

Research Article

An Iteration to a Common Point of Solution of Variational Inequality and Fixed Point-Problems in Banach Spaces

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We introduce an iterative process which converges strongly to a common point of solution of variational inequality problem for a monotone mapping and fixed point of uniformly Lipschitzian relatively asymptotically nonexpansive mapping in Banach spaces. As a consequence, we provide a scheme that converges strongly to a common zero of finite family of monotone mappings under suitable conditions. Our theorems improve and unify most of the results that have been proved for this important class of nonlinear operators.

1. Introduction

Let E be a smooth Banach space. Throughout this paper, we denote by $\phi : E \times E \rightarrow \mathbb{R}$ the function defined by

$$\phi(y, x) = \|y\|^2 - 2\langle y, Jx \rangle + \|x\|^2, \quad \text{for } x, y \in E, \quad (1.1)$$

which was studied by Alber [1], Kamimura and Takahashi [2], and Reich [3], where J is the normalized duality mapping from E to 2^{E^*} defined by

$$Jx := \{f^* \in E^* : \langle x, f^* \rangle = \|x\|^2 = \|f^*\|^2\}, \quad (1.2)$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing. It is well known that if E is smooth, then J is single-valued, and, if E has uniformly *Gâteaux differentiable* norm, then J is uniformly continuous on bounded subsets of E . Moreover, if E is a reflexive and strictly convex Banach space with a strictly convex dual, then J^{-1} is single valued, one-to-one, surjective, and it is the duality mapping from E^* into E , and thus $JJ^{-1} = I_{E^*}$ and $J^{-1}J = I_E$ (see [4]).

It is obvious from the definition of the function ϕ that

$$(\|x\| - \|y\|)^2 \leq \phi(x, y) \leq (\|x\| + \|y\|)^2 \quad \text{for } x, y \in E, \quad (1.3)$$

and, in a Hilbert space H , (1.1) reduces to $\phi(x, y) = \|x - y\|^2$, for $x, y \in H$.

Let E be a reflexive, strictly convex, and smooth Banach space, and let C be a nonempty closed and convex subset of E . The *generalized projection mapping*, introduced by Alber [1], is a mapping $\Pi_C : E \rightarrow C$ that assigns an arbitrary point $x \in E$ to the minimizer, \bar{x} , of $\phi(\cdot, x)$ over C , that is, $\Pi_C x = \bar{x}$, where \bar{x} is the solution to the minimization problem

$$\phi(\bar{x}, x) = \min\{\phi(y, x), y \in C\}. \quad (1.4)$$

Let E be a real Banach space with dual E^* . A mapping $A : D(A) \subset E \rightarrow E^*$ is said to be *monotone* if, for each $x, y \in D(A)$, the following inequality holds:

$$\langle x - y, Ax - Ay \rangle \geq 0. \quad (1.5)$$

A is said to be γ -*inverse strongly monotone* if there exists positive real number γ such that

$$\langle x - y, Ax - Ay \rangle \geq \gamma \|Ax - Ay\|^2, \quad \forall x, y \in K. \quad (1.6)$$

If A is γ -inverse strongly monotone, then it is *Lipschitz continuous* with constant $1/\gamma$, that is, $\|Ax - Ay\| \leq (1/\gamma)\|x - y\|$, for all $x, y \in D(A)$, and it is called *strongly monotone* if there exists $k > 0$ such that, for all $x, y \in D(A)$,

$$\langle x - y, Ax - Ay \rangle \geq k \|x - y\|^2. \quad (1.7)$$

Clearly, the class of monotone mappings include the class of strongly monotone and γ -inverse strongly monotone mappings.

Suppose that A is monotone mapping from C into E^* . The variational inequality problem is formulated as finding a point $u \in C$ such that $\langle v - u, Au \rangle \geq 0$, for all $v \in C$. The set of solutions of the variational inequality problems is denoted by $VI(C, A)$.

The notion of monotone mappings was introduced by Zarantonello [5], Minty [6], and Kacurovskii [7] in Hilbert spaces. Monotonicity conditions in the context of variational methods for nonlinear operator equations were also used by Vainberg and Kachurovsky [8]. Variational inequalities were initially studied by Stampacchia [9, 10] and ever since have been

widely studied in general Banach spaces (see, e.g., [2, 11–13]). Such a problem is connected with the convex minimization problem, the complementarity problem, the problem of finding point $u \in C$ satisfying $0 \in Au$.

If $E = H$, a Hilbert space, one method of solving a point $u \in \text{VI}(C, A)$ is the projection algorithm which starts with any point $x_1 = x \in C$ and updates iteratively as x_{n+1} according to the formula

$$x_{n+1} = P_C(x_n - \alpha_n Ax_n), \quad \text{for any } n \geq 1, \quad (1.8)$$

where P_C is the metric projection from H onto C and $\{\alpha_n\}$ is a sequence of positive real numbers. In the case that A is γ -inverse strongly monotone, Iiduka et al. [14] proved that the sequence $\{x_n\}$ generated by (3.35) *converges weakly* to some element of $\text{VI}(C, A)$.

In the case that E is a 2-uniformly convex and uniformly smooth Banach space, Iiduka and Takahashi [15] introduced the following iteration scheme for finding a solution of the variational inequality problem for an inverse strongly monotone operator A :

$$x_{n+1} = \Pi_C J^{-1}(Jx_n - \alpha_n Ax_n), \quad \text{for any } n \geq 1, \quad (1.9)$$

where Π_C is the generalized projection from E onto C , J is the normalized duality mapping from E into E^* , and $\{\alpha_n\}$ is a sequence of positive real numbers. They proved that the sequence $\{x_n\}$ generated by (1.9) *converges weakly* to some element of $\text{VI}(C, A)$ provided that A satisfies $\|Ax\| \leq \|Ax - Ap\|$, for $x \in C$ and $p \in \text{VI}(C, A)$.

It is worth to mention that the convergence is *weak convergence*.

To obtain strong convergence, when $E = H$, a Hilbert space and A is γ -inverse strongly monotone; Iiduka et al. [14] studied the following iterative scheme:

$$\begin{aligned} x_0 &\in C, \text{ chosen arbitrary,} \\ y_n &= P_C(x_n - \alpha_n Ax_n), \\ C_n &= \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n &= \{z \in C : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \\ x_{n+1} &= P_{C_n \cap Q_n}(x_0), \quad n \geq 1, \text{ for } n \geq 1, \end{aligned} \quad (1.10)$$

where $\{\alpha_n\}$ is a sequence in $[0, 2\gamma]$. They proved that the sequence $\{x_n\}$ generated by (1.10) converges strongly to $P_{\text{VI}(C, A)}(x_0)$, where $P_{\text{VI}(C, A)}$ is the metric projection from H onto $\text{VI}(C, A)$ provided that A satisfies $\|Ax\| \leq \|Ax - Ap\|$, for $x \in C$ and $p \in \text{VI}(C, A)$.

