*Research Article*

# On a Functional Equation Associated with ($a, k$)-Regularized Resolvent Families

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Let $a \in L^1_{loc}(\mathbb{R}_+)$ and $k \in C(\mathbb{R}_+)$ be given. In this paper, we study the functional equation $R(s)(a * R)(t) - (a * R)(s)R(t) = k(s)(a * R)(t) - k(t)(a * R)(s)$, for bounded operator valued functions $R(t)$ defined on the positive real line $\mathbb{R}_+$. We show that, under some natural assumptions on $a(\cdot)$ and $k(\cdot)$, every solution of the above mentioned functional equation gives rise to a commutative $(a, k)$-resolvent family $R(t)$ generated by $Ax = \lim_{t \to 0^+}(R(t)x - k(t)x/(a * k)(t))$ defined on the domain $D(A) := \{x \in X : \lim_{t \to 0^+}(R(t)x - k(t)x/(a * k)(t))$ exists in $X\}$ and, conversely, that each $(a, k)$-resolvent family $R(t)$ satisfy the above mentioned functional equation. In particular, our study produces new functional equations that characterize semigroups, cosine operator families, and a class of operator families in between them that, in turn, are in one to one correspondence with the well-posedness of abstract fractional Cauchy problems.

## 1. Introduction

Functional equations arise in most parts of mathematics. Well-known examples are Cauchy's equation, the functional equations for the Riemann zeta function, the equation for entropy, and numerous equations in combinatorics. Still other examples arise in probability theory, geometry, and operator theory [1].

The theory of functional equations for bounded operators emerged after the book of Hille and Phillips [2] in 1957. A strongly continuous semigroup $T(t)$ of bounded and linear operators on a Banach space $X$ is defined by means of Abel's functional equation:

$$T(t)T(s) = T(t + s), \quad t \geq 0,$$
$$T(0) = I,$$

(1.1)

which, in turn, characterizes the well-posedness of the abstract Cauchy problem of first order:

$$u'(t) = Au(t), \quad t \geq 0,$$
$$u(0) = u_0, \tag{1.2}$$

where $Ax = \lim_{t \to 0^+}(T(t)x - x/t)$ is defined on the domain $D(A) := \{x \in X : \lim_{t \to 0^+}((T(t)x - x)/t)$ exists in $X\}$. In 1966, Sova [3] introduces the concept of strongly continuous cosine operator functions, $C(t)$, by means of D'Alembert's functional equation:

$$C(t+s) + C(t-s) = 2C(t)C(s), \quad t, s \in \mathbb{R},$$
$$C(0) = I, \tag{1.3}$$

which characterizes the well-posedness of the abstract Cauchy problem of second order:

$$u''(t) = Au(t), \quad t \geq 0,$$
$$u(0) = u_0, \tag{1.4}$$
$$u'(0) = u_1,$$

where now $Ax = 2\lim_{t \to 0^+}((C(t)x - x)/t^2)$ is defined on $D(A) := \{x \in X : \lim_{t \to 0^+}((C(t)x - x)/t^2)$ exists in $X\}$. Let $A$ be a linear operator defined on a Banach space $X$. In [4] Prüss proved that the Volterra equation of scalar type:

$$u(t) = \int_0^t a(t-s)Au(s)ds + f(t) \tag{1.5}$$

is well posed if and only if it admits a resolvent family, that is, a strongly continuous family $S(t)$ of bounded and linear operators which commutes with $A$ and satisfies the so-called resolvent equation [4, Definition 1.3]:

$$S(t)x = x + \int_0^t a(t-s)AS(s)x\,ds, \quad t \geq 0, x \in X. \tag{1.6}$$

Resolvent families of operators have been known for a long time. They have many applications in the study of abstract differential and integral equations. However, at present there are associated functional equations for resolvent families only in special cases of the scalar kernel $a(t)$: for example $a(t) = 1$ or $a(t) = t$, which corresponds to the well-known cases of strongly continuous semigroups and cosine operator functions, respectively. Recently, Peng and Li in [5] have proposed an interesting functional equation for resolvent families in case $a(t) = g_\alpha(t) := t^{\alpha-1}/\Gamma(\alpha)$ which works for $0 < \alpha < 1$ (see also [6] for the scalar case). A functional equation for the kernel $a(t) = g_\alpha(t)$ in case $\alpha > 0$ has been proposed by Chen and Li in [7] (see [7, Definition 3.1 and Theorem 3.4]).

