Research Article

A Proximal Analytic Center Cutting Plane Algorithm for Solving Variational Inequality Problems

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Under the condition that the values of mapping F are evaluated approximately, we propose a proximal analytic center cutting plane algorithm for solving variational inequalities. It can be considered as an approximation of the earlier cutting plane method, and the conditions we impose on the corresponding mappings are more relaxed. The convergence analysis for the proposed algorithm is also given at the end of this paper.

1. Introduction

According to [1], the history of algorithms for solving finite-dimensional variational inequalities is relatively short. A recent development of such methods is the analytic center method based on cutting plane methods. It combines the feature of the newly developed interior point methods with the classical cutting plane scheme to achieve polynomial complexity in theory and quick convergence in practice. More details can be found in [2, 3]. Specifically, Goffin et al. [4] developed a convergent framework for finding a solution x^* of the variational inequality $\mathrm{VI}_P(F, X)$ associated with the continuous mapping F from X to R^n and the polyhedron $X = \{x \in R^n \mid Ax \leq b\}$ under an assumption slightly stronger than pseudomonotonicity. Again, Marcotte and Zhu [5] extended this algorithm to quasimonotone variational inequalities that satisfy a weak additional assumption. Such methods are effective in practice.

Note that the facts in optimization problems, see [6–8], some functions from R^n to R are themselves defined through other minimization problems. For example, consider the Lagrangian relaxation, see [9–12], the primal problem is

$$\max\{q(\xi) \mid \xi \in P, h(\xi) = 0\}, \tag{1.1}$$

where P is a compact subset of R^m and $q : R^m \to R$, $h : R^m \to R^n$ are two functions. Lagrangian relaxation in this problem leads to the problem $\min\{f(x) \mid x \in R^n\}$, where

$$f(x) = \max_{\xi \in P}\{q(\xi) + \langle x, h(\xi)\rangle\} \tag{1.2}$$

is the dual function. Trying to solve problem (1.1) by means of solving its dual problem $\min\{f(x) \mid x \in R^n\}$ becomes more difficult since in this case evaluating the function value $f(x)$ requires solving exactly another optimization problem (1.2). Let us see another example: consider the problem

$$\min\{f(x) \mid x \in C\}, \tag{1.3}$$

where f is convex (not necessarily differentiable), $C \subset R^n$ is a nonempty closed convex set, F is called the Moreau-Yosida regularization of f on C, that is,

$$F(x) = \min_{z \in C}\left\{f(z) + \frac{1}{2\alpha}\|z - x\|^2\right\}, \tag{1.4}$$

where α is a positive parameter. A point $x \in C$ is a solution to (1.3) if and only if it is a solution to the problem:

$$\min_{x \in R^n} F(x). \tag{1.5}$$

The problem (1.5) is easier to deal with than (1.3), see [13]. But in this case, computing the exact function value of F at an arbitrary point x is difficult or even impossible since F itself is defined through a minimization problem involving another function f. Intuitively, we consider the approximate computation of function F.

The above-mentioned phenomenon also exists for mappings from X to Y, where X and Y are two subspaces of any two finite-dimensional spaces, respectively. Once a mapping, and more specifically, a continuous mapping is defined implicitly rather than explicitly, the approximation of the mapping becomes inevitable, see [14]. In this paper we try to solve $\text{VI}_P(F, X)$ by assuming the values of the mapping F from X to R^n can be only computed approximately. Under the assumption, we construct an algorithm for solving the approximate variational inequality problem $\text{AVI}_P(F, X)$ and we also prove that there exists a cluster point of the iteration points generated by the proposed algorithm, it is a solution to the original problem $\text{VI}_P(F, X)$.

This paper is organized as follows. Some basic concepts and results are introduced in Section 2. In Section 3, a proximal analytic center cutting plane algorithm for solving the variational inequality problems is given. The convergence analysis of the proposed algorithm is addressed in Section 4. In the last section, we give some conclusions.

