Moments for Generating Functions of Al-Salam-Carlitz Polynomials

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Received 31 March 2012; Accepted 24 May 2012

Academic Editor: Natig Atakishiyev

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We employ the moment representations for Al-Salam-Carlitz polynomials and show how to deduce bilinear, trilinear, and multilinear generating functions for Al-Salam-Carlitz polynomials. Moreover, we obtain two terminating generating functions for Al-Salam-Carlitz polynomials by the method of moments.

Dedicated to Professor H. M. Srivastava

1. Introduction

The Al-Salam-Carlitz polynomials [1, equations (4.1) and (4.2)]

\[
\sum_{n=0}^{\infty} U_n(x) \frac{t^n}{(q;q)_n} = \frac{(t, at; q)_\infty}{(xt; q)_\infty}, \quad \sum_{n=0}^{\infty} V_n(x) \frac{t^n}{(q;q)_n} = \frac{(xt; q)_\infty}{(t, at; q)_\infty} \tag{1.1}
\]

are important $q$-orthogonal polynomials whose applications varies in many fields such as the $q$-harmonic oscillator, theta function, quantum groups, and coding theory [2–4]. The generalized Al-Salam-Carlitz polynomials $U_n(x, y, a \mid q)$ [5, equation (1.2)] and $V_n(x, y, a \mid q)$ are defined by the following

\[
\sum_{n=0}^{\infty} U_n(x, y, a \mid q) \frac{t^n}{(q;q)_n} = \frac{(at, yt; q)_\infty}{(xt; q)_\infty}, \quad \sum_{n=0}^{\infty} V_n(x, y, a \mid q) \frac{t^n}{(q;q)_n} = \frac{(xt; q)_\infty}{(at, yt; q)_\infty}, \tag{1.2}
\]
where \( U_n(x, 1, a \mid q) = U_n^{(a)}(x) \) and \( V_n(x, 1, a \mid q) = V_n^{(a)}(x) \) and

\[
U_n(x, y, a \mid q) = (-a)^n q^{n(n+1)/2} \binom{n}{\frac{n}{2}}_2 \Phi_1 \left[ q^{-n}, \frac{y}{x}; q, \frac{q^x}{a} \right],
\]

\[
V_n(x, y, a \mid q) = a^n \binom{n}{\frac{n}{2}}_2 \Phi_0 \left[ q^{-n}, \frac{x}{y}; q, \frac{y^{n+1}}{a} \right] .
\]

(1.3)

There are close relationships between Al-Salam-Carlitz polynomials and other polynomials, such as \( q \)-Bessel polynomials [6], Stieltjes-Wigert polynomials [7], and Rogers-Szeg"o polynomials [5]. For more information, please refer to [5–7].

Al-Salam and Carlitz [1] defined moments of two discrete distribution \( da^{(a)}(x) \) and \( d\beta^{(a)}(x) \) by Rogers-Szeg"o polynomials as follows

\[
\int_{\infty}^{-\infty} x^n da^{(a)}(x) = h_n(a \mid q) \triangleq V_n(0, 1, a \mid q),
\]

\[
\int_{\infty}^{-\infty} x^n d\beta^{(a)}(x) = g_n(a \mid q) \triangleq (-1)^n q^{n(n+1)/2} U_n(0, 1, a \mid q),
\]

(1.4)

where \( \alpha^{(a)}(x) \) is a step function whose jumps occur at the points \( q^k \) and \( aq^k \) for \( k \in \mathbb{N} \), while the jumps of \( \beta^{(a)}(x) \) occur at the points \( q^{-k} \) for \( k \in \mathbb{N} \). These jumps are given by the following

\[
da^{(a)}(q^k) = \frac{q^k}{(a; q)_\infty (q, q/a; q)_k}, \quad da^{(a)}(aq^k) = \frac{q^k}{(1/a; q)_\infty (q, aq; q)_k},
\]

\[
d\beta^{(a)}(q^{-k}) = \frac{a^{k+1}q^{k+1} (aq^{k+1}; q)_\infty}{(q; q)_k}.
\]

(1.5)

In view of the fact that the discrete probability measure \( \alpha^{(a)} \) of moments in (1.4) on \([a, 1]\) may be given by the sum of two terms [8, equation (3.3)],

\[
\alpha^{(a)} = \sum_{k=0}^\infty \left[ \frac{q^k}{(a; q)_\infty (q, q/a; q)_k} \varepsilon_{q^k} + \frac{q^k}{(1/a; q)_\infty (q, aq; q)_k} \varepsilon_{aq^k} \right].
\]

(1.6)

However, \( \beta^{(a)} \) is given by [8, equation (3.15)],

\[
\beta^{(a)} = (aq; q) \sum_{k=0}^\infty \frac{a^k q^{k^2}}{(aq; q)_k} \varepsilon_{q^{-k}},
\]

(1.7)

where \( k \in \mathbb{N} \) and \( \varepsilon_y \) denotes a unit mass supported at \( y \), see also [9, 10].
Liu gained the following expression of bivariate Rogers-Szegő polynomials by the technique of partial fraction [11, equation (4.20)],

\[
    h_n(a, b \mid q) = \frac{a^n}{(b/a; q)_\infty} \sum_{k=0}^{\infty} \frac{q^{(n+1)k}}{(q, aq/b; q)_k} \frac{q^k}{(a/b; q)_\infty} + \frac{b^n}{(a/b; q)_\infty} \sum_{k=0}^{\infty} \frac{q^{(n+1)k}}{(q, qb/a; q)_k}.
\]  

(1.8)

So it’s natural to define the generalized discrete probability measure \( \alpha^{(a,b)} \) and \( \beta^{(a,b)} \) by the following

\[
    \alpha^{(a,b)} = \sum_{k=0}^{\infty} \left[ \frac{q^k}{(a/b; q)_\infty} \frac{q^k}{(q, qb/a; q)_k} \frac{q^k}{(b/a; q)_\infty} \frac{q^k}{(q, aq/b; q)_k} \frac{q^k}{(a/b; q)_\infty} \right],
\]  