In the case that E is 2-uniformly convex and uniformly smooth Banach space, Iiduka and Takahashi [11] studied the following iteration scheme for a variational inequality

problem for γ -inverse strongly monotone mapping:

$$\begin{aligned} x_0 &\in K, \text{ chosen arbitrary,} \\ y_n &= \Pi_C J^{-1}(Jx_n - \alpha_n Ax_n), \\ C_n &= \{z \in E : \phi(z, y_n) \leq \phi(z, x_n)\}, \\ Q_n &= \{z \in E : \langle x_n - z, Jx_0 - Jx_n \rangle \geq 0\}, \\ x_{n+1} &= \Pi_{C_n \cap Q_n}(x_0), n \geq 1, \quad \text{for } n \geq 1, \end{aligned} \quad (1.11)$$

where $\Pi_{C_n \cap Q_n}$ is the generalized projection from E onto $C_n \cap Q_n$, J is the normalized duality mapping from E into E^* , and $\{\alpha_n\}$ is a positive real sequence satisfying certain condition. Then, they proved that the sequence $\{x_n\}$ converges strongly to an element of $\text{VI}(C, A)$ provided that $\text{VI}(C, A) \neq \emptyset$ and A satisfies $\|Ax\| \leq \|Ax - Ap\|$ for all $x \in C$ and $p \in \text{VI}(C, A)$.

Remark 1.1. We remark that the computation of x_{n+1} in Algorithms (1.10) and (1.11) is not simple because of the involvement of computation of C_{n+1} from C_n and Q_n for each $n \geq 1$.

Let T be a mapping from C into itself. We denote by $F(T)$ the fixed points set of T . A point p in C is said to be an *asymptotic fixed point* of T (see [3]) if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0$. The set of asymptotic fixed points of T will be denoted by $\hat{F}(T)$. A mapping T from C into itself is said to be *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for each $x, y \in C$ and is called *relatively nonexpansive* if (R1) $F(T) \neq \emptyset$; (R2) $\phi(p, Tx) \leq \phi(p, x)$ for $x \in C$ and (R3) $F(T) = \hat{F}(T)$. T is called *relatively quasi-nonexpansive* if $F(T) \neq \emptyset$ and $\phi(p, Tx) \leq \phi(p, x)$ for all $x \in C$, and $p \in F(T)$.

A mapping T from C into itself is said to be *asymptotically nonexpansive* if there exists $\{k_n\} \subset [1, \infty)$ such that $k_n \rightarrow 1$ and $\|T^n x - T^n y\| \leq k_n \|x - y\|$ for each $x, y \in C$ and is called *relatively asymptotically nonexpansive* if there exists $\{k_n\} \subset [1, \infty)$ such that (N1) $F(T) \neq \emptyset$; (N2) $\phi(p, T^n x) \leq k_n \phi(p, x)$ for $x \in C$ and $p \in F(T)$, and (N3) $F(T) = \hat{F}(T)$, where $k_n \rightarrow 1$ as $n \rightarrow \infty$. A self mapping on C is called *uniformly L-Lipschitzian* if there exists $L > 0$ such that $\|T^n x - T^n y\| \leq L \|x - y\|$ for all $x, y \in C$. T is called *closed* if $x_n \rightarrow x$ and $Tx_n \rightarrow y$, then $Tx = y$.

Clearly, we note that the class of relatively nonexpansive mappings is contained in a class of relatively asymptotically nonexpansive mappings but the converse is not true. Now, we give an example of relatively asymptotically nonexpansive mapping which is not relatively nonexpansive.

Example 1.2 (see [16]). Let $X = l^p$, where $1 < p < \infty$, and $C = \{x = (x_1, x_2, \dots) \in X; x_n \geq 0\}$. Then C is closed and convex subset of X . Note that C is not bounded. Obviously, X is uniformly convex and uniformly smooth. Let $\{\lambda_n\}$ and $\{\bar{\lambda}_n\}$ be sequences of real numbers satisfying the following properties:

- (i) $0 < \lambda_n < 1$, $\bar{\lambda}_n > 1$, $\lambda_n \uparrow 1$, and $\bar{\lambda}_n \downarrow 1$,
 - (ii) $\lambda_{n+1} \bar{\lambda}_n = 1$ and $\lambda_{j+1} \bar{\lambda}_{n+j} < 1$ for all n and j (e.g., $\lambda_n = 1 - 1/(n+1)$, $\bar{\lambda}_n = 1 + 1/(n+1)$).
- Then, the map $T : C \rightarrow C$ defined by

$$Tx := (0, \bar{\lambda}_1 |\sin x_1|, \lambda_2 x_2, \bar{\lambda}_2 x_3, \lambda_3 x_4, \bar{\lambda}_3 x_5, \dots), \quad (1.12)$$

for all $x = (x_1, x_2, \dots) \in C$, is uniformly Lipschitzian which is relatively asymptotically nonexpansive but not relatively nonexpansive (see [16] for the details). Note also that $F(T) = \{0\}$.

In 2005, Matsushita and Takahashi [17] proposed the following hybrid iteration method with generalized projection for relatively nonexpansive mapping T in a Banach space E :

$$\begin{aligned} x_0 &\in C, \text{ chosen arbitrary,} \\ y_n &= J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JT x_n), \\ C_n &= \{z \in C : \phi(z, y_n) \leq \phi(z, x_n)\}, \\ Q_n &= \{z \in C; \langle x_n - z, Jx_0 - Jx_n \rangle \geq 0\}, \\ x_{n+1} &= \Pi_{C_n \cap Q_n}(x_0), \quad n \geq 1. \end{aligned} \tag{1.13}$$

They proved that, if the sequence $\{\alpha_n\}$ is bounded above from one, then the sequence $\{x_n\}$ generated by (1.13) converges strongly to $\Pi_{F(T)}x_0$.

Recently, many authors have considered the problem of finding a common element of the fixed-point set of relatively nonexpansive mapping and the solution set of variational inequality problem for γ -inverse monotone mapping (see, e.g., [12, 13, 18–20]).

In [21], Iiduka and Takahashi studied the following iterative scheme for a common point of solution of a variational inequality problem for γ -inverse strongly monotone mapping A and fixed point of nonexpansive mapping T in a Hilbert space H :

$$\begin{aligned} x_1 &= x \in C, \\ x_{n+1} &= \alpha_n x + (1 - \alpha_n)SP_C(x_n - \lambda_n Ax_n), \quad n \geq 1, \end{aligned} \tag{1.14}$$

where $\{\alpha_n\}$ is sequences satisfying certain condition. They proved that the sequence $\{x_n\}$ converges strongly to an element of $F := F(S) \cap VI(C, A)$ provided that $F \neq \emptyset$.

In the case that E is a Banach space more general than Hilbert spaces, Zegeye et al. [12] studied the following iterative scheme for a common point of solution of a variational inequality problem for γ -inverse strongly monotone mapping A and fixed point of a closed relatively quasi-nonexpansive mapping T in a 2-uniformly convex and uniformly smooth Banach space E :

$$\begin{aligned} C_1 &= C, \text{ chosen arbitrary,} \\ z_n &= \Pi_C(x_n - \lambda_n Ax_n), \\ y_n &= J^{-1}(\beta Jx_n + (1 - \beta)JT z_n), \\ C_{n+1} &= \{z \in C_n : \phi(z, y_n) \leq \phi(z, x_n)\}, \\ x_{n+1} &= \Pi_{C_{n+1}}(x_0), \quad n \geq 1, \end{aligned} \tag{1.15}$$

where $\{\lambda_n\}$ is sequences satisfying certain condition. They proved that the sequence $\{x_n\}$ converges strongly to an element of $F := F(S) \cap VI(C, A) \neq \emptyset$ provided that $F \neq \emptyset$ and A satisfies $\|Ax\| \leq \|Ax - Ap\|$ for all $x \in C$ and $p \in F$.