2. Basic Concepts and Results

Let $X = \{x \in R^n \mid Ax \leq b\}$ be a polyhedron and F a continuous mapping from X to R^n. A vector x^* is a solution to the variational inequality $\text{VI}_P(F, X)$ if and only if it satisfies the system of nonlinear inequalities:

$$\langle F(x^*), x - x^* \rangle \geq 0, \quad \forall x \in X. \tag{2.1}$$

The vector x^* is a solution to the dual variational inequality $\text{VI}_D(F, X)$ of $\text{VI}_P(F, X)$ if and only if it satisfies

$$\langle F(x), x - x^* \rangle \geq 0, \quad \forall x \in X. \tag{2.2}$$

We denote by X_P^* the solution set of $\text{VI}_P(F, X)$, and X_D^* the solution set of $\text{VI}_D(F, X)$, respectively. Whenever F is continuous, we have $X_D^* \subseteq X_P^*$, see [15]. If F is pseudomonotone on X, then $X_D^* = X_P^*$, see [15]. If F is quasimonotone at $x^* \in X_P^*$ and $F(x^*)$ is not normal to X at x^*, then X_D^* is nonempty, see Proposition 1 in [5]. For the definitions of monotone, pseudomonotone and quasimonotone, see [5, 15].

Definition 2.1. The gap functions $g_P(x)$ and $g_D(x)$ of $\text{VI}_P(F, X)$ and $\text{VI}_D(F, X)$ are, respectively, defined by

$$g_P(x) = \max_{y \in X} \langle F(x), x - y \rangle,$$

$$g_D(x) = \max_{y \in X} \langle F(y), x - y \rangle. \tag{2.3}$$

Note that $g_P(x) \geq 0$, $g_D(x) \geq 0$, and $g_P(x^*) = 0$ if and only if x^* is a solution to $\text{VI}_P(F, X)$, $g_D(x^*) = 0$ if and only if x^* is a solution to $\text{VI}_D(F, X)$. Thus, $X_P^* = \{x \in X \mid g_P(x) = 0\}$, $X_D^* = \{x \in X \mid g_D(x) = 0\}$.

Definition 2.2. A point $\tilde{x} \in X$ is called an ε-solution to $\text{VI}_P(F, X)$ if $g_P(\tilde{x}) \leq \varepsilon$ for given $\varepsilon > 0$.

Definition 2.3. For $x, y \in X$, we say $F(x) \preceq F(y)$ if and only if $F_i(x) \leq F_i(y)$, for $i = 1, 2, \ldots, n$, where $F(x) = (F_1(x), F_2(x), \ldots, F_n(x))^T$.

Assumptions 2.4. Throughout this paper, we make the following assumptions: for each $x, y \in X$, given any $\bar{\varepsilon} = (\varepsilon, \varepsilon, \ldots, \varepsilon)$, $\bar{\delta} = (\delta, \delta, \ldots, \delta)$, where $\varepsilon, \delta \in (0, 1)$, we can always find a $\overline{F}_x \in R^n$ and a $\overline{F}_y \in R^n$ such that

(a) $F(x) \preceq \overline{F}_x \preceq F(x) + \bar{\varepsilon}$, $F(y) \preceq \overline{F}_y \preceq F(y) + \bar{\delta}$,

that is, we can compute the approximate value of F with any precision;

(b) $\overline{F}_y \longrightarrow \overline{F}_x$ if $y \longrightarrow x$, no matter the relationship between $\bar{\varepsilon}$ and $\bar{\delta}$;

(c) $\left\| \overline{F}_y - \overline{F}_x \right\| \leq L \|y - x\|$, where $L > 0$ is a constant.