(1.9)

\[
    \beta^{(a,b)} = \left( \frac{aq}{b} ; q \right) \sum_{k=0}^{\infty} \frac{a^k b^{-k} q^{k^2}}{(aq/b, q; q)_k} \frac{q^k}{(a/b; q)_\infty} \frac{q^k}{(q, aq/b; q)_k} \frac{q^k}{(a/b; q)_\infty},
\]  

(1.10)

where the bivariate Rogers-Szegő polynomials expressed by the following

\[
    h_n(a, b \mid q) = \int_{-\infty}^{\infty} x^n d\alpha^{(a,b)}(x) \triangleq V_n(0, a, b \mid q),
\]  

(1.11)

\[
    g_n(a, b \mid q) = \int_{-\infty}^{\infty} x^n d\beta^{(a,b)}(x) \triangleq U_n(0, a, b \mid q),
\]  

and their generating functions given by [11, equations (2.3) and (2.13)],

\[
    \sum_{n=0}^{\infty} h_n(a, b \mid q) \left( \frac{t^n}{(q; q)_n} \right) = \frac{1}{(at, bt; q)_\infty}, \quad \sum_{n=0}^{\infty} g_n(a, b \mid q) \left( \frac{(-1)^n q^{(\frac{n}{2})} t^n}{(q; q)_n} \right) = (at, bt; q)_\infty.
\]  

(1.12)

Al-Salam and Ismail [9] built relations between multilinear generating functions for Rogers-Szegő polynomials and \( _8W_7 \) by the method of moments. Berg and Ismail [8] gained orthogonality relations of generating functions for Rogers-Szegő polynomials by moments. Ismail and Masson [12] obtained several \( q \)-beta integrals and Bailey’s \( _6\phi_5 \) from \( q \)-Hermite polynomials by moments. Ismail and Stanton [13, 14] derived generating functions for Al-Salam-Chihara and continuous \( q \)-Hermite polynomials by moments. For more information, please refer to [8, 9, 12–18].


One may ask naturally a question: can Al-Salam-Carlitz polynomials be expressed by moments and some related problems deduced by moments? In this paper, we would like to represent Al-Salam-Carlitz polynomials in terms of moments (see Theorem 2.3 below) and
show how to deduce generating functions for Al-Salam-Carlitz polynomials by the method of moments.

This paper is organized as follows. In Section 3, two bilinear generating functions for Al-Salam-Carlitz polynomials are obtained by the method of moments. In Section 4, a trilinear generating function for Al-Salam-Carlitz polynomials is gained and a corollary is achieved. In Section 5, a multilinear generating function for Al-Salam-Carlitz polynomials is supplied naturally. In Section 6, two miscellaneous generating functions for Al-Salam-Carlitz polynomials are given.

2. Notations and Moments for Al-Salam-Carlitz Polynomials

In this paper, we follow the notations and terminology in [23] and suppose that $0 < q < 1$. The $q$-series and its compact factorials are defined, respectively, by the following

$$
(a; q)_0 = 1, \quad (a; q)_n = \prod_{k=0}^{n-1} (1 - aq^k), \quad (a; q)_\infty = \prod_{k=0}^{\infty} (1 - aq^k) \quad (2.1)
$$

and $(a_1, a_2, \ldots, a_m; q)_n = (a_1; q)_n (a_2; q)_n \cdots (a_m; q)_n$, where $m$ is a positive integer and $n$ is a nonnegative integer or $\infty$. In the context, convergence of basic hypergeometric series is no issue at all because they are the terminating $q$-series.

The $q$-Chu-Vandermonde formula reads that [23, equations (II.6) and (II.7)],

$$
\phi_1 \left[ q^{-n}, a; q, c \right] = \frac{(c/a; q)_n a^n}{(c; q)_n}, \quad \phi_1 \left[ q^{-n}, a; q, cq^n/a \right] = \frac{(c/a; q)_n}{(c; q)_n}. \quad (2.2)
$$

The $q$-difference operators $D_a$ and $\theta_a$ defined by [23, 24],

$$
D_a \{ f(a) \} = \frac{f(a) - f(aq)}{a}, \quad \theta_a \{ f(a) \} = \frac{f(aq^{-1}) - f(a)}{aq^{-1}}. \quad (2.3)
$$

Before the proof of main results, we need the following lemmas.

Lemma 2.1 ($q$-Leibniz formula [25, 26]). For $n \in \mathbb{N}$, one has

$$
\theta^n_a \{ f(a)g(a) \} = \sum_{k=0}^{n} \binom{n}{k} \theta^k_a \{ f(a) \} \theta^{n-k}_a \{ g(aq^{-k}) \}, \quad (2.4)
$$

$$
D^n_a \{ f(a)g(a) \} = \sum_{k=0}^{n} \binom{n}{k} q^{k(k-n)} D^k_a \{ f(a) \} D^{n-k}_a \{ g(aq^k) \}. \quad (2.5)
$$
Lemma 2.2 (equations (III.12) and (III.13)). For \( n \in \mathbb{N} \), one has

\[
3\Phi_2\left[ q^{-n}, b, c ; q, q \right] = \frac{(e/c; q)_n c^n}{(e; q)_n} 3\Phi_2\left[ q^{-n}, c, d \frac{b}{b} ; q, \frac{b q}{e} \right],
\]

(2.6)

\[
3\Phi_2\left[ q^{-n}, b, c ; q, q \right] = \left( \frac{(e/c; q)_n c^n}{(e; q)_n} \right) 3\Phi_1\left[ q^{-n}, c \frac{c q^{-n}}{e} ; q, \frac{b q}{e} \right],
\]

(2.7)

for \( d \to 0 \) in (2.6) and \( d \to \infty \) in (2.7), one has

\[
3\Phi_2\left[ q^{-n}, b, c ; q, q \right] = \frac{(e/c; q)_n c^n}{(e; q)_n} 2\Phi_1\left[ q^{-n}, c \frac{c q^{-n}}{e} ; q, \frac{b q}{e} \right],
\]