Furthermore, Zegeye and Shahzad [22] studied the following iterative scheme for common point of solution of a variational inequality problem for γ -inverse strongly

monotone mapping A and fixed point of a relatively asymptotically nonexpansive mapping on a closed convex and bounded set C which is a subset of a real Hilbert space H :

$$\begin{aligned} C_1 &= C, \text{ chosen arbitrary,} \\ z_n &= P_C(x_n - \lambda_n A x_n), \\ y_n &= \alpha_n x_n + (1 - \alpha_n) S^n z_n, \\ C_{n+1} &= \left\{ z \in C_n : \|z - u_n\|^2 \leq \|z - x_n\|^2 + \theta_n \right\}, \\ x_{n+1} &= P_{C_{n+1}}(x_0), \quad n \geq 1, \end{aligned} \tag{1.16}$$

where P_{C_n} is the metric projection from H into C_n and $\theta_n = (1 - \alpha_n)(k_n^2 - 1)(\text{diam}(C))^2$ and $\{\alpha_n\}, \{\lambda_n\}$ are sequences satisfying certain condition. Then, they proved that the sequence $\{x_n\}$ converges strongly to an element of $F := F(S) \cap \text{VI}(C, A) \neq \emptyset$ provided that $F \neq \emptyset$ and A satisfies $\|Ax\| \leq \|Ax - Ap\|$ for all $x \in C$ and $p \in F$.

Remark 1.3. We again remark that the computation of x_{n+1} in Algorithms (1.13), (1.15), and (1.16) is not simple because of the involvement of computation of C_{n+1} from C_n for each $n \geq 1$.

It is our purpose in this paper to introduce an iterative scheme $\{x_n\}$ which converges strongly to a common point of solution of variational inequality problem for a monotone operator $A : C \rightarrow E^*$ satisfying appropriate conditions, for some nonempty closed convex subset C of a Banach space E and fixed points of uniformly L -Lipschitzian relatively asymptotically nonexpansive mapping in Banach spaces. As a consequence, we provide a scheme which converges strongly to a common zero of finite family of monotone mappings. Our scheme does not involve computation of C_{n+1} from C_n or Q_n , for each $n \geq 1$. Our theorems improve and unify most of the results that have been proved for this important class of nonlinear operators.

2. Preliminaries

Let E be a normed linear space with $\dim E \geq 2$. The *modulus of smoothness* of E is the function $\rho_E : [0, \infty) \rightarrow [0, \infty)$ defined by

$$\rho_E(\tau) := \sup \left\{ \frac{\|x + y\| + \|x - y\|}{2} - 1 : \|x\| = 1; \|y\| = \tau \right\}. \tag{2.1}$$

The space E is said to be *smooth* if $\rho_E(\tau) > 0$, for all $\tau > 0$, and E is called *uniformly smooth* if and only if $\lim_{t \rightarrow 0^+} (\rho_E(t)/t) = 0$.

The *modulus of convexity* of E is the function $\delta_E : (0, 2] \rightarrow [0, 1]$ defined by

$$\delta_E(\epsilon) := \inf \left\{ 1 - \left\| \frac{x + y}{2} \right\| : \|x\| = \|y\| = 1; \epsilon = \|x - y\| \right\}. \tag{2.2}$$

E is called *uniformly convex* if and only if $\delta_E(\epsilon) > 0$, for every $\epsilon \in (0, 2]$.

In the sequel, we will need the following results.

Lemma 2.1 (see [23]). *Let C be a nonempty closed and convex subset of a real reflexive, strictly convex, and smooth Banach space E . If $A : C \rightarrow E^*$ is continuous monotone mapping, then $VI(C, A)$ is closed and convex.*

Lemma 2.2. *Let C be a closed convex subset of a uniformly convex and smooth Banach space E , and let S be continuous relatively asymptotically nonexpansive mapping from C into itself. Then, $F(S)$ is closed and convex.*

Lemma 2.3 (see [1]). *Let K be a nonempty closed and convex subset of a real reflexive, strictly convex, and smooth Banach space E , and let $x \in E$. Then, for all $y \in K$,*

$$\phi(y, \Pi_K x) + \phi(\Pi_K x, x) \leq \phi(y, x). \quad (2.3)$$

Lemma 2.4 (see [2]). *Let E be a real smooth and uniformly convex Banach space, and let $\{x_n\}$ and $\{y_n\}$ be two sequences of E . If either $\{x_n\}$ or $\{y_n\}$ is bounded and $\phi(x_n, y_n) \rightarrow 0$ as $n \rightarrow \infty$, then $x_n - y_n \rightarrow 0$, as $n \rightarrow \infty$.*

We make use of the function $V : E \times E^* \rightarrow \mathbb{R}$ defined by

$$V(x, x^*) = \|x\|^2 - 2\langle x, x^* \rangle + \|x\|^2, \quad \forall x \in E, x^* \in E^*, \quad (2.4)$$

studied by Alber [1]. That is, $V(x, y) = \phi(x, J^{-1}x^*)$ for all $x \in E$ and $x^* \in E^*$. We know the following lemma.

Lemma 2.5 (see [1]). *Let E be reflexive strictly convex and smooth Banach space with E^* as its dual. Then,*

$$V(x, x^*) + 2\langle J^{-1}x^* - x, y^* \rangle \leq V(x, x^* + y^*), \quad (2.5)$$

for all $x \in E$ and $x^*, y^* \in E^*$.

Lemma 2.6 (see [1]). *Let C be a convex subset of a real smooth Banach space E . Let $x \in E$. Then $x_0 = \Pi_C x$ if and only if*

$$\langle z - x_0, Jx - Jx_0 \rangle \leq 0, \quad \forall z \in C. \quad (2.6)$$

Lemma 2.7 (see [12]). *Let E be a uniformly convex Banach space and $B_R(0)$ a closed ball of E . Then, there exists a continuous strictly increasing convex function $g : [0, \infty) \rightarrow [0, \infty)$ with $g(0) = 0$ such that*

$$\|\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3\|^2 \leq \alpha_1 \|x_1\|^2 + \alpha_2 \|x_2\|^2 + \alpha_3 \|x_3\|^2 - \alpha_i \alpha_j g(\|x_i - x_j\|), \quad (2.7)$$

for $\alpha_i \in (0, 1)$ such that $\alpha_1 + \alpha_2 + \alpha_3 = 1$ and $x_i \in B_R(0) := \{x \in E : \|x\| \leq R\}$, for $i = 1, 2, 3$.

Let E be a smooth and strictly convex Banach space, C a nonempty closed convex subset of E , and $A : C \rightarrow E^*$ a monotone operator satisfying

$$D(A) \subseteq C \subseteq J^{-1}(\cap_{r>0} R(J + rA)), \quad (2.8)$$

for $r > 0$. Then, we can define the resolvent $Q_r : C \rightarrow D(A)$ of A by

$$Q_r x := \{z \in D(A) : Jx \in Jz + rAz\}, \quad \forall x \in C. \quad (2.9)$$

In other words, $Q_r x = (J + rA)^{-1} Jx$ for $x \in C$. We know that $Q_r x$ is single-valued mapping from C into $D(A)$, for all $x \in C$ and $r > 0$ and $F(Q_r) = A^{-1}(0)$, where $F(Q_r)$ is the set of fixed points of Q_r (see, [4]).