$$\tag{2.4}$$

These assumptions are realistic in practice, see [16, 17]. By using the given architecture in [16, 17], we can approximate the mapping F arbitrary well since neural networks are capable of approximating any function from one finite-dimensional real vector space to another one arbitrary well, see [18]. Specifically, let us consider the case of univariate function. If f is a min-type function of the form

$$f(x) = \inf\{N_z(x) \mid z \in Z\}, \tag{2.5}$$

where each $N_z(x)$ is convex and Z is an infinite set, then it may be impossible to calculate $f(x)$. However, we may still consider two cases. In the first case of controllable accuracy, for each positive $\varepsilon > 0$ one can find an ε-minimizer of (2.5), that is, an element $z_x \in Z$ satisfying $N_{z_x}(x) \leq f(x) + \varepsilon$; in the second case, this may be possible only for some fixed (any possibly unknown) $\varepsilon < \infty$. In both cases, we may set $\overline{f}_x = N_{z_x}(x) \leq f(x) + \varepsilon$. A special case of (2.5) arises from Lagrangian relaxation [15], where the problem $\max\{f(x) \mid x \in S\}$ with $S = R_+^n$ is the Lagrangian dual of the primal problem

$$\inf \ \psi_0(x) \quad \text{s.t.} \ \psi_j(x) \geq 0, \quad j = 1, 2, \ldots, n, \ x \in X, \tag{2.6}$$

with $N_z(x) = \psi_0(z) + \langle x, \psi(z) \rangle$ for $\psi = (\psi_1, \psi_2, \ldots, \psi_n)$. Then, for each multiplier $x \geq 0$, we need only to find $z_x \in Z$ such that $\overline{f}_x = N_{z_x}(x) \leq f(x) + \varepsilon$, see [19].

   Under the above assumptions (2.4), we introduce an approximate problem $\text{AVI}_P(F, X)$ associated with $\text{VI}_P(F, X)$: finding $x^* \in X$ such that

$$\left\langle \overline{F}_{x^*}, x - x^* \right\rangle \geq 0, \quad \forall x \in X, \tag{2.7}$$

where $\overline{F}_{x^*}$ satisfies $F(x^*) \leq \overline{F}_{x^*} \leq F(x^*) + \overline{\varepsilon}$ for arbitrary $\overline{\varepsilon} > 0$. Its dual problem $\text{AVI}_D(F, X)$ is to find $x^* \in X$ such that

$$\left\langle \overline{F}_x, x - x^* \right\rangle \geq 0, \quad \forall x \in X, \tag{2.8}$$

where $\overline{F}_x$ satisfies $F(x) \leq \overline{F}_x \leq F(x) + \overline{\varepsilon}$ for arbitrary $\overline{\varepsilon} > 0$.

Definition 2.5. The gap function of $\text{AVI}_P(F, X)$ is defined by $\overline{g}_P(x) = \max_{y \in X} \langle \overline{F}_x, x - y \rangle$.

Definition 2.6. A point $\tilde{x} \in X$ is called an ε-solution to $\text{AVI}_P(F, X)$ if $\overline{g}_P(\tilde{x}) \leq \varepsilon$ for given $\varepsilon > 0$.

   The optimal solution sets of $\text{AVI}_P(F, X)$ and $\text{AVI}_D(F, X)$ are denoted by AX_P^* and AX_D^*, respectively. The following proposition ensures that AX_D^* is nonempty.

Proposition 2.7. *If there exists a point $x^* \in AX_P^*$ such that*

$$\left\langle \overline{F}_{x^*}, y - x^* \right\rangle > 0 \Longrightarrow \left\langle \overline{F}_y, y - x^* \right\rangle \geq 0, \quad \forall y \in X, \tag{2.9}$$

and $\overline{F}_{x^*}$ is not normal to X at x^*, then AX_D^* is nonempty.