(2.8)

\[
3\Phi_1\left[ q^{-n}, b, c ; q, \frac{e q^{-n}}{e} \right] = \frac{(e/c; q)_n c^n}{(e; q)_n} 2\Phi_1\left[ q^{-n}, c \frac{c q^{-n}}{e} ; q, \frac{b q}{e} \right].
\]

(2.9)

Theorem 2.3. For \( n \in \mathbb{N} \), one has

\[
U_n(x, y, a | q) = \frac{(-1)^n q^{\frac{n}{2}}}{(q x/y, q x/a; q)_\infty} \int_{-\infty}^{\infty} \mu^n \left( \frac{q x \mu}{a y}; q \right)_\infty d\beta^{(y,a)}(\mu),
\]

(2.10)

\[
V_n(x, y, a | q) = \left( \frac{x}{y}, \frac{x}{a} \right)_\infty \int_{-\infty}^{\infty} \mu^n \left( \frac{x \mu}{(a y)}; q \right)_\infty d\alpha^{(y,a)}(\mu),
\]

(2.11)

where \( \alpha^{(y,a)}(\mu) \) and \( \beta^{(y,a)}(\mu) \) are defined by (1.6) and (1.7).

Theorem 2.4. For \( f \in \mathbb{N} \) and \( s/r = q^{-f} \), if \( \max(|ar|, |at|, |bt|, |aw|, |bw|) < 1 \), one has

\[
\int_{-\infty}^{\infty} \frac{(s; q)_\infty}{(r; q)_\infty} d\alpha^{(a,b)}(\mu) = \frac{(as, abtw; q)_\infty}{(ar, at, aw, bw; q)_\infty} 3\Phi_2\left[ \frac{s}{r}, \frac{q}{at} \frac{q}{aw}; q, q \right].
\]

(2.12)

For \( r/s = q^{-f} \), if \( \max(|as|, |abtw/q|) < 1 \), one has

\[
\int_{-\infty}^{\infty} \frac{(r; q)_\infty}{(s; q)_\infty} d\beta^{(a,b)}(\mu) = \frac{(as, abtw/q; q)_\infty}{(ar, at, aw, bw; q)_\infty} 3\Phi_2\left[ \frac{q}{as}, \frac{q^2}{btw}; q, q \right].
\]

(2.13)
Proof of Theorem 2.3. By (1.11) and (1.12), we have

$$\int_{-\infty}^{\infty} \frac{1}{(t\mu; q)_{\infty}} da^{(a,b)}(\mu) = \frac{1}{(at, bt; q)_{\infty}}. \quad (2.14)$$

Differentiating by $D^n_t$ and using $q$-Leibniz formula (2.5) for $D_t$ on both sides of (2.14), we have

$$\int_{-\infty}^{\infty} \frac{\mu^n}{(t\mu; q)_{\infty}} da^{(a,b)}(\mu) = a^n \frac{a^n}{(at, bt; q)_{\infty}} \sum_{k=0}^{n} \left[ q^{-n}, \frac{q}{a} \right]. \quad (2.15)$$

Comparing (1.2), (1.3), and (2.15), we have (2.11). Similarly, by $q$-Leibniz formula (2.4) for $\theta_t$, we obtain

$$\int_{-\infty}^{\infty} \frac{\mu^n}{(t\mu; q)_{\infty}} d\beta^{(a,b)}(\mu) = a^n \frac{a^n}{(at, bt; q)_{\infty}} \sum_{k=0}^{n} \left[ q^{-n}, \frac{q}{a} \right]. \quad (2.16)$$

so we achieve (2.10) by (1.3). The proof is complete. \qed

Proof of Theorem 2.4. Multiplying $\omega^n/(q; q)_n$ and summing $n$ over $0 \leq n \leq \infty$ on both sides of (2.15) yields

$$\int_{-\infty}^{\infty} \frac{1}{(t\mu, w\mu; q)_{\infty}} da^{(a,b)}(\mu) = \frac{(abtw; q)_{\infty}}{(at, bt, aw, bw; q)_{\infty}}. \quad (2.17)$$

Differentiating by $D^n_t$ and using (2.5) on both sides of (2.17), we have

$$D^n_t \left[ \frac{(abtw; q)_{\infty}}{(at, bt; q)_{\infty}} \right] = \sum_{k=0}^{n} q^{k(n-k)} \left[ \begin{array}{c} n \\ k \end{array} \right] D^k_t \left[ \frac{(abtw; q)_{\infty}}{(bt; q)_{\infty}} \right] D^{n-k} T \left[ \frac{1}{(atq^k; q)_{\infty}} \right]$$

$$= \sum_{k=0}^{n} q^{k(n-k)} \left[ \begin{array}{c} n \\ k \end{array} \right] q^k (aw; q)_k (abtwq^k; q)_{\infty} \frac{(aq^k)^{n-k}}{(atq^k; q)_{\infty}}, \quad (2.18)$$

that is

$$\int_{-\infty}^{\infty} \frac{\mu^n}{(t\mu, w\mu; q)_{\infty}} da^{(a,b)}(\mu) = a^n \frac{(abtw; q)_{\infty}}{(at, bt, aw, bw; q)_{\infty}} \sum_{k=0}^{n} \left[ q^{-n}, \frac{aw, at}{abtw} \right]. \quad (2.19)$$

Multiplying $(s/r; q)_n r^n/(q; q)_n$ and summing on both sides of (2.19), we obtain (2.12) after simplification. Similarly, we have

$$\int_{-\infty}^{\infty} \frac{1}{(t\mu, w\mu; q)_{\infty}} d\beta^{(a,b)}(\mu) = \frac{(at, bt, aw, bw; q)_{\infty}}{(abtw; q)_{\infty}}. \quad (2.20)$$
by $q$-Leibniz formula for $\theta_t$, we gain

$$
\theta^n_t \left\{ \frac{(at, bt; q)_\infty}{(abtw / q; q)_\infty} \right\} = \sum_{k=0}^{n} \binom{n}{k} q^{\frac{k+1}{2}} (abw)^k (q/(aw); q) (bt; q)_\infty (-aq^{-k})^{n-k} (atq^{-k}; q)_\infty,
$$

(2.21)

so we have

$$
\int_{-\infty}^{\infty} \mu^n_t(\mu, w; q)_\infty d\beta_{\{a,b\}}(\mu) = \frac{a^n (aw, bw, at, bt; q)_\infty}{(abtw / q; q)_\infty} \left\{ \begin{array}{c}
\frac{q^{-n}}{3}\frac{q}{(aw)} (at) \\
\frac{q^2}{(abtw)}
\end{array}\right. 0 , q, q \right.,
$$

(2.22)

which is (2.13) after multiplying $(r/s; q)_n s^n/(q; q)_n$ and summing on both sides of (2.22). The proof is complete.