Lemma 2.8 (see [24]). *Let E be a smooth and strictly convex Banach space, C a nonempty closed convex subset of E , and $A \subset E \times E^*$ a monotone operator satisfying (2.8) and $A^{-1}(0)$ is nonempty. Let Q_r be the resolvent of A . Then, for each $r > 0$,*

$$\phi(u, Q_r x) + \phi(Q_r x, x) \leq \phi(u, x) \quad (2.10)$$

for all $u \in A^{-1}(0)$ and $x \in C$, that is, Q_r is relatively nonexpansive.

Lemma 2.9 (see [25]). *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:*

$$a_{n+1} \leq (1 - \beta_n) a_n + \beta_n \delta_n, \quad n \geq n_0, \quad (2.11)$$

where $\{\beta_n\} \subset (0, 1)$ and $\{\delta_n\} \subset \mathbb{R}$ satisfying the following conditions: $\lim_{n \rightarrow \infty} \beta_n = 0$, $\sum_{n=1}^{\infty} \beta_n = \infty$, and $\limsup_{n \rightarrow \infty} \delta_n \leq 0$. Then, $\lim_{n \rightarrow \infty} a_n = 0$.

Lemma 2.10 (see [26]). *Let $\{a_n\}$ be sequences of real numbers such that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $a_{n_i} < a_{n_i+1}$ for all $i \in \mathbb{N}$. Then, there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \rightarrow \infty$, and the following properties are satisfied by all (sufficiently large) numbers $k \in \mathbb{N}$:*

$$a_{m_k} \leq a_{m_k+1}, \quad a_k \leq a_{m_k+1}. \quad (2.12)$$

In fact, $m_k = \max\{j \leq k : a_j < a_{j+1}\}$.

3. Main Result

We note that, as it is mentioned in [27], if C is a subset of a real Banach space E and $A : C \rightarrow E^*$ is a mapping satisfying $\|Ax\| \leq \|Ax - Ap\|$, for all $x \in C$ and $p \in \text{VI}(C, A)$, then

$$\text{VI}(C, A) = A^{-1}(0) = \{p \in C : Ap = 0\}. \quad (3.1)$$

In fact, clearly, $A^{-1}(0) \subseteq \text{VI}(C, A)$. Now, we show that $\text{VI}(C, A) \subseteq A^{-1}(0)$. Let $p \in \text{VI}(C, A)$, then we have by hypothesis that $\|Ap\| \leq \|Ap - Ap\| = 0$ which implies that $p \in A^{-1}(0)$. Hence, $\text{VI}(C, A) \subseteq A^{-1}(0)$. Therefore, $\text{VI}(C, A) = A^{-1}(0)$. Now we prove the main theorem of our paper.

Theorem 3.1. *Let C be a nonempty, closed, and convex subset of a uniformly convex and uniformly smooth real Banach space E . Let $A : C \rightarrow E^*$ be a monotone mapping satisfying (2.8) and $\|Ax\| \leq \|Ax - Ap\|$, for all $x \in C$ and $p \in \text{VI}(C, A)$. Let $T : C \rightarrow C$ be a uniformly L -Lipschitzian relatively asymptotically nonexpansive mapping with sequence $\{k_n\}$. Assume that $F := \text{VI}(C, A) \cap F(S)$ is nonempty. Let Q_r be the resolvent of A and $\{x_n\}$ a sequence generated by*

$$\begin{aligned} x_0 &= w \in C, \text{ chosen arbitrarily,} \\ y_n &= \Pi_C J^{-1}(\alpha_n Jw + (1 - \alpha_n)Jx_n), \\ x_{n+1} &= J^{-1}(\beta_n Jx_n + \gamma_n JT^n y_n + \theta_n JQ_r y_n), \end{aligned} \quad (3.2)$$

where $\alpha_n \in (0, 1)$ such that $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\lim_{n \rightarrow \infty} ((k_n - 1)/\alpha_n) = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\{\beta_n\}, \{\gamma_n\}, \{\theta_n\} \subset [c, d] \subset (0, 1)$ such that $\beta_n + \gamma_n + \theta_n = 1$. Then, $\{x_n\}$ converges strongly to an element of F .

Proof. Let $p := \Pi_F w$. Then, from (3.2), Lemma 2.3, and property of ϕ , we get that

$$\begin{aligned} \phi(p, y_n) &= \phi(p, \Pi_C J^{-1}(\alpha_n Jw + (1 - \alpha_n)Jx_n)) \\ &\leq \phi(p, J^{-1}(\alpha_n Jw + (1 - \alpha_n)Jx_n)) \\ &= \|p\|^2 - 2\langle p, \alpha_n Jw + (1 - \alpha_n)Jx_n \rangle + \|\alpha_n Jw + (1 - \alpha_n)Jx_n\|^2 \\ &\leq \|p\|^2 - 2\alpha_n \langle p, Jw \rangle - 2(1 - \alpha_n) \langle p, Jx_n \rangle + \alpha_n \|Jw\|^2 + (1 - \alpha_n) \|Jx_n\|^2 \\ &= \alpha_n \phi(p, w) + (1 - \alpha_n) \phi(p, x_n). \end{aligned} \quad (3.3)$$

Now, from (3.2) and relatively asymptotically nonexpansiveness of T , relatively nonexpansiveness of Q_r , property of ϕ , and (3.3), we get that

$$\begin{aligned} \phi(p, x_{n+1}) &= \phi(p, J^{-1}(\beta_n Jx_n + \gamma_n JT^n y_n + \theta_n JQ_r y_n)) \\ &\leq \beta_n \phi(p, x_n) + \gamma_n \phi(p, T^n y_n) + \theta_n \phi(p, Q_r y_n) \\ &\leq \beta_n \phi(p, x_n) + \gamma_n k_n \phi(p, y_n) + \theta_n \phi(p, y_n) \\ &\leq \beta_n \phi(p, x_n) + (\gamma_n k_n + \theta_n) \phi(p, y_n) \\ &\leq \beta_n \phi(p, x_n) + (\gamma_n k_n + \theta_n) [\alpha_n \phi(p, w) + (1 - \alpha_n) \phi(p, x_n)] \\ &\leq (\gamma_n k_n + \theta_n) \alpha_n \phi(p, w) + [\beta_n + (\gamma_n k_n + \theta_n)(1 - \alpha_n)] \phi(p, x_n) \\ &\leq (\gamma_n k_n + \theta_n) \alpha_n \phi(p, w) + [(1 - \alpha_n)(\gamma_n k_n + \theta_n)) + \gamma_n(k_n - 1)] \times \phi(p, x_n) \\ &\leq c_n \phi(p, w) + [1 - (1 - \epsilon)c_n] \phi(p, x_n), \end{aligned} \quad (3.4)$$

where $c_n = \alpha_n(\gamma_n k_n + \theta_n)$, since there exists $N_0 > 0$ such that $\gamma_n(k_n - 1)/\alpha_n \leq \epsilon(\gamma_n k_n + \theta_n)$ for all $n \geq N_0$ and for some $\epsilon > 0$ satisfying $(1 - \epsilon)c_n \leq 1$. Thus, by induction,

$$\phi(p, x_{n+1}) \leq \max\{\phi(p, x_0), (1 - \epsilon)^{-1}\phi(p, w)\}, \quad \forall n \geq N_0 \quad (3.5)$$

which implies that $\{x_n\}$, and hence $\{y_n\}$ is bounded. Now, let $z_n = J^{-1}(\alpha_n Jw + (1 - \alpha_n)Jx_n)$. Then we have that $y_n = \Pi_C z_n$. Using Lemmas 2.3, 2.5, and property of ϕ , we obtain that

$$\begin{aligned} \phi(p, y_n) &\leq \phi(p, z_n) = V(p, Jz_n) \\ &\leq V(p, Jz_n - \alpha_n(Jw - Jp)) - 2\langle z_n - p, -\alpha_n(Jw - Jp) \rangle \\ &= \phi(p, J^{-1}(\alpha_n Jp + (1 - \alpha_n)Jw_n)) + 2\alpha_n\langle z_n - p, Jw - Jp \rangle \\ &\leq \alpha_n\phi(p, p) + (1 - \alpha_n)\phi(p, w_n) + 2\alpha_n\langle z_n - p, Jw - Jp \rangle \\ &= (1 - \alpha_n)\phi(p, w_n) + 2\alpha_n\langle z_n - p, Jw - Jp \rangle \\ &\leq (1 - \alpha_n)\phi(p, x_n) + 2\alpha_n\langle z_n - p, Jw - Jp \rangle. \end{aligned} \quad (3.6)$$