Proof. Since $\overline{F}_{x^*}$ is not normal to X at x^*, there exists a point $x_0 \in X$ such that $\langle \overline{F}_{x^*}, x_0 - x^* \rangle > 0$. $\forall x \in X$, set $x_t = tx_0 + (1-t)x$ for $t \in (0, 1]$, then we have $\langle \overline{F}_{x^*}, x - x^* \rangle \geq 0$ and $\langle \overline{F}_{x^*}, x_t - x^* \rangle > 0$. Note the condition (2.9), we obtain $\langle \overline{F}_{x_t}, x_t - x^* \rangle \geq 0$. Letting $t \to 0$, it follows from the condition (b) in (2.4) that $\langle \overline{F}_x, x - x^* \rangle \geq 0$, that is, $x^* \in AX_D^*$. □

In the following part, we focus our attention on solving $\text{AVI}_P(F, X)$. Let $\Gamma(y, x) : R^n \times R^n \to R^n$ denote an auxiliary mapping, continuous in x and y, strongly monotone in y, that is,

$$\langle \Gamma(y, x) - \Gamma(z, x), y - z \rangle \geq \beta \|y - z\|^2, \quad \forall y, z \in X, \tag{2.10}$$

for some $\beta > 0$. We consider the auxiliary variational inequality associated with Γ, whose solution $w(x)$ satisfies

$$\left\langle \Gamma(w(x), x) - \Gamma(x, x) + \overline{F}_x, y - w(x) \right\rangle \geq 0, \quad \forall y \in X. \tag{2.11}$$

In view of the strong monotonicity of $\Gamma(y, x)$ with respect to y, this auxiliary variational inequality has a unique solution $w(x)$.

Proposition 2.8. *The mapping $w : X \to X$ is continuous on X. Furthermore, x is a solution to $AVI_P(F, X)$ if and only if $x = w(x)$.*

Proof. The first part of the proposition follows from Theorem 5.4 in [1]. To prove the second part, we first suppose that $x = w(x)$. This yields $\langle \overline{F}_x, y - x \rangle \geq 0$, $\forall y \in X$, that is, x solves $\text{AVI}_P(F, X)$. Conversely, suppose that x solves $\text{AVI}_P(F, X)$, then

$$\left\langle \overline{F}_x, w(x) - x \right\rangle \geq 0, \tag{2.12}$$

and from (2.11), we have

$$\left\langle \Gamma(w(x), x) - \Gamma(x, x) + \overline{F}_x, x - w(x) \right\rangle \geq 0. \tag{2.13}$$

Adding the two preceding inequalities, one obtains

$$\langle \Gamma(w(x), x) - \Gamma(x, x), x - w(x) \rangle \geq 0, \tag{2.14}$$

and we conclude, from the strong monotonicity of Γ with respect to y, that $x = w(x)$. □

Let $\rho < 1$, $\alpha < \beta$ be two positive numbers. Let l be the smallest nonnegative integer satisfying

$$\left\langle \overline{F}_{x + \rho^l(w(x) - x)}, x - w(x) \right\rangle \geq \alpha \|w(x) - x\|^2, \tag{2.15}$$

where $\overline{F}_{x+\rho^l(w(x)-x)}$ satisfies $F(x + \rho^l(w(x) - x)) \leq \overline{F}_{x+\rho^l(w(x)-x)} \leq F(x + \rho^l(w(x) - x)) + \overline{\varepsilon}$ for arbitrary $\overline{\varepsilon} > 0$. The existence of a finite l will be proved in Proposition 2.9. The composite mapping G is defined, for every $x \in X$, by

$$G(x) = \overline{G}_x = \overline{F}_{x+\rho^l(w(x)-x)}. \tag{2.16}$$

If x^* is a solution to $\mathrm{AVI}_P(F, X)$, then we have $w(x^*) = x^*$, $l = 0$ and $\overline{G}_{x^*} = \overline{F}_{x^*}$.

Proposition 2.9. *The operator G is well defined for every $x \in X$. Moreover, we have*

$$l \leq \frac{\ln((\beta - \alpha)/L)}{\ln \rho}, \tag{2.17}$$

where L is the number given in (2.4)-(c).