3. Bilinear Generating Function for Al-Salam-Carlitz Polynomials

Chen et al. [5] gave the following bilinear generating functions for Al-Salam-Carlitz polynomials.

**Theorem 3.1** (see [5, Theorem 5.3]). Assume that $\max\{|xq/a|, |uq/b|\} < 1$. One has

$$
\sum_{k=0}^{\infty} (-1)^{m+n+k} q^{-kn} \left( \binom{n}{2} \right)^{-mk} U_{n+k}(x, y, a | q) U_{m+k}(u, v, b | q) \frac{x^k}{(q, q)_k}
$$

$$
= \frac{(yq/a, vq/b, abz; q)_\infty}{(xq/a, uq/b; q)_\infty} a^n b^m \sum_{k=0}^{\infty} \frac{(y/x, q/(abz); q)_k}{(q, yq/a; q)_k} \left( \frac{xbz}{q^n} \right)
$$

$$
\times \left[ \begin{array}{c}
u \\
\frac{q}{u'} (abz) \\
\frac{u}{vq} \\
b
\end{array}\right] \left[ \begin{array}{c}
u \\
a uz \frac{q^{m+k}}{q^n} \\
b
\end{array}\right],
$$

where $y/x = q^{-f}$ and $v/u = q^{-g}$ for nonnegative integers $f, g$ and $|auzq^{-(m+k)}| < 1$. 


In this section, we gain (3.1) and the following dual ones by the method of moments.

**Theorem 3.2.** Assume that \( \max(|yq/a|, |vq/b|) < 1 \). One has

\[
\sum_{k=0}^{\infty} V_{n+k}(x, y, a | q)V_{m+k}(u, v, b | q) \frac{(zq)^k}{(q; q)_k} = \frac{(x/a, u/b; q)_{\infty}a^n b^m}{(y/a, abz, v/b; q)_{\infty}} \sum_{k=0}^{\infty} \frac{(x/y, abz; q; q)^{(1+n)k}}{(q, aq/y; q)_k} \times 2\phi_1 \left[ \begin{array}{c} \frac{u}{v} abzq^k \\ \frac{bq}{v} \end{array} \right] ; q, q^{1+m}, \tag{3.2} \]

where \( x/y = q^{-f} \) and \( u/v = q^{-g} \) for nonnegative integers \( f, g \).

**Corollary 3.3** (see [5, Theorem 4.5]). For \( f \in \mathbb{N} \), one has

\[
\sum_{k=0}^{\infty} (-1)^k q^{-k} \left( \begin{array}{c} k+1 \end{array} \right) U_k(x, y, a | q)U_k(u, v, b | q) \frac{(zq)^k}{(q; q)_k} = \frac{(abzq, avzq, yq/a; q)_{\infty}}{(auzq, xq/a; q)_{\infty}} 3\phi_2 \left[ \begin{array}{c} y' \frac{1}{(abz)}, \frac{1}{(avz)} \\ \frac{1}{(auz)', aq} \end{array} \right] ; q, q^{1+m}, \tag{3.3} \]

provided that \( y/x = q^{-f} \) or \( u/v = q^{-f} \) and \( \max(|auzq^{-f}|, |xq/a|, |xbvz/u|) < 1 \).

**Corollary 3.4.** For \( f \in \mathbb{N} \), one has

\[
\sum_{k=0}^{\infty} V_k(x, y, a | q)V_k(u, v, b | q) \frac{z^k}{(q; q)_k} = \frac{(auz, x/a; q)_{\infty}}{(abz, avz, y/a; q)_{\infty}} 3\phi_2 \left[ \begin{array}{c} abz, x/y, avz \\ auz, aq/y \end{array} \right] ; q, q, \tag{3.4} \]

provided that \( x/y = q^{-f} \) or \( u/v = q^{-f} \) and \( \max(|abz|, |avz|, |y/a|) < 1 \).

**Remark 3.5.** For \( m = n = 0 \), formula (3.1) and (3.2) reduce to (3.3) and (3.4), respectively, by \( q \)-Gauss sum [23, equation (II.8)],

\[
2\phi_1 \left[ \begin{array}{c} a, b \\ c \end{array} \right] ; q, c/ab = \frac{(c/a, c/b; q)_{\infty}}{(c, c/(ab); q)_{\infty}} \tag{3.5} \]

and \( q \)-Chu-Vandermonde formula (2.2).
Proof of Theorems 3.1 and 3.2. The left-hand side of (3.1) is equal to