In this paper, we will be concerned with a commutative one parameter family of strongly continuous operators $R_{a,k}(t)$, depending on two scalar kernels $a(t)$ and $k(t)$, satisfying $R_{a,k}(0) = k(0)I$ and the functional equation:

$$R_{a,k}(s)(a * R_{a,k})(t) - (a * R_{a,k})(s)R_{a,k}(t) = k(s)(a * R_{a,k})(t) - k(t)(a * R_{a,k})(s), \quad t, s \geq 0.$$

(1.7)

In case $k(t) \equiv 1$ and $a(t)$ positive, one of our main results in this paper shows that the functional equation (1.7) characterizes a resolvent family and hence the well-posedness of the Volterra equation (1.5). Moreover, we have the representation:

$$Ax = \lim_{t \to 0^+} \frac{R_{a,1}(t)x - x}{\int_0^t a(s)ds},$$

(1.8)

for all $x \in D(A) := \{x \in X : \lim_{t \to 0^+}((R_{a,1}(t)x - x)/(1 * a)(t))$ exists in $X\}$, which includes the case of semigroups, cosine operator functions, and resolvent families for $a(t) = g_\alpha(t)$, $\alpha > 0$. Our discussion will not be restricted to resolvent families; the more general case of $(a, k)$-regularized resolvent families [8] is included in our results. Indeed, in such case we will see that in addition to (1.7) the condition:

$$\sup_{t>0} \frac{\int_0^t |a(s)|ds}{\left|\int_0^t k(t-s)a(s)ds\right|} < +\infty$$

(1.9)

is necessary to characterize an $(a, k)$-regularized resolvent family. A remarkable consequence is that the domain of $A$ must be necessarily dense on the Banach space $X$. In particular, setting $k(t) \equiv 1$ we obtain a new (but equivalent) functional equation for strongly continuous semigroups (i.e., the case $a(t) \equiv 1$) and strongly continuous cosine operator functions (i.e., $a(t) \equiv t$), respectively. On the other hand, we prove that the condition:

$$\lim_{s \to 0^+} \frac{(a * a * k)(s)}{(a * k)(s)} = 0,$$

(1.10)

that include, for example, the theory of $\alpha$-times integrated semigroups, is also necessary to characterize an $(a, k)$-regularized resolvent family. However, the immediate denseness of $D(A)$ is not automatically obtained in such case, in concordance with the theory of integrated semigroups [9]. We remark that our results recover and extend Chen and Li [7, Theorem 3.4] and [7, Theorem 3.12], where the case $k(t) = g_{\beta+1}(t)$, $a(t) = g_\alpha(t)$, $\alpha > 0$, $\beta \geq 0$ was considered (see [7, Definition 3.7]). In particular, our formula extend the functional equations stated recently by Peng and Li [5, 6] (see Remark 3.11 below).

## 2. Preliminaries

In this section we review some of the main results in the literature about the theory of $(a, k)$-regularized resolvent families. This notion was introduced in [8] and studied, as well as extended, in subsequent papers (see e.g., [10–16]).

Let us fix some notations. From now on, we take $X$ to be a complex Banach space with norm $\|\cdot\|$. We denote by $\mathcal{B}(X)$ the Banach algebra of all bounded linear operators on $X$ endowed with the operator norm, which again is denoted by $\|\cdot\|$. The identity operator on $X$ is denoted by $I \in \mathcal{B}(X)$, and $\mathbb{R}_+$ denotes the interval $[0, \infty)$. For a closed operator $A$, we denote by $\sigma(A)$, $\sigma_p(A)$, $\sigma_r(A)$, and $\sigma_a(A)$ the spectrum, the point spectrum, the residual spectrum, and the approximate spectrum of $A$, respectively.