Furthermore, from (3.2), Lemma 2.7, relatively asymptotically nonexpansiveness of T , relatively nonexpansiveness of Q_r , and (3.6), we have that

$$\begin{aligned} \phi(p, x_{n+1}) &= \phi(p, J^{-1}(\beta_n Jx_n + \gamma_n JT^n y_n + \theta_n JQ_r y_n)) \\ &= \|p\|^2 - \langle p, \beta_n Jx_n + \gamma_n JT^n y_n + \theta_n JQ_r y_n \rangle + \|\beta_n Jx_n + \gamma_n JT^n y_n + \theta_n JQ_r y_n\|^2 \quad (3.7) \\ &\leq \|p\|^2 - 2\beta_n\langle p, Jx_n \rangle - 2\gamma_n\langle p, JT^n y_n \rangle - 2\theta_n\langle p, JQ_r y_n \rangle \\ &\quad + \beta_n\|Jx_n\|^2 + \gamma_n\|JT^n y_n\|^2 + \theta_n\|JQ_r y_n\|^2 - \gamma_n\beta_n g(\|Jx_n - JT^n y_n\|) \\ &\leq \beta_n\phi(p, x_n) + \gamma_n\phi(p, T^n y_n) + \theta_n\phi(p, Q_r y_n) - \gamma_n\beta_n g(\|Jx_n - JT^n y_n\|) \\ &\leq \beta_n\phi(p, x_n) + (\gamma_n + \theta_n)\phi(p, y_n) + \gamma_n(k_n - 1)\phi(p, y_n) - \gamma_n\beta_n g(\|Jx_n - JT^n y_n\|) \\ &\leq \beta_n\phi(p, x_n) + (\gamma_n + \theta_n)[(1 - \alpha_n)\phi(p, x_n) + 2\alpha_n\langle z_n - p, Jw - Jp \rangle] \\ &\quad + \gamma_n(k_n - 1)\phi(p, y_n) - \gamma_n\beta_n g(\|Jx_n - JT^n y_n\|) \\ &\leq [\beta_n + (\gamma_n + \theta_n)(1 - \alpha_n)]\phi(p, x_n) + 2\alpha_n(\gamma_n + \theta_n)\langle z_n - p, Jw - Jp \rangle \\ &\quad + \gamma_n(k_n - 1)\phi(p, y_n) - \gamma_n\beta_n g(\|Jx_n - JT^n y_n\|) \\ &\leq (1 - \delta_n)\phi(p, x_n) + 2\delta_n\langle z_n - p, Jw - Jp \rangle + (k_n - 1)M - \gamma_n\beta_n g(\|Jx_n - JT^n y_n\|) \quad (3.8) \\ &\leq (1 - \delta_n)\phi(p, x_n) + 2\delta_n\langle z_n - p, Jw - Jp \rangle + (k_n - 1)M, \quad (3.9) \end{aligned}$$

for some $M > 0$, where $\delta_n = (\gamma_n + \theta_n)\alpha_n$.

Similarly, from (3.7), we obtain that

$$\begin{aligned}
 \phi(p, x_{n+1}) &= \phi\left(p, J^{-1}(\beta_n Jx_n + \gamma_n JT^n y_n + \theta_n JQ_r y_n)\right) \\
 &\leq (1 - \delta_n)\phi(p, x_n) + 2\delta_n \langle z_n - p, Jw - Jp \rangle + (k_n - 1)M - \theta_n \beta_n g(\|Jx_n - JQ_r y_n\|) \\
 &\leq (1 - \delta_n)\phi(p, x_n) + 2\delta_n \langle z_n - p, Jw - Jp \rangle + (k_n - 1)M,
 \end{aligned} \tag{*}$$

for some $M > 0$. Note that $\{\delta_n\}$ satisfies that $\lim_{n \rightarrow \infty} \delta_n = 0$ and $\sum \delta_n = \infty$.

Now, the rest of the proof is divided into two parts.

Case 1. Suppose that there exists $n_0 \in N > N_0$ such that $\{\phi(p, x_n)\}$ is nonincreasing for all $n \geq n_0$. In this situation, $\{\phi(p, x_n)\}$ is then convergent. Then, from (3.8) and (*), we have that

$$\gamma_n \beta_n g(\|Jx_n - JT^n y_n\|) \rightarrow 0, \quad \theta_n \beta_n g(\|Jx_n - JQ_r y_n\|) \rightarrow 0, \tag{3.10}$$

which implies, by the property of g , that

$$Jx_n - JT^n y_n \rightarrow 0, \quad Jx_n - JQ_r y_n \rightarrow 0, \quad \text{as } n \rightarrow \infty, \tag{3.11}$$

and, hence, since J^{-1} is uniformly continuous on bounded sets, we obtain that

$$x_n - T^n y_n \rightarrow 0, \quad x_n - Q_r y_n \rightarrow 0, \quad \text{as } n \rightarrow \infty. \tag{3.12}$$

Furthermore, Lemma 2.3, property of ϕ , and the fact that $\alpha_n \rightarrow 0$ as $n \rightarrow \infty$ imply that

$$\begin{aligned}
 \phi(x_n, y_n) &= \phi(x_n, \Pi_C z_n) \leq \phi(x_n, z_n) \\
 &= \phi(x_n, J^{-1}(\alpha_n Jw + (1 - \alpha_n)Jx_n)) \\
 &\leq \alpha_n \phi(x_n, w) + (1 - \alpha_n)\phi(x_n, x_n) \\
 &\leq \alpha_n \phi(x_n, w) + (1 - \alpha_n)\phi(x_n, x_n) \rightarrow 0, \quad \text{as } n \rightarrow \infty,
 \end{aligned} \tag{3.13}$$

and hence

$$x_n - y_n \rightarrow 0, \quad x_n - z_n \rightarrow 0, \quad \text{as } n \rightarrow \infty. \tag{3.14}$$

Therefore, from (3.12) and (3.14), we obtain that

$$y_n - z_n \rightarrow 0, \quad y_n - T^n y_n \rightarrow 0, \quad y_n - Q_r y_n \rightarrow 0, \quad \text{as } n \rightarrow \infty. \tag{3.15}$$