\[
\sum_{k=0}^{\infty} (-1)^{m+n+k} q^{\left(\frac{n+k}{2}\right)} \frac{(-1)^{n+k} q^{\left(\frac{n+k}{2}\right)}}{(qx/y, qx/a; q)_{\infty}} \int_{-\infty}^{\infty} \mu^{n+k} \left(\frac{qx}{(ay)}, q\right) \mu^{(a,v)}(w) \frac{w}{\mu^{(b,v)}(w)} \\
\times \frac{(1/2,q; q)}{\mu^{(b,v)}(w) \mu^{(a,v)}(w)} \frac{1}{(qx/y, qx/a; q)_{\infty}} \int_{-\infty}^{\infty} \mu^{n} \left(\frac{1}{2}, q\right) \mu^{(a,v)-1}(w) \mu^{(b,v)}(w) \frac{w}{\mu^{(b,v)}(w)} \frac{w}{\mu^{(a,v)}(w)} \\
= \frac{1}{(qx/y, qx/a; q)_{\infty}} \int_{-\infty}^{\infty} \left\{ a^n (awz, ywz; q) \frac{q^n, q}{1} \frac{y}{x} ; q, q \right\} \frac{1}{3\phi_2} \left( \frac{\mu^{(b,v)}(w), \mu^{(a,v)}(w)}{\mu^{(b,v)}(w), \mu^{(a,v)}(w)} \right) \\
\times w^{m} \left( \frac{q^{\nu}}{1} ; q, q \right) \mu^{(b,v)}(w) \\
= \frac{1}{(qx/y, qx/a; q)_{\infty}} \int_{-\infty}^{\infty} \left\{ a^n (awz, ywz; q) \frac{q^n, q}{1} \frac{y}{x} ; q, q \right\} \frac{1}{3\phi_2} \left( \frac{\mu^{(b,v)}(w), \mu^{(a,v)}(w)}{\mu^{(b,v)}(w), \mu^{(a,v)}(w)} \right) \\
\times w^{m} \left( \frac{q^{\nu}}{1} ; q, q \right) \mu^{(b,v)}(w) \\
= \frac{a^n (qy/a; q)_{\infty}}{(qx/y, qx/a; q)_{\infty}} \sum_{k=0}^{\infty} (y/x; q)_{k} (-1)^{k} q^{\left(\frac{k+1}{2}\right)} \left( \frac{x}{aq^n} \right)^{k} \\
\times w^{m} \left( \frac{q^{\nu}}{1} ; q, q \right) \mu^{(b,v)}(w) \text{ by (2.22)} \\
= \frac{a^n (qy/a; q)_{\infty}}{(qx/y, qx/a; q)_{\infty}} \sum_{k=0}^{\infty} (y/x; q)_{k} (-1)^{k} q^{\left(\frac{k+1}{2}\right)} \left( \frac{x}{aq^n} \right)^{k} \\
\times b^{m} (qy/a,qwz^{-k}, awz^{-k}; q)_{\infty} \frac{q^{m}, q^{m} \nu}{(uq^{m}z^{-k}; q)_{\infty}} \frac{1}{3\phi_2} \left( \frac{\mu^{(a,v)}(w), \mu^{(b,v)}(w)}{\mu^{(a,v)}(w), \mu^{(b,v)}(w)} \right) \frac{w}{\mu^{(b,v)}(w), \mu^{(a,v)}(w)} \text{ by (2.8)}
\]
which is the right hand side of (3.1) after simplification. Similarly, the right hand side of (3.2) is equivalent to

\[
\sum_{k=0}^{\infty} \left( \frac{x}{q^n} \right)^k \int_{-\infty}^{\infty} \frac{\mu^{n+k}}{(x\mu/(ay);q)_{\infty}} \text{da}^{(a,y)}(\mu) \left( \frac{u}{b}, \frac{u}{b}, q \right)_{\infty} \\
\times \int_{-\infty}^{\infty} \frac{w^{m+k}}{(u\omega/(bv);q)_{\infty}} \text{da}^{(b,v)}(w) \left( \frac{zq}{q^n} \right)_{\infty} \\
= \left( \frac{u}{b}, \frac{u}{b}, q \right)_{\infty} \int_{-\infty}^{\infty} \left( \frac{x}{y}, \frac{x}{z}; q \right)_{\infty} \int_{-\infty}^{\infty} \frac{\mu^{n}}{(x\mu/(ay), \mu\omega z; q)_{\infty}} \text{da}^{(a,y)}(\mu) \\
\times \frac{w^{m}}{(u\omega/(bv);q)_{\infty}} \text{da}^{(b,v)}(w) \text{ by (2.9)} \\
= \left( \frac{u}{b}, \frac{u}{b}, q \right)_{\infty} \int_{-\infty}^{\infty} \frac{a^{n}(x\omega z; q)_{\infty}}{(awz, y\omega z; q)_{\infty}} \phi_{1} \left[ q^{-n}, awz, \frac{x}{y}, \frac{yq^{n}}{awz}; q, q^{1+n} \right] \\
\times \frac{w^{m}}{(u\omega/(bv);q)_{\infty}} \text{da}^{(b,v)}(w) \text{ by (2.19)} \\
= \frac{a^{n}(u/v, u/b, x/a; q)_{\infty}}{(y/a; q)_{\infty}} \int_{-\infty}^{\infty} \frac{w^{m}}{(u\omega/(bv), awzq^{k}; q)_{\infty}} \text{da}^{(b,v)}(w) 
\]
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\[
= a^n \left( \frac{u/v, u/b, x/a; q}{(y/a; q)_\infty} \right) \sum_{k=0}^{\infty} \frac{(x/y, q)_k q^{(1+n)k}}{(q, aq/y; q)_k} \frac{b^m(auzq^k; q)_\infty}{(u/v, u/b, abzq^k, avzq^k; q)_\infty} \times \phi_1 \left[ \frac{q^{-m}, abzq^k, u/v}{auzq^k, u/v, \frac{vq^m}{y/b} ; q} \right],
\]

which is the right hand side of (3.2) after using (2.9). The proof is complete.

4. Trilinear Generating Function for Al-Salam-Carlitz Polynomials

Verma and Jain deduced the following trilinear generating functions for Rogers-Szegö polynomials.

Proposition 4.1 (see [27, equation (2.14)]). For \( s \in \mathbb{N} \), one has

\[
\sum_{n=0}^{\infty} h_{n+s}(x \mid q) h_n(y \mid q) h_n(z \mid q) = \frac{w^n}{(q; q)_n} \times \phi_3 \left[ \frac{w, yw, zw, yzw, 0, 0}{x, w, yw, zw, yzw, q, q^{1+s}} \right]
\]

\[
+ \frac{(yzw^2; q)_\infty}{(x, w, yz, zw, yzw; q)_\infty} \times \phi_3 \left[ \frac{xw, yz, zw, yzw, 0, 0}{xq, xzw, yxw, xzw, q, q^{1+s}} \right].
\]

In this section, we obtain the following trilinear generating functions for Al-Salam-Carlitz polynomials.