*Definition 2.1.* Let $k \in C(\mathbb{R}_+)$, $k \neq 0$, and $a \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$, $a \neq 0$ be given. Assume that $A$ is a linear operator with domain $D(A)$. A strongly continuous family $\{R(t)\}_{t \geq 0} \subset \mathcal{B}(X)$ is called an $(a, k)$-regularized resolvent family on $X$ having $A$ as a generator if the following properties hold:

(i) $R(0) = k(0)I$;

(ii) $R(t)x \in D(A)$ and $R(t)Ax = AR(t)x$ for all $x \in D(A)$ and $t \geq 0$;

(iii) $R(t)x = k(t)x + \int_0^t a(t-s)AR(s)x\, ds$, $t \geq 0$, $x \in D(A)$.

We emphasize that the main properties of this theory admit very clear and simple proofs, and what is more interesting, it is easy to associate a suitable regularized resolvent family to a wide class of linear evolution equations, including, for example, fractional abstract differential equations (see [17]).

Assume that $a$ and $k$ are both positive and one of them is nondecreasing. Let $\{R(t)\}_{t \geq 0}$ be an $(a, k)$-regularized resolvent family with generator $A$ such that

$$\|R(t)\| \leq Mk(t), \quad t \geq 0, \tag{2.1}$$

for some constant $M > 0$. Then we have

$$Ax = \lim_{t \to 0^+} \frac{R(t)x - k(t)x}{(a * k)(t)}, \quad x \in D(A). \tag{2.2}$$

Here we denote $(a * k)(t) := \int_0^t k(t-s)a(s)ds$ the finite convolution between $a$ and $k$. The above representation of $A$ in terms of $R(t)$ was established in [16] (see also [15]). We note that there is a one-to-one correspondence between $(a, k)$-regularized resolvent families and their generators. Moreover, we can prove that an $(a, k)$-regularized resolvent family is uniformly continuous if and only if its generator is a bounded linear operator [8].

We say that $\{R(t)\}_{t \geq 0}$ is of type $(M, \omega)$ if there exist constants $M \geq 0$ and $\omega \in \mathbb{R}$ such that

$$\|R(t)\| \leq Me^{\omega t} \quad \forall\, t \geq 0. \tag{2.3}$$

The next result corresponds to the generation theorem for the theory. We assume that the Laplace transform for $a(t)$ and $k(t)$ exists for all $\lambda > \omega$.

**Theorem 2.2** (see [8]). *Let $A$ be a closed linear densely defined operator on $X$. Then $\{R(t)\}_{t \geq 0}$ is $(a, k)$-regularized resolvent family of type $(M, \omega)$ if and only if the following conditions hold:*

(1) $\widehat{a}(\lambda) \neq 0$ *and* $1/\widehat{a}(\lambda) \in \rho(A)$ *for all* $\lambda > \omega$,

(2) $H(\lambda) := \widehat{k}(\lambda)(I - \widehat{a}(\lambda)A)^{-1}$ satisfies the estimates:

$$\left\| H^{(n)}(\lambda) \right\| \leq \frac{Mn!}{(\lambda - \omega)^{n+1}}, \quad \lambda > \omega, \; n \in \mathbb{N}_0. \tag{2.4}$$

In the case where $k(t) \equiv 1$, Theorem 2.2 is well known. In fact, if $a(t) \equiv 1$, then it is just the Hille-Yosida theorem; if $a(t) \equiv t$, then it is the generation theorem for generators of cosine functions due to Sova and Fattorini; for arbitrary $a(t)$, it is the generation theorem due essentially to Prüss [4]. In the case where $k(t) = t^n/n!$ and $a(t) \equiv 1$, it is the generation theorem for $n$-times integrated semigroups [18]; if $k(t) = t^n/n!$ and $a(t)$ is arbitrary, it corresponds to the generation theorem for integrated solutions of Volterra equations due to Arendt and Kellermann [19].