But observe that from (3.2) and (3.11), we have

$$\|Jx_{n+1} - Jx_n\| \leq \gamma_n \|JT^n y_n - Jx_n\| + \theta_n \|JQ_r y_n - Jx_n\| \rightarrow 0, \tag{3.16}$$

as $n \rightarrow \infty$. Thus, as J^{-1} is uniformly continuous on bounded sets, we have that $x_{n+1} - x_n \rightarrow 0$ which implies from (3.14) that $x_{n+1} - y_n \rightarrow 0$, as $n \rightarrow \infty$, and that

$$\|y_{n+1} - y_n\| \leq \|y_{n+1} - x_{n+1}\| + \|x_{n+1} - y_n\| \rightarrow 0, \quad \text{as } n \rightarrow \infty. \quad (3.17)$$

Furthermore, since

$$\begin{aligned} \|y_n - Ty_n\| &\leq \|y_n - T^n y_n\| + \|T^n y_n - T^{n+1} y_n\| + \|T^{n+1} y_n - Ty_n\|, \\ &\leq \|y_n - T^n y_n\| + \|T^{n+1} y_n - T^{n+1} y_{n+1}\| + \|T^{n+1} y_{n+1} - y_{n+1}\| \\ &\quad + \|y_{n+1} - y_n\| + \|y_n - T^n y_n\| + \|T(T^n y_n) - Ty_n\| \\ &\leq \|y_n - T^n y_n\| + L\|y_n - y_{n+1}\| + \|T^{n+1} y_{n+1} - y_{n+1}\| \\ &\quad + \|y_{n+1} - y_n\| + \|y_n - T^n y_n\| + \|T(T^n y_n) - Ty_n\| \\ &\leq 2\|y_n - T^n y_n\| + (1+L)\|y_n - y_{n+1}\| + \|T^{n+1} y_{n+1} - y_{n+1}\| + \|T(T^n y_n) - Ty_n\|, \end{aligned} \quad (3.18)$$

we have from (3.17), (3.15), and uniform continuity of T that

$$\|y_n - Ty_n\| \rightarrow 0, \quad \text{as } n \rightarrow \infty. \quad (3.19)$$

Since $\{z_n\}$ is bounded and E is reflexive, we choose a subsequence $\{z_{n_i}\}$ of $\{z_n\}$ such that $z_{n_i} \rightharpoonup z$ and $\limsup_{n \rightarrow \infty} \langle z_n - p, Jw - Jp \rangle = \lim_{i \rightarrow \infty} \langle z_{n_i} - p, Jw - Jp \rangle$. Then, from (3.14) and (3.15) we get that

$$y_{n_i} \rightharpoonup z, x_{n_i} \rightharpoonup z, \quad \text{as } i \rightarrow \infty. \quad (3.20)$$

Thus, since T satisfies condition (N3), we obtain from (3.19) that $z \in F(T)$ and the fact that Q_r is relatively nonexpansive and $y_{n_i} \rightharpoonup z$ implies that $z \in F(Q_r) = A^{-1}(0)$, and, hence, using (3.1), we obtain that $z \in \text{VI}(C, A)$.

Therefore, from the above discussions, we obtain that $z \in F := F(T) \cap \text{VI}(C, A)$. Hence, by Lemma 2.6, we immediately obtain that $\limsup_{n \rightarrow \infty} \langle z_n - p, Jw - Jp \rangle = \lim_{i \rightarrow \infty} \langle z_{n_i} - p, Jw - Jp \rangle = \langle z - p, Jw - Jp \rangle \leq 0$. It follows from Lemma 2.9 and (3.9) that $\phi(p, x_n) \rightarrow 0$, as $n \rightarrow \infty$. Consequently, $x_n \rightarrow p$.

Case 2. Suppose that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that

$$\phi(p, x_{n_i}) < \phi(p, x_{n_i+1}) \quad (3.21)$$

for all $i \in N$. Then, by Lemma 2.10, there exists a nondecreasing sequence $\{m_k\} \subset N$ such that $m_k \rightarrow \infty$, $\phi(p, x_{m_k}) \leq \phi(p, x_{m_k+1})$ and $\phi(p, x_k) \leq \phi(p, x_{m_k+1})$ for all $k \in N$. Then, from (3.8), (*) and the fact $\delta_n \rightarrow 0$, we have

$$g(\|Jx_{m_k} - JT^{m_k} y_{m_k}\|) \rightarrow 0, \quad g(\|Jx_{m_k} - JQ_r y_{m_k}\|) \rightarrow 0, \quad \text{as } k \rightarrow \infty. \quad (3.22)$$

Thus, using the same proof as in Case 1, we obtain that $y_{m_k} - Ty_{m_k} \rightarrow 0$, $y_{m_k} - Q_r y_{m_k} \rightarrow 0$, as $k \rightarrow \infty$, and, hence, we obtain that

$$\limsup_{k \rightarrow \infty} \langle z_{m_k} - p, Jw - Jp \rangle \leq 0. \quad (3.23)$$

Then, from (3.9), we have that

$$\phi(p, x_{m_k+1}) \leq (1 - \delta_{m_k})\phi(p, x_{m_k}) + 2\delta_{m_k} \langle z_{m_k} - p, Jw - Jp \rangle + (k_{m_k} - 1)M. \quad (3.24)$$

Since $\phi(p, x_{m_k}) \leq \phi(p, x_{m_k+1})$, (3.24) implies that

$$\begin{aligned} \delta_{m_k} \phi(p, x_{m_k}) &\leq \phi(p, x_{m_k}) - \phi(p, x_{m_k+1}) + 2\delta_{m_k} \langle z_{m_k} - p, Jw - Jp \rangle + (k_{m_k} - 1)M \\ &\leq 2\delta_{m_k} \langle z_{m_k} - p, Jw - Jp \rangle + (k_{m_k} - 1)M. \end{aligned} \quad (3.25)$$

In particular, since $\delta_{m_k} > 0$, we get

$$\phi(p, x_{m_k}) \leq 2 \langle z_{m_k} - p, Jw - Jp \rangle + \frac{(k_{m_k} - 1)}{\delta_{m_k}} M. \quad (3.26)$$

Then, from (3.23) and the fact that $(k_{m_k} - 1)/\delta_{m_k} \rightarrow 0$, we obtain $\phi(p, x_{m_k}) \rightarrow 0$, as $k \rightarrow \infty$. This together with (3.24) gives $\phi(p, x_{m_k+1}) \rightarrow 0$, as $k \rightarrow \infty$. But $\phi(p, x_k) \leq \phi(p, x_{m_k+1})$, for all $k \in N$, thus we obtain that $x_k \rightarrow p$. Therefore, from the above two cases, we can conclude that $\{x_n\}$ converges strongly to p and the proof is complete. \square

If, in Theorem 3.1, we assume that T is relatively nonexpansive, we get the following corollary.

Corollary 3.2. *Let C be a nonempty, closed, and convex subset of a uniformly convex and uniformly smooth real Banach space E . Let $A : C \rightarrow E^*$ be a monotone mapping satisfying (2.8) and $\|Ax\| \leq \|Ax - Ap\|$, for all $x \in C$ and $p \in \text{VI}(C, A)$. Let $T : C \rightarrow C$ be a relatively nonexpansive mapping. Assume that $F := \text{VI}(C, A) \cap F(S)$ is nonempty. Let $\{x_n\}$ be a sequence generated by*

$$\begin{aligned} x_0 &= w \in C, \text{ chosen arbitrarily,} \\ y_n &= \Pi_C J^{-1}(\alpha_n Jw + (1 - \alpha_n) Jx_n), \\ x_{n+1} &= J^{-1}(\beta_n Jx_n + \gamma_n JTy_n + \theta_n JQ_r y_n), \end{aligned} \quad (3.27)$$

where $\alpha_n \in (0, 1)$ such that $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\{\beta_n\}, \{\gamma_n\}, \{\theta_n\} \subset [c, d] \subset (0, 1)$ such that $\beta_n + \gamma_n + \theta_n = 1$. Then, $\{x_n\}$ converges strongly to an element of F .