Theorem 4.2. For \( k \in \mathbb{N} \), one has

\[
\sum_{n=0}^{\infty} V_n(x, y, a \mid q) V_n(u, v, b \mid q) V_{n+k}(r, s, c \mid q) = \frac{w^n}{(q; q)_n} \times \phi_2 \left[ \frac{x, u/v, abwsq^j, c/s, abwcq^j, bxwcq^j ; q, yvwsq^j}{auwsq^j, bxwcq^j, bwysq^j,_bywsq^j ; q} \right]
\]

\[
+ \frac{(r/s, abws, bwys; q)_\infty c^k}{(c/s, abws, bwys, byws; q)_\infty} \times \phi_2 \left[ \frac{x, u/v, abwsq^j, c/s, abwcq^j, bxwcq^j ; q, yvwsq^j}{auwsq^j, bxwcq^j, bwysq^j, bywsq^j ; q} \right]
\]

\[
+ \frac{(r/s, abwc, bwoc; q)_\infty c^k}{(s/c, aboc, bwoc, bywc; q)_\infty} \times \phi_2 \left[ \frac{x, u/v, abwcq^j, c/s, abocq^j, bxocq^j ; q, yvocq^j}{auwcq^j, bxocq^j, bywcq^j, byocq^j ; q} \right],
\]

provided that the right hand side of (4.2) is convergent.
Remark 4.3. Ismail and Stanton [14] developed a method for deriving integral representations of certain orthogonal polynomials as moments and obtained [14, equation (4.13)] trilinear generating functions for $q$-Hermite polynomials $H_n(\cos \theta | q)$, which is equivalent to Proposition 4.1. For more information, please refer to [14].

Proof of Theorem 4.2. The left hand side of (4.2) is equal to

\[
\left( \frac{r}{s}, \frac{r}{c}; q \right) \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} \frac{\mu^{n+k}}{(r \mu/(cs); q)^{\infty}} \frac{d(a^{(c,s)}(\mu))}{(q;q)_n} V_n(x, y, a | q)V_n(u, v, b | q) \frac{\theta^n}{(q;q)_n} \\
= \left( \frac{r}{s}, \frac{r}{c}; q \right) \int_{-\infty}^{\infty} \frac{\mu^k}{(r \mu/(cs); q)^{\infty}} \sum_{n=0}^{\infty} V_n(x, y, a | q)V_n(u, v, b | q) \frac{\theta^n}{(q;q)_n} d(a^{(c,s)}(\mu)) \text{ by (3.4)} \\
= \left( \frac{r}{s}, \frac{r}{c}; q \right) \int_{-\infty}^{\infty} \frac{\mu^k(a \omega \mu, x/a; q)^{\infty}}{(r \mu/(cs), ab \omega \mu, a \omega \mu, y/a; q)^{\infty}} \\
\times \left[ \frac{x}{y}, \frac{u}{v}, \frac{a \omega \mu}{a \omega \mu}, \frac{b \omega \mu}{b \omega \mu}; q, q \right] d(a^{(c,s)}(\mu)) \text{ by (1.9)} \\
= \left( \frac{r}{s}, \frac{r}{c}; q \right) \sum_{j=0}^{\infty} \frac{q^j}{(c/s; q)^{\infty}(q,q_s/c; q)^{\infty}} \frac{(s q^{i})^{k}(a \omega s q^{i}, b \omega s q^{i}; q)^{\infty}}{(r q^{i}/c, ab \omega s q^{i}, a \omega s q^{i}, b \omega s q^{i}; q)^{\infty}} \\
\times 3 \phi_2 \left[ \frac{x}{y}, \frac{u}{v}, a \omega s q^{i}; q, q \right] + \left( \frac{r}{s}, \frac{r}{c}; q \right) \sum_{j=0}^{\infty} \frac{q^j}{(s/c; q)^{\infty}(q,q_c/s; q)^{\infty}} \\
\times \frac{(c q^{i})^{k}(a \omega c q^{i}, b \omega c q^{i}; q)^{\infty}}{(r q^{i}/s, ab \omega c q^{i}, a \omega c q^{i}, b \omega c q^{i}; q)^{\infty}} 3 \phi_2 \left[ \frac{x}{y}, \frac{u}{v}, a \omega c q^{i}; q, q \right], 
\]

(4.3)

which is the right hand of (4.2) after simplification. The proof is complete. \qed

Proof of Proposition 4.1. Noting the fact that $V_n(0, y, 1 | q) = h_n(y | q)$ and

\[
\left( \frac{\nu \omega^2 y; q}{2^j} \right) = (w \sqrt{\nu y}, -w \sqrt{\nu y}, w \sqrt{\nu y}, -w \sqrt{\nu y}; q)_{2^j}, \\
\left( \frac{s^2 \nu \omega^2 y; q}{2^j} \right) = (s w \sqrt{\nu y}, -s w \sqrt{\nu y}, s w \sqrt{\nu y}, -s w \sqrt{\nu y}; q)_{2^j},
\]

(4.4)
If letting \( x = u = r = 0 \) and \( a = b = c = 1 \) in (4.2), the right hand side of (4.2) is equal to

\[
\frac{s^k(s^2vw^2y;q)_\infty}{(1/s, vw, yw, yv; q)_\infty} \sum_{j=0}^{\infty} \frac{(w, vw, yw, yv; q)_j (s^2vw^2y;q)_{2j}}{(q, q/s; q)_j (vw^2y;q)_{2j}},
\]

which reduces to the right hand side of (4.1) after setting \((s, v) = (x, z)\). The proof is complete.

\[\square\]

5. Multilinear Generating Functions for Al-Salam-Carlitz Polynomials

Ismail and Stanton gained the following multilinear generating functions for Rogers-Szegő polynomials.

**Proposition 5.1** (see [28, equation (5.14)]). One has

\[
\sum_{m_1, \ldots, m_k=0}^{\infty} h_{m_1, \ldots, m_k} (a | q) h_{m_1} \left( \frac{t_1}{t_2} | q \right) \cdots h_{m_k} \left( \frac{t_{2k-1}}{t_{2k}} | q \right) \prod_{j=1}^{k} \frac{t_{ij}}{(q; q)_{m_j}}
\]

\[
= 2k \phi_{2k-1} \left\{ (a; q)_{\infty} \prod_{j=1}^{2k} (t_j; q)_{\infty} \right\} + 2k \phi_{2k-1} \left\{ (1/a; q)_{\infty} \prod_{j=1}^{2k} (at_j; q)_{\infty} \right\}.
\]

In this section, we obtain the following multilinear generating functions for Al-Salam-Carlitz polynomials.

