q/(2s)-1}(a, b)]^{(s-1)q/(2s)-1} L(a, b) - b^{(s-1)q/(2s)} \right\}^{1/q}, \\
 & |f'(a)f'(b)|^{1/(2s)} \left[g_2 \left(\alpha \left(\frac{q}{2s}, \frac{q}{2s} \right) \right) \right]^{1/q} \\
 & = (b-a)^{-1/q} b^{(s-1)/(2s)} \left(\frac{2sL(a, b)}{(1-s)q} \right)^{1/q} \left\{ a^{(s-1)q/(2s)} - [L_{(s-1)q/(2s)-1}(a, b)]^{(s-1)q/(2s)-1} L(a, b) \right\}^{1/q}, \\
 & |f'(a)f'(b)|^{1/(2s)} \left[g_1 \left(\alpha \left(\frac{q}{2s}, \frac{q}{2s} \right) \right) \right]^{1/q} \\
 & = (b-a)^{-1/q} b^{(s-1)/(2s)} \left(\frac{2sL(a, b)}{(1-s)q} \right)^{1/q} \left\{ [L_{(s-1)q/(2s)-1}(a, b)]^{(s-1)q/(2s)-1} L(a, b) - b^{(s-1)q/(2s)} \right\}^{1/q}, \\
 & |f'(a)|^{1/s} \left[g_2 \left(\alpha \left(\frac{q}{2s}, \frac{q}{2s} \right) \right) \right]^{1/q} \\
 & = (b-a)^{-1/q} a^{(s-1)/(2s)} \left(\frac{2sL(a, b)}{(1-s)q} \right)^{1/q} \left[a^{(s-1)q/(2s)} - [L_{(s-1)q/(2s)-1}(a, b)]^{(s-1)q/(2s)-1} L(a, b) \right]^{1/q}.
 \end{aligned} \tag{4.6}$$

By Theorem 3.1, Theorem 4.1 is thus proved. $\square$

Theorem 4.2. Let $0 < a < b \leq 1$, $s \in (0, 1)$, and $q > 1$. Then one has

$$\begin{aligned}
 & |[A(a, b)]^s - [L_s(a, b)]^s| \leq \frac{(b-a)s}{2} \left(\frac{q-1}{2q-1} \right)^{1-1/q} A \left(a^{(s-1)/(2s)}, b^{(s-1)/(2s)} \right) \\
 & \quad \times \left[[L_{(s-1)q/(2s)-1}(a, b)]^{(s-1)q/2s-1} L(a, b) \right]^{1/q}
 \end{aligned}$$

$$\begin{aligned}
|A(a^s, b^s) - [L_s(a, b)]^s| &\leq \frac{(b-a)s}{2} \left(\frac{q-1}{2q-1} \right)^{1-1/q} A\left(a^{(s-1)/(2s)}, b^{(s-1)/(2s)}\right) \\
&\quad \times \left[[L_{(s-1)q/(2s)-1}(a, b)]^{(s-1)q/(2s)-1} L(a, b) \right]^{1/q}.
\end{aligned} \tag{4.7}$$

Proof. Let $f(x) = x^s/s$, $x \in (0, 1]$, $0 < s < 1$. Then $|f'(a)| = a^{s-1} > b^{s-1} = |f'(b)| \geq 1$ and

$$g_3\left(\alpha\left(\frac{q}{2s}, \frac{q}{2s}\right)\right) = a^{-(s-1)q/(2s)} [L_{(s-1)q/(2s)-1}(a, b)]^{(s-1)q/(2s)-1} L(a, b). \tag{4.8}$$

Using Theorem 3.3, Theorem 4.2 is thus proved. $\square$

Acknowledgments

The research was supported by Science Research Funding of Inner Mongolia University for Nationalities (Project no. NMD1103).

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