Although the generation theorem characterizes all generators of $(a, k)$-regularized resolvent families, it is difficult to verify the estimate of all derivatives of the operator $H(\lambda)$ in concrete applications. Thus, one tries to build up the operator from simpler ones using perturbation techniques. The following is the main result available until now. It corresponds to the extension of the Miyadera-Voigt perturbation theorem in the theory of $C_0$ semigroups. In this theorem, the perturbation $B$ is bounded only from the domain of the generator $D(A)$, endowed with the graph norm $\|x\|_A := \|x\| + \|Ax\|$.

**Theorem 2.3** (see [16]). *Let $A$ be a closed operator on X. Assume that $A$ generates an $(a, k)$-regularized resolvent family $\{R(t)\}_{t \geq 0}$ of type $(M, \omega)$ and suppose that*

(i) *there exists $b \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$ such that $(k * b)(t) = a(t)$ for all $t \geq 0$;*

(ii) *there exists constants $\mu > \omega$ and $\gamma \in [0, 1)$ such that*

$$\int_0^\infty e^{-\mu r} \left\| B \int_0^r b(r - s)R(s)x \, ds \right\| dr \leq \gamma \|x\| \quad \forall \, x \in D(A). \tag{2.5}$$

*Then $A + B$ generates an $(a, k)$-regularized resolvent family $\{R(t)\}_{t \geq 0}$ on X such that $\|R(t)\| \leq (M/(1 - \gamma))e^{\mu t}$. In addition*

$$R(t)x = R(t)x + \int_0^t R(t - r)B \int_0^r b(r - s)R(s)x \, ds \, dr, \quad x \in X. \tag{2.6}$$

The next result shows, roughly speaking, the continuous dependence of an $(a, k)$-regularized resolvent family $R(t)$ on its generator $A$. More precisely, the theorem below show that the convergence, in an appropriate sense, of a sequence of generators is equivalent to the convergence of the corresponding $(a, k)$-regularized resolvent families.

**Theorem 2.4** (see [20]). *Let $\{k_n\}_{n \geq 0} \in L^1_{\mathrm{loc}}(\mathbb{R}^+)$ and $\{a_n\}_{n \geq 0} \in AC_{\mathrm{loc}}(\mathbb{R}^+)$ be of type $(M, \omega)$, $\omega \geq 0$, such that $\widehat{a}_n(\lambda) \neq 0$ for $\lambda > \omega$ and $\int_0^\infty e^{-\omega s}|a'_n(s)|ds < \infty$. Let $A_n$ be closed and linear operators in X such that $A_0$ is densely defined. For each fixed $n \in \mathbb{N}_0$, assume that $R_n(t)$ is an $(a_n, k_n)$-regularized resolvent family generated by $A_n$ in X, and that there exists constants $M > 0$ and $\omega \in \mathbb{R}$, independent of n, such that*

$$\|R_n(t)\| \leq Me^{\omega t}, \quad \forall \, t \geq 0. \tag{2.7}$$

*Suppose also $a_n(t) \to a_0(t)$ and $k_n(t) \to k_0(t)$ as $n \to \infty$. Then the following statements are equivalent:*

(1) $\lim_{n \to \infty} k_n(\lambda)(I - a_n(\lambda)A_n)^{-1} = k_0(\lambda)(I - a_0(\lambda)A_0)^{-1}$ *for all* $\lambda \geq \omega$;

(2) $\lim_{n \to \infty} R_n(t)x = R_0(t)x$ *for all* $x \in X$, $t \geq 0$. *Moreover the convergence is uniform in $t$ on every compact subset of* $\mathbb{R}^+$.

Note that the above theorem is the extension of the Trotter-Kato theorem for the theory of $C_0$ semigroups, which follows in case $a(t) \equiv k(t) \equiv 1$.

In our next result, of concern are ergodic type theorems. Here the contributions to the theory are contained in [14, 21]. We below cite only a simple, but typical, result.