**Theorem 5.2.** For \( s \in \mathbb{N} \), one has

\[
\sum_{m_1, \ldots, m_k=0}^{\infty} V_{m_1, \ldots, m_k+s} (y, x, a | q) V_{m_1} (y_1, x_1, a_1 | q) \cdots V_{m_k} (y_k, x_k, a_k | q) \frac{t_{1}^{m_1} \cdots t_{k}^{m_k}}{(q; q)_{m_1} \cdots (q; q)_{m_k}}
\]

\[
= a^s \left( \frac{y}{a}, y_1 t_1 a, \ldots, y_k t_k a; q \right)_{\infty} \left( \frac{x}{a}, a_1 t_1 a, x_1 t_1 a, \ldots, a_k t_k a, x_k t_k a; q \right)_{\infty}
\]

\[
\times 2k+1 \phi_{2k} \left[ \frac{y}{x}, a_1 t_1 a, x_1 t_1 a, \ldots, a_k t_k a, x_k t_k a \ ; q, q^{1+s} \right].
\]
provided that the right hand side of (5.2) is convergent.

**Corollary 5.3** (see [20, equation (1.12)]). For \( s \in \mathbb{N} \), one has

\[
\sum_{n_1, \ldots, n_k=0}^{\infty} h_{n_1, \ldots, n_k, s}(x | q) h_{n_1}(x_1 | q) \cdots h_{n_k}(x_k | q) \frac{t_1^{n_1}}{(q; q)_{n_1}} \cdots \frac{t_k^{n_k}}{(q; q)_{n_k}} \\
= \frac{1}{(x, t_1, x_1 t_1, \ldots, t_k, x_k t_k; q)_{\infty}} 2k \phi_{2k-1} \left[ \frac{t_1, x_1 t_1, \ldots, t_k, x_k t_k}{q x, 0, \ldots, 0} ; q, q^{1+s} \right] \\
+ \frac{x^s}{(1/x, xt_1, xx_1 t_1, \ldots, xt_k, xx_k t_k; q)_{\infty}} 2k \phi_{2k-1} \left[ \frac{x t_1, xx_1 t_1, \ldots, xt_k, xx_k t_k}{qx, 0, \ldots, 0} ; q, q^{1+s} \right] .
\]

(5.3)

**Remark 5.4.** Letting \( y = y_1 = \cdots = y_k = s = 0 \), \( a = a_1 = \cdots = a_k = 1 \) and taking \((x, x_1, t_1, \ldots, t_k, t_2)\) in Theorem 5.2, formula (5.2) reduces to (5.1). Setting \( y = y_1 = \cdots = y_k = 0 \) and \( a = a_1 = \cdots = a_k = 1 \) in Theorem 5.2, formula (5.2) reduces to (5.3).

**Proof.** The left hand side of (5.2) is equal to

\[
\left( \frac{y}{x}, \frac{y}{a} ; q \right)_{\infty} \int_{-\infty}^{\infty} \sum_{m_1, \ldots, m_k=0}^{\infty} V_{m_1}(y_1 t_1, a_1 t_1, \ldots) V_{m_k}(y_k t_k, a_k t_k, \ldots) \frac{t_1^{m_1}}{(q; q)_{m_1}} \cdots \frac{t_k^{m_k}}{(q; q)_{m_k}} \\
\times \frac{\mu^s da^{(a,x)}(\mu)}{(y \mu / (ax); q)_{\infty}} \\
= \left( \frac{y}{x}, \frac{y}{a} ; q \right)_{\infty} \int_{-\infty}^{\infty} \frac{\mu^s (y_1 t_1, \ldots, y_k t_k, \ldots)}{(a_1 t_1, x_1 t_1, \ldots, a_k t_k, x_k t_k, \ldots, y \mu / (ax); q)_{\infty}} da^{(a,x)}(\mu) \\
= \left( \frac{y}{x}, \frac{y}{a} ; q \right)_{\infty} \sum_{r=0}^{\infty} \frac{q^r}{(x/a; q)_{\infty} (q, qa/x; q)_r} \\
\times \frac{(aq^r)^s (y_1, y_k a q^r, \ldots)}{(a_1 t_1, a q^r, x_1 t_1, a q^r, \ldots, a_k t_k, a q^r, x_k t_k, \ldots, y q^r / x; q)_\infty} \\
+ \sum_{r=0}^{\infty} \frac{q^r}{(a/x; q)_{\infty} (q, qx/a; q)_r} \\
\times \frac{(x q^r)^s (y_1, x q^r, y_k x q^r, \ldots)}{(a_1 t_1, x q^r, x_1 t_1, x q^r, \ldots, a_k t_k, x q^r, x_k t_k, x q^r, y q^r / a; q)_\infty}
\]
\[
\begin{align*}
= \frac{\binom{y/a}{y/a_1,q} \cdots \binom{y/a_k}{y/a_1,q} \cdots \binom{y/a_k}{y/a_1,q} \cdots}{\binom{x/a}{x/a_1,q} \cdots \binom{x/a_k}{x/a_1,q} \cdots} \\
\times 2k+1 \Phi_{2k} \left[ \frac{y/x, a_1 x_1, \ldots, a_k x_k x}{aq/x, y_1 a_1, \ldots, y_k x_k} ; q, q^{1+s} \right] \\
\times 2k+1 \Phi_{2k} \left[ \frac{y/x, a_1 x_1, \ldots, a_k x_k x}{aq/x, y_1 a_1, \ldots, y_k x_k} ; q, q^{1+s} \right], \\
\times 2k+1 \Phi_{2k} \left[ \frac{y/x, a_1 x_1, \ldots, a_k x_k x}{aq/x, y_1 a_1, \ldots, y_k x_k} ; q, q^{1+s} \right], \\
\times 2k+1 \Phi_{2k} \left[ \frac{y/x, a_1 x_1, \ldots, a_k x_k x}{aq/x, y_1 a_1, \ldots, y_k x_k} ; q, q^{1+s} \right], \\
\times 2k+1 \Phi_{2k} \left[ \frac{y/x, a_1 x_1, \ldots, a_k x_k x}{aq/x, y_1 a_1, \ldots, y_k x_k} ; q, q^{1+s} \right],
\end{align*}
\]

which is the right hand side of (5.2) after simplification. The proof is complete. \qed

6. Some Miscellaneous Generating Functions for Al-Salam-Carlitz Polynomials

In this section, we deduce the following terminating generating functions for Al-Salam-Carlitz polynomials.