Proof. We note that the method of proof of Theorem 3.1 provides the required assertion. \square

If $E = H$, a real Hilbert space, then E is uniformly convex and uniformly smooth real Banach space. In this case, $J = I$, identity map on H and $\Pi_C = P_C$, projection mapping from H onto C . Thus, the following corollary holds.

Corollary 3.3. Let C be a nonempty, closed, and convex subset of a real Hilbert space H . Let $A : C \rightarrow H$ be a monotone mapping satisfying (2.8) and $\|Ax\| \leq \|Ax - Ap\|$, for all $x \in C$ and $p \in VI(C, A)$. Let $T : C \rightarrow C$ be a uniformly L -Lipschitzian relatively asymptotically nonexpansive mapping with sequence $\{k_n\}$. Assume that $F := VI(C, A) \cap F(S)$ is nonempty. Let $\{x_n\}$ be a sequence generated by

$$\begin{aligned} x_0 &= w \in C, \text{ chosen arbitrarily,} \\ y_n &= P_C(\alpha_n w + (1 - \alpha_n)x_n), \\ x_{n+1} &= \beta_n x_n + \gamma_n T^n y_n + \theta_n Q_r y_n, \end{aligned} \quad (3.28)$$

where $\alpha_n \in (0, 1)$ such that $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\lim_{n \rightarrow \infty} ((k_n - 1)/\alpha_n) = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\{\beta_n\}, \{\gamma_n\}, \{\theta_n\} \subset [c, d] \subset (0, 1)$ such that $\beta_n + \gamma_n + \theta_n = 1$. Then, $\{x_n\}$ converges strongly to an element of F .

Now, we state the second main theorem of our paper.

Theorem 3.4. Let C be a nonempty, closed, and convex subset of a uniformly convex and uniformly smooth real Banach space E . Let $A : C \rightarrow E^*$ be a monotone mapping satisfying (2.8). Let $T : C \rightarrow C$ be a uniformly L -Lipschitzian relatively asymptotically nonexpansive mapping with sequence $\{k_n\}$. Assume that $F := A^{-1}(0) \cap F(S)$ is nonempty. Let Q_r be the resolvent of A and $\{x_n\}$ a sequence generated by

$$\begin{aligned} x_0 &= w \in C, \text{ chosen arbitrarily,} \\ y_n &= \Pi_C J^{-1}(\alpha_n Jw + (1 - \alpha_n)Jx_n), \\ x_{n+1} &= J^{-1}(\beta_n Jx_n + \gamma_n JT^n y_n + \theta_n JQ_r y_n), \end{aligned} \quad (3.29)$$

where $\alpha_n \in (0, 1)$ such that $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\lim_{n \rightarrow \infty} ((k_n - 1)/\alpha_n) = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\{\beta_n\}, \{\gamma_n\}, \{\theta_n\} \subset [c, d] \subset (0, 1)$ such that $\beta_n + \gamma_n + \theta_n = 1$. Then, $\{x_n\}$ converges strongly to an element of F .

Proof. Similar method of proof of Theorem 3.1 provides the required assertion. \square

If, in Theorem 3.4, $A = 0$, then we have the following corollary. Similar proof of Theorem 3.1 provides the assertion.

Corollary 3.5. Let C be a nonempty, closed, and convex subset of a uniformly convex and uniformly smooth real Banach space E . Let $T : C \rightarrow C$ be a uniformly L -Lipschitzian relatively asymptotically nonexpansive mapping with sequence $\{k_n\}$. Assume that $F := F(S)$ is nonempty. Let $\{x_n\}$ be a sequence generated by

$$\begin{aligned} x_0 &= w \in C, \text{ chosen arbitrarily,} \\ y_n &= \Pi_C J^{-1}(\alpha_n Jw + (1 - \alpha_n)Jx_n), \\ x_{n+1} &= J^{-1}(\beta_n Jx_n + (1 - \beta_n)JT^n y_n), \end{aligned} \quad (3.30)$$

where $\alpha_n \in (0, 1)$ such that $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\lim_{n \rightarrow \infty} ((k_n - 1)/\alpha_n) = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\{\beta_n\} \subset [c, d] \subset (0, 1)$. Then, $\{x_n\}$ converges strongly to an element of F .

If, in Theorem 3.4, $T = I$, identity mapping on C , then we have the following corollary.

Corollary 3.6. *Let C be a nonempty, closed, and convex subset of a uniformly smooth and uniformly convex real Banach space E . Let $A : C \rightarrow E^*$ be a monotone mapping satisfying (2.8). Assume that $F := A^{-1}(0)$ is nonempty. Let Q_r be the resolvent of A and $\{x_n\}$ a sequence generated by*

$$\begin{aligned} x_0 &= w \in C, \text{ chosen arbitrarily,} \\ y_n &= \Pi_C J^{-1}(\alpha_n Jw + (1 - \alpha_n)Jx_n), \\ x_{n+1} &= J^{-1}(\beta_n Jx_n + (1 - \beta_n)JQ_r y_n), \end{aligned} \quad (3.31)$$

where $\alpha_n \in (0, 1)$ such that $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\{\beta_n\} \subset [c, d] \subset (0, 1)$. Then, $\{x_n\}$ converges strongly to an element of F .

If, in Theorem 3.4, we assume that T is relatively nonexpansive, we get the following corollary.

Corollary 3.7. *Let C be a nonempty, closed, and convex subset of a uniformly convex and uniformly smooth real Banach space E . Let $A : C \rightarrow E^*$ be a monotone mapping satisfying (2.8). Let $T : C \rightarrow C$ be a relatively nonexpansive mapping. Assume that $F := A^{-1}(0) \cap F(S)$ is nonempty. Let $\{x_n\}$ be a sequence generated by*

$$\begin{aligned} x_0 &= w \in C, \text{ chosen arbitrarily,} \\ y_n &= \Pi_C J^{-1}(\alpha_n Jw + (1 - \alpha_n)Jx_n), \\ x_{n+1} &= J^{-1}(\beta_n Jx_n + \gamma_n JT y_n + \theta_n JQ_r y_n), \end{aligned} \quad (3.32)$$

where $\alpha_n \in (0, 1)$ such that $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\{\beta_n\}, \{\gamma_n\}, \{\theta_n\} \subset [c, d] \subset (0, 1)$ such that $\beta_n + \gamma_n + \theta_n = 1$. Then, $\{x_n\}$ converges strongly to an element of F .

We may also get the following corollary for a common zero of monotone mappings.