**Theorem 2.5.** *Let $A$ be the generator of an $(a, k)$-regularized resolvent family $\{R(t)\}_{t \geq 0}$ such that*

$$\|R(t)\| \leq Mk(t) \quad \forall \, t \geq 0. \tag{2.8}$$

*Suppose that*

(i) $a(t)$ *is positive, and $k(t)$ is nondecreasing and positive as well;*

(ii) $\lim_{t \to \infty}(k(t)/(k * a)(t)) = 0$;

(iii) $\sup_{t>0}(k(t)(1 * k)(t)/(k * a)(t)) < \infty$;

(iv) $\lim_{t \to \infty}((a * a * k)(t)/(a * k)(t)) = \infty$.

*Define*

$$A_t x := \frac{1}{k * a(t)} \int_0^t a(t - s)R(s)x \, ds; \quad x \in X, \; t > 0, \tag{2.9}$$

*then the following holds.*

(1) *The mapping $Px := \lim_{t \to \infty} A_t x$ is a bounded linear projection with $\mathrm{Ran}(P) = \mathrm{Ker}(A)$, $\mathrm{Ker}(P) = \overline{\mathrm{Ran}(A)}$, and*

$$D(P) = \mathrm{Ker}(A) \oplus \overline{\mathrm{Ran}(A)}. \tag{2.10}$$

(2) *For $0 < \beta \leq 1$ and $x \in \mathrm{Ker}(A) \oplus \overline{\mathrm{Ran}(A)}$, one has*

$$\|A_t x - Px\| = O\left(\left[\frac{k(t)}{a * k(t)}\right]^\beta\right) \quad \text{as } |t| \longrightarrow \infty. \tag{2.11}$$

(3) *If $X$ is reflexive then $\mathrm{Ker}(A) \oplus \overline{\mathrm{Ran}(A)} = X$.*

Note that in the case $k(t) = t^\beta/\Gamma(\beta + 1)$, $a(t) = (t^{\alpha-1}/\Gamma(\alpha))$, $\alpha > 0$, $\beta \geq 0$ the conditions (i)–(iv) are automatically satisfied.

In the next result, we are interested in the relation between the spectrum of $A$ and the spectrum of each one of the operators $R(t)$, $t \geq 0$. We denote by $s(t, \lambda)$ the unique solution of the convolution equation:

$$s(t, \lambda) := a(t) + \lambda \int_0^t a(t - \tau) s(\tau, \lambda) d\tau. \tag{2.12}$$

We also define

$$r(t, \lambda) := k(t) + \lambda \int_0^t s(t - \tau, \lambda) k(\tau) d\tau. \tag{2.13}$$

From a purely formal point of view one would expect the relation $\sigma(R(t)) = r(t, \sigma(A))$. This, however, is not true in general. The following result corresponds to the inclusion theorem.

**Theorem 2.6** (see [15]). *Let $A$ be a closed operator on $X$ and let $R(t)$ be an $(a, k)$-regularized resolvent family with generator $A$. Then*

(i) $\sigma(R(t)) \supset r(t, \sigma(A))$, $t \geq 0$,

(ii) $\sigma_p(R(t)) \supset r(t, \sigma_p(A))$, $t \geq 0$,

(iii) *If $A$ is densely defined then* $\sigma_r(R(t)) \supset r(t, \sigma_r(A))$, $t \geq 0$,

(iv) $\sigma_a(R(t)) \supset r(t, \sigma_a(A))$, $t \geq 0$.

*Remark 2.7.* In the case $k(t) = t^\beta / \Gamma(\beta + 1)$, $\beta \geq 0$, $a(t) = (t^{\alpha-1} / \Gamma(\alpha))$, $\alpha > 0$, we have that

$$r_{\alpha,\beta}(t, \lambda) = t^\beta E_{\alpha,\beta+1}(\lambda t^\alpha), \tag{2.14}$$

where $E_{\alpha,\beta+1}$ denotes the Mittag-Leffler function, defined as follows:

$$E_{\alpha,\beta+1}(z) := \sum_{n=0}^\infty \frac{z^n}{\Gamma(n\alpha + \beta + 1)}, \quad \alpha > 0, \ \beta > -1. \tag{2.15}$$