Proof. From the definition of $w(x)$, we have

$$\langle \overline{F}_x, x - w(x) \rangle \geq \langle \Gamma(w(x), x) - \Gamma(x, x), w(x) - x \rangle \geq \beta \|x - w(x)\|^2. \tag{2.18}$$

Suppose (2.15) does not hold for any finite integer l, that is,

$$\langle \overline{F}_{x+\rho^l(w(x)-x)}, x - w(x) \rangle < \alpha \|x - w(x)\|^2. \tag{2.19}$$

Note the assumption (2.4)-(b), we obtain

$$\overline{F}_{x+\rho^l(w(x)-x)} \longrightarrow \overline{F}_x \quad \text{as } l \longrightarrow +\infty, \tag{2.20}$$

therefore,

$$\left\langle \overline{F}_x, x - w(x) \right\rangle \leq \alpha \|x - w(x)\|^2. \tag{2.21}$$

Since $\alpha < \beta$, (2.21) is in contradiction with (2.18). To prove the second part, we notice that

$$\begin{aligned}
\left\langle \overline{F}_{x+\rho^l(w(x)-x)}, x - w(x) \right\rangle &= \left\langle \overline{F}_x, x - w(x) \right\rangle + \left\langle \overline{F}_{x+\rho^l(w(x)-x)} - \overline{F}_x, x - w(x) \right\rangle \\
&\geq \beta \|w(x) - x\|^2 - L\rho^l \|w(x) - x\|^2 \\
&\geq \alpha \|w(x) - x\|^2
\end{aligned} \tag{2.22}$$

if $\alpha \leq \beta - L\rho^l$, which means the second conclusion of Proposition 2.9 holds. $\quad\square$

Proposition 2.10. *If $x \notin AX_P^*$, then $\forall y^* \in AX_D^*$, we have*

$$\left\langle \overline{G}_x, x - y^* \right\rangle > 0. \tag{2.23}$$

Proof. Let $y(x) = x + \rho^l(w(x) - x)$, then $\overline{G}_x = \overline{F}_{y(x)}$ and

$$\left\langle \overline{F}_{y(x)}, w(x) - x \right\rangle \leq -\alpha \|w(x) - x\|^2. \tag{2.24}$$

Since $x \notin AX_P^*$, so $w(x) \neq x$. Therefore,

$$\left\langle \overline{F}_{y(x)}, y(x) - x \right\rangle = \rho^l \left\langle \overline{F}_{y(x)}, w(x) - x \right\rangle < 0. \tag{2.25}$$

For all $y^* \in AX_D^*$, there holds

$$\left\langle \overline{F}_{y(x)}, y(x) - y^* \right\rangle \geq 0. \tag{2.26}$$

By combining (2.25) with (2.26), we obtain $\langle \overline{F}_{y(x)}, x - y^* \rangle > 0$, that is, $\langle \overline{G}_x, x - y^* \rangle > 0$. $\square$

3. A Proximal Analytic Center Cutting Plane Algorithm

Algorithm 3.1 offered in this section is a modification of the algorithm in [5]. Algorithm 3.1 is described as follows.

Algorithm 3.1. Let β be the strong monotonicity constant of $\Gamma(x, y)$ with respect to y, and let $\alpha \in (0, \beta)$, $\varepsilon \in (0, 1)$ be two constants. Set $k = 0$, $A^k = A \in R^{m \times n}$, $b^k = b$, and $\varepsilon^k = \varepsilon$.

Step 1 (computation of the center). Find an approximate analytic center x^k of $X^k = \{x \in R^n \mid A^k x \leq b^k\}$.

Step 2 (stopping criterion). If $g_P(x^k) \leq \varepsilon$, stop.

Step 3 (solving the approximate auxiliary variational inequality problem). Find $w(x^k)$, such that

$$\left\langle \overline{F}_{x^k} + \Gamma\left(w\left(x^k\right), x^k\right) - \Gamma\left(x^k, x^k\right), y - w\left(x^k\right) \right\rangle \geq 0, \quad \forall y \in X, \tag{3.1}$$

where $\overline{F}_{x^k}$ satisfies $F(x^k) \leq \overline{F}_{x^k} \leq F(x^k) + \overline{\varepsilon}_{x^k}, \overline{\varepsilon}_{x^k} = (\varepsilon^k, \varepsilon^k, \ldots, \varepsilon^k)^T$.