Corollary 3.8. *Let C be a nonempty, closed, and convex subset of a uniformly convex and uniformly smooth real Banach space E . Let $A, B : C \rightarrow E^*$ be monotone mappings satisfying (2.8). Suppose that $T_1 = (J + rA)^{-1}J$ and $T_2 = (J + rB)^{-1}J$. Assume that $F := A^{-1}(0) \cap B^{-1}(0)$ is nonempty. Let $\{x_n\}$ be a sequence generated by*

$$\begin{aligned} x_0 &= w \in C, \text{ chosen arbitrarily,} \\ y_n &= \Pi_C J^{-1}(\alpha_n Jw + (1 - \alpha_n)Jx_n), \\ x_{n+1} &= J^{-1}(\beta_n Jx_n + \gamma_n JT_1 y_n + \theta_n JT_2 y_n), \end{aligned} \quad (3.33)$$

where $\alpha_n \in (0, 1)$ such that $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\{\beta_n\}, \{\gamma_n\}, \{\theta_n\} \subset [c, d] \subset (0, 1)$ such that $\beta_n + \gamma_n + \theta_n = 1$. Then, $\{x_n\}$ converges strongly to an element of F .

Proof. Clearly, from Lemma 2.8, we know that T_1 and T_2 are relatively nonexpansive mappings. We also have that $F(T_1) = A^{-1}(0)$ and $F(T_2) = B^{-1}(0)$. Thus, the conclusion follows from Corollary 3.7. \square

Remark 3.9. We remark that from Corollary 3.8 the scheme converges strongly to a common zero of two monotone operators. We may also have the following theorem for a common zero of finite family of monotone mappings.

Theorem 3.10. Let C be a nonempty, closed, and convex subset of a uniformly convex and uniformly smooth real Banach space E . Let $A_i : C \rightarrow E^*$, $i = 1, 2, \dots, N$ be monotone mappings satisfying (2.8). Suppose that $T_i = (J + rA_i)^{-1}J$, and assume that $F := \bigcap_{i=1}^N A_i^{-1}(0)$ is nonempty. Let $\{x_n\}$ be a sequence generated by

$$\begin{aligned} x_0 &= w \in C, \text{ chosen arbitrarily,} \\ y_n &= \Pi_C J^{-1}(\alpha_n Jw + (1 - \alpha_n)Jx_n), \\ x_{n+1} &= J^{-1}(\beta_{n,0}Jx_n + \beta_{n,1}JT_1y_n + \dots + \beta_{n,N}JT_Ny_n), \end{aligned} \quad (3.34)$$

where $\alpha_n \in (0, 1)$ such that $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\{\beta_{n,i}\} \subset [c, d] \subset (0, 1)$, for $i = 0, 1, 2, \dots, N$, such that $\sum_{i=0}^N \beta_{n,i} = 1$. Then, $\{x_n\}$ converges strongly to an element of F .

A monotone mapping $A : C \rightarrow E^*$ is said to be *maximal monotone* if its graph is not properly contained in the graph of any monotone operator. We know that if A is maximal monotone operator, then $A^{-1}(0)$ is closed and convex: see [4] for more details. The following Lemma is well known.

Lemma 3.11 (see [28]). Let E be a smooth and strictly convex and reflexive Banach space, let C be a nonempty closed convex subset of E , and let $A : C \rightarrow E^*$ be a monotone operator. Then A is maximal if and only if $R(J + rA) = E^*$ for all $r > 0$.

We note from the above lemma that if A is maximal then it satisfies condition (2.8) and hence we have the following corollary.

Corollary 3.12. Let C be a nonempty, closed, and convex subset of a uniformly convex and uniformly smooth real Banach space E . Let $A : C \rightarrow E^*$ be a maximal monotone mapping. Let $T : C \rightarrow C$ be a uniformly L -Lipschitzian relatively asymptotically nonexpansive mapping with sequence $\{k_n\}$. Assume that $F := A^{-1}(0) \cap F(S)$ is nonempty. Let $\{x_n\}$ be a sequence generated by

$$\begin{aligned} x_0 &= w \in C, \text{ chosen arbitrarily,} \\ y_n &= \Pi_C J^{-1}(\alpha_n Jw + (1 - \alpha_n)Jx_n), \\ x_{n+1} &= J^{-1}(\beta_n Jx_n + \gamma_n JT^n y_n + \theta_n JQ_r y_n), \end{aligned} \quad (3.35)$$

where $\alpha_n \in (0, 1)$ such that $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\lim_{n \rightarrow \infty} ((k_n - 1)/\alpha_n) = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\{\beta_n\}, \{\gamma_n\}, \{\theta_n\} \subset [c, d] \subset (0, 1)$ such that $\beta_n + \gamma_n + \theta_n = 1$. Then, $\{x_n\}$ converges strongly to an element of F .

4. Application

In this section, we study the problem of finding a minimizer of a lower semicontinuous continuously convex functional in Banach spaces.

Theorem 4.1. Let E be a uniformly convex and uniformly smooth real Banach space. Let $f, g : E \rightarrow (-\infty, \infty)$ be a proper lower semicontinuous convex functions. Assume that $F := (\partial f)^{-1}(0) \cap (\partial g)^{-1}(0)$ is nonempty. Let $\{x_n\}$ be a sequence generated by

$$\begin{aligned} x_0 &= w \in C, \text{ chosen arbitrarily,} \\ y_n &= \Pi_C J^{-1}(\alpha_n Jw + (1 - \alpha_n) Jx_n), \\ h_n &= \operatorname{argmin} \left\{ f(z) + \frac{1}{2r} \phi(z, y_n), z \in E \right\}, \\ t_n &= \operatorname{argmin} \left\{ g(z) + \frac{1}{2r} \phi(z, y_n), z \in E \right\}, \\ x_{n+1} &= J^{-1}(\beta_n Jx_n + \gamma_n Jh_n + \theta_n Jt_n), \end{aligned} \quad (4.1)$$

where $\alpha_n \in (0, 1)$ such that $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\{\beta_n\}, \{\gamma_n\}, \{\theta_n\} \subset [c, d] \subset (0, 1)$ such that $\beta_n + \gamma_n + \theta_n = 1$. Then, $\{x_n\}$ converges strongly to an element of F .

Proof. Let A and B be operators defined by $A = \partial f$ and $B = \partial g$ and $Q_r = (J + rA)^{-1}J$, $Q_r^B = (J + rB)^{-1}J$ for all $r > 0$. Then, by Rockafellar [29], A and B are maximal monotone mappings. We also have that

$$\begin{aligned} h_n &= Q_r^A y = \operatorname{argmin} \left\{ f(z) + \frac{1}{2r} \phi(z, y), z \in E \right\}, \\ t_n &= Q_r^B y = \operatorname{argmin} \left\{ g(z) + \frac{1}{2r} \phi(z, y), z \in E \right\}, \end{aligned} \quad (4.2)$$

for all $y \in E$ and $r > 0$. Furthermore, we have that $F(Q_r^A) = A^{-1}(0)$ and $F(Q_r^B) = B^{-1}(0)$. Thus, by Corollary 3.8, we obtain the desired result. \square

Remark 4.2. Consider the following.

- (1) Theorem 3.1 improves and extends the corresponding results of Zegeye et al. [12] and Zegeye and Shahzad [22] in the sense that either our scheme does not require computation of C_{n+1} for each $n \geq 1$ or the space considered is more general.
- (2) Corollary 3.5 improves the corresponding results of Nakajo and Takahashi [30] and Matsushita and Takahashi [17] in the sense that either our scheme does not require computation of C_{n+1} for each $n \geq 1$ or the class of mappings considered in our corollary is more general.
- (3) Corollary 3.6 improves the corresponding results of Iiduka and Takahashi [11], Iiduka et al. [14], and Alber [1] in the sense that our scheme does not require computation of C_{n+1} for each $n \geq 1$ or the class of mappings considered in our corollary is more general.

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