**Theorem 6.1.** For \( n \in \mathbb{N} \), one has

\[
\sum_{k=0}^{n} U_k \binom{cq^{k-1}}{d, b} q^{n-k} \binom{(q^{-n}, a)}{c/d, c/b, q} k \left( \frac{c}{abcd} \right)^k = \binom{c/(ad)}{c/d, c/b, q}^n \left( \frac{dq^{1-n}}{c} \right) \left( q, cq^n / b \right).
\]

**Theorem 6.2.** For \( n \in \mathbb{N} \), one has

\[
\sum_{k=0}^{n} V_{n-k} \binom{aq^{k}}{d, b} q^{n-k} \binom{(q^{n}, a/d, a/b)}{q, c, q} k \left( \frac{bcdn^n}{a} \right)^k = \binom{cd/a}{c}^n \left( \frac{aq}{c} \right) \left( q, \frac{aq}{bc} \right).
\]
Corollary 6.3. For $n \in \mathbb{N}$, one has

$$
\sum_{k=0}^{n} g_k(d | q) \left( \frac{q^{-n}; q}{(q; q)_k} \right) \left( \frac{cq^n}{d} \right)^k = \left( \frac{c}{d}; q \right)_n \phi_1 \left[ \frac{q^{-n}}{d q^{1-n}; q, d q} \right].
$$

(6.3)

$$
\sum_{k=0}^{n} h_{n-k}(d | q) \left( \frac{q^{-n}; q}{(q; q)_k} \right) (cdq^n)^k = (cd; q)_n \phi_1 \left[ \frac{q^{-n}, 0}{q^{1-n}; q, \frac{q}{c}} \right].
$$

Remark 6.4. Letting $b \to \infty$ in Theorems 6.1 and 6.2, noting the fact that

$$
\lim_{b \to \infty} (-1)^k q \left( \frac{q}{2} \right)^b U_k \left( cq^{-1}, d, b | q \right) = \lim_{b \to \infty} b^{k-n} V_{n-k} \left( aq^k, d, b | q \right) = 1.
$$

(6.4)

Formula (6.1) and (6.2) reduce to $q$-Chu-Vandermonde formula (2.2), respectively. Replacing $c$ by $ac$ and letting $(a, b) = (0, 1)$ in Theorems 6.1 and 6.2, formula (6.1) and (6.2) reduce to (6.3), respectively.

Proof of Theorems 6.1 and 6.2. Replacing $c$ by $c \mu / (bd)$ on both sides of the second formula in (2.2), we have

$$
\sum_{k=0}^{n} \left( \frac{q^{-n}, a; q}{(q; q)_k} \right) \left( \frac{cq^n}{abd} \right)^k \cdot \mu^k \left( \frac{c \mu q^k}{(bd); q} \right) = \frac{c \mu / (abd), c \mu q^n / (bd); q}_\infty (c \mu q^n / (bd); q)_\infty.
$$

(6.5)

Applying moments $\int_{-\infty}^{\infty} d \beta^{(b, d)}(\mu)$ in (6.5), using (2.10) and (2.13), we gain

$$
\sum_{k=0}^{n} \left( \frac{q^{-n}, a; q}{(q; q)_k} \right) \left( \frac{cq^n}{abd} \right)^k \left( \frac{cq^k}{d}, \frac{c q^k}{b}; q \right)_\infty U_k \left( cq^{-1}, d, b | q \right)
$$

$$
= \frac{c / (ad), cq^n / d, cq^n / b; q}_\infty \phi_2 \left[ \frac{q^{-n}, dq^{1-n}}{c} ; q, cq^n / b \right].
$$

(6.6)

which is (6.1) after simplification. Similarly, setting $a$ by $aw / (bd)$ on both sides of the second formula in (2.2), we obtain

$$
\sum_{k=0}^{n} \left( \frac{q^{-n}; q}{(q; q)_k} \right) \left( \frac{bcdq^n}{a} \right)^k \cdot \frac{aw^{-k}}{awq^n / (bd); q}_\infty
$$

$$
= \frac{(-1)^n q \left( \frac{q}{2} \right) (bcd / a)^n}{(c; q)_n} \cdot \frac{(awq^{1-n} / (bcd); q)_\infty}{(awq / (bcd), aw / (bd); q)_\infty}.
$$

(6.7)
Utilizing moments $\int_{-\infty}^{\infty} da^{(b,d)}(w)$ in (6.7), by (2.11) and (2.12), we have

$$
\sum_{k=0}^{n} \frac{(q^{-n}; q)_k}{(q, c; q)_k} \left( \frac{bcdq^n}{a} \right)^k \frac{1}{(aq^k/d, aq^k/b; q)_\infty} V_{n-k}(aq^k, d, b | q) 
= (-1)^n q^{\left(\frac{n}{2}\right)} (bcd/a)^n \frac{(aq^{1-n}/(cd); q)_\infty}{(aq/(cd), a/d, a/b; q)_\infty} {}_2\phi_1 \left[ \begin{array}{c} q^{-n} \frac{a}{d} \\ q^{-1-n} \frac{aq}{bc} \end{array} \left( \frac{cd}{aq} \right) \right],
$$

which becomes (6.2) after simplification. The proof is complete. $\square$

**Acknowledgments**

The author would like to express deep appreciation to the referee and the editor for their helpful suggestions.

**References**


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