Step 4 (construction of the approximate cutting plane). Let $y^k = x^k + \rho^{l_k}(w(x^k) - x^k)$ and $\overline{G}_{x^k} = \overline{F}_{y^k}$, where l_k is the smallest integer that satisfies

$$\left\langle \overline{F}_{x^k + \rho^{l_k}(w(x^k) - x^k)}, x^k - w\left(x^k\right) \right\rangle \geq \alpha \left\| w(x^k) - x^k \right\|^2, \tag{3.2}$$

where $\overline{F}_{x^k + \rho^{l_k}(w(x^k) - x^k)}$ satisfies $F(x^k + \rho^{l_k}(w(x^k) - x^k)) \leq \overline{F}_{x^k + \rho^{l_k}(w(x^k) - x^k)} \leq F(x^k + \rho^{l_k}(w(x^k) - x^k)) + \overline{\varepsilon}_{x^k + \rho^{l_k}(w(x^k) - x^k)}, \overline{\varepsilon}_{x^k + \rho^{l_k}(w(x^k) - x^k)} = (\varepsilon^k, \varepsilon^k, \ldots, \varepsilon^k)^T$.

Let $H^k = \{x \mid \langle \overline{G}_{x^k}, x - x^k \rangle = 0\}$,

$$A^{k+1} = \begin{pmatrix} A^k \\ \overline{G}_{x^k} \end{pmatrix}, \qquad b^{k+1} = \begin{pmatrix} b^k \\ \overline{G}_{x^k}^T x^k \end{pmatrix}. \tag{3.3}$$

Increase k by one and go to Step 1.
End of Algorithm 3.1

4. Convergence Analysis

In [20], the authors proposed a column generation scheme to generate the polytope X^k, and they proved if k satisfies the following inequality

$$\frac{\varepsilon^2}{m} \geq \frac{1/2 + 2m\ln\left(1 + (k+1)/8m^2\right)}{2m + k + 1} e^{-2\alpha((k+1)/(2m+k+1))}, \tag{4.1}$$

where $\varepsilon < 1/2$ is a constant, the scheme will stop with a feasible solution, that is, they can find a vector a_{k+1} such that $\{y \mid a_{k+1}^T y \leq a_{k+1}^T y^k\} \supset \Gamma$ with $\|a_{k+1}\| = 1$, Γ contains a full-dimensional closed ball with $\varepsilon < 1/2$ radius. In other words, there exists the smallest $k(\varepsilon)$ such that $X^{k(\varepsilon)}$ generated by the column generation scheme does not contain the ball with $\varepsilon < 1/2$ radius, and it is known as the finite cut property. It is easy to know that the result of Theorem 6.6 in [20] also holds without much change for our Algorithm 3.1 using the approximate centers. That is, by using the row generation scheme, there exists the smallest $k(\rho)$ such that $X^{k(\rho)}$ generated in Step 4 in Algorithm 3.1 does not contain the ball with ρ radius lying inside the polytope X. This result plays an important role in proving the convergence of the described Algorithm 3.1 in Section 3.

Theorem 4.1. *Let the polyhedron X have nonempty interior, and let AX_D^* be nonempty. Assumption (2.4) holds. Then either Algorithm 3.1 stops with a solution to $AVI_P(F, X)$ after a finite number of iterations, or there exists a subsequence of the infinite sequence $\{x^k\}$ that converges to a point in AX_P^*.*

Proof. Assume that $x^k \notin AX_P^*$ for every iteration k, and let $y^* \in AX_D^*$. From Proposition 2.10, we know that $y^* \in X^k$ never lies on H^k for any k. Let $\{\overline{y}_i\}_{i \in N}$ be an arbitrary sequence of point in the interior of X converging to y^* and δ_i a sequence of positive numbers such that $\lim_{i \to +\infty} \delta_i = 0$ and that the sequence of closed balls $\{B(\overline{y}_i, \delta_i)\}_{i \in N}$ lies in the interior of X. Note that $\lim_{i \to +\infty} B(\overline{y}_i, \delta_i) = \{y^*\}$. From finite cut property, we know that there exists the smallest index $k(i)$ and a point $\widetilde{y}_i \in B(\overline{y}_i, \delta_i)$ such that $\widetilde{y}_i$ satisfies

$$\langle \overline{G}_{x^{k(i)}}, x^{k(i)} - \widetilde{y}_i \rangle < 0. \tag{4.2}$$

As $\langle \overline{G}_{x^{k(i)}}, x^{k(i)} - y^* \rangle > 0$, there exists a point $\widehat{y}_i$ on the segment $[\widetilde{y}_i, y^*]$ such that $\langle \overline{G}_{x^{k(i)}}, x^{k(i)} - \widehat{y}_i \rangle = 0$. Since X is compact, we can extract from $\{x^{k(i)}\}_{i \in N}$ a convergent subsequence $\{x^{k(i)}\}_{i \in S}$. Denote by $\widehat{x}$ its limit point, we have

$$\langle \overline{G}_{x^{k(i)}}, \widehat{y}_i - x^{k(i)} \rangle = 0, \quad \forall i \in S. \tag{4.3}$$

From Proposition 2.9, we know that $l_{k(i)}$ is bounded. Consequently, we can extract form $\{l_{k(i)}\}_{i \in S}$ a constant subsequence $l_{k^*(i)} = k^*$. Now from the continuity of $w(x)$ for fixed k and the relations (2.15) and (4.3), it follows by taking the limit in (4.3) that

$$\langle \overline{G}_{\widehat{x}}, y^* - \widehat{x} \rangle = 0. \tag{4.4}$$

By Proposition 2.10, we conclude that $\widehat{x} \in AX_P^*$. $\square$

Theorem 4.2. *Under the conditions of Theorem 4.1, either Algorithm 3.1 stops with a solution to $AVI_P(F, X)$ after a finite number of iterations, or there exists a subsequence of the infinite sequence $\{x^k\}$ that converges to a point in X_P^*.*

Proof. Since $\varepsilon \in (0, 1)$, $\varepsilon^k \in (0, 1)$. At the end of Step 4 in Algorithm 3.1 we increase k by one, so we have $\varepsilon^{k+1} < \varepsilon^k$, $\varepsilon^k \to 0$ as $k \to \infty$. Moreover, $\overline{\varepsilon}_{x^k} \to \overline{0}$ as $k \to \infty$ in Algorithm 3.1, where $\overline{0}$ denotes the zero vector. This means $\overline{F}_{\widehat{x}} = F(\widehat{x})$ as $k \to \infty$. Therefore, from the second result of Theorem 4.1, we know $\widehat{x}$ is the solution to the problem $VI_P(F, X)$. $\square$

5. Conclusions

In [5], the authors proposed a cutting plane method for solving the quasimonotone variational inequalities, but throughout the paper they employed the exact information of the mapping F from X to R^n. Just like the discussion in the first part of our paper, sometimes, it is not so easy or even impossible to compute the exact values of the mapping F. Motivated by this fact, we consider constructing an approximate problem $AVI_P(F, X)$ of $VI_P(F, X)$, and try out a proximal analytic center cutting plane algorithm for solving $AVI_P(F, X)$. In contrast to [5], our algorithm can be viewed as an approximation algorithm, and it is easier to implement than [5] since it only requires the inexact information of the corresponding mapping. At the same time, the conditions we impose on the corresponding mappings are more relaxed, for example, [5] requires the mapping F satisfies the Lipschitz condition, but we only require that the so-called approximate Lipschitz condition (2.4)-(c) holds.

Acknowledgments

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