In particular: $\alpha = 1$, $\beta = 0$ gives $E_{1,1}(z) = e^z$ and then $r_{1,0}(t, \lambda) = e^{\lambda t}$. Here $R(t)$ is the $C_0$ semigroup generated by $A$ and therefore we recover the well known inclusion:

$$e^{\sigma(A)t} \subset \sigma(R(t)), \quad t > 0. \tag{2.16}$$

If $\alpha = 2$, $\beta = 0$ we have $E_{2,1}(z^2) = \cosh(z)$ and then $r_{1,0}(t, \lambda) = \cosh \sqrt{(\lambda)} t$. Here we recover the inclusion [22]:

$$\cosh \sqrt{\sigma(A)} t \subset \sigma(R(t)), \quad t > 0. \tag{2.17}$$

In general, let $\alpha > 0$ and suppose that the fractional Cauchy problem:

$$D_t^\alpha u(t) = Au(t), \quad t > 0 \tag{2.18}$$

is well posed, where $D_t^\alpha$ denotes Caputo's fractional derivative. Then $A$ generates an $(\alpha, 0)$-regularized resolvent family $R_\alpha(t)$ and we conclude that

$$E_{\alpha,1}(\sigma(A)t^\alpha) \subset \sigma(R_\alpha(t)), \quad t > 0. \tag{2.19}$$

This result was first proved by Li and Zheng [23].

Recent results on the theory of $(a, k)$-regularized resolvent families include conditions under which the complex inversion formula for the Laplace transform holds for $(a, k)$-regularized families [24] and Kallman-Rota-type inequalities [13]. However, following the analogy with the theories of $C_0$ semigroups and cosine operator functions, many problems are still to be solved.

## 3. A Functional Equation

Let $X$ be a complex Banach space and $f \in L^1_{loc}(\mathbb{R}^+, X)$. The Laplace integral is defined by

$$\widehat{f}(\lambda) := \int_0^\infty e^{-\lambda t} f(t) dt := \lim_{\tau \to \infty} \int_0^\tau e^{-\lambda t} f(t) dt. \tag{3.1}$$

Also define $\mathrm{abs}(f) := \inf\{\mathrm{Re}\,\lambda : \widehat{f}(\lambda)\text{ exist}\}$. Recall that a function $f$ is called Laplace transformable if $\mathrm{abs}(f) < \infty$. Note that a locally Bochner integrable function $f$ is Laplace transformable if and only if its antiderivative $F(t) = \int_0^t f(s) ds$ is exponentially bounded, see [9, Section 1.4]. The following theorem is the main result in this section.

**Theorem 3.1.** *Let $\{R(t)\}_{t \geq 0} \subset \mathcal{B}(X)$ be an $(a, k)$-regularized resolvent family generated by a closed operator $A$ such that $\rho(A) \neq \emptyset$. Then, for all $t, s \geq 0$ one has $R(t)R(s) = R(s)R(t)$, and the functional equation:*

$$\text{(FE)} \quad R(s)(a * R)(t) - (a * R)(s)R(t) = k(s)(a * R)(t) - k(t)(a * R)(s) \tag{3.2}$$

*holds.*

*Proof.* Note that by [8, Lemma 2.2] we have that for all $x \in X$, $(a * R)(t)x \in D(A)$ and

$$R(t)x = k(t)x + A(a * R)(t)x. \tag{3.3}$$

Hence

$$
\begin{aligned}
(a * R)(s)R(t)x &= (a * R)(s)k(t)x + (a * R)(s)A(a * R)(t)x \\
&= (a * R)(s)k(t)x + A(a * R)(s)(a * R)(t)x \\
&= (a * R)(s)k(t)x + R(s)(a * R)(t)x - k(s)(a * R)(t)x,
\end{aligned}
\tag{3.4}
$$

where we used the item (ii) in Definition 2.1 and (3.3). This shows that (FE) holds for all $x \in X$ and $t, s \geq 0$.

Now we show that $R(t)R(s) = R(s)R(t)$ for all $t, s \geq 0$. Let $F(t) := R(t)R(s)$ and $H(t) := R(s)R(t)$ then for all $x \in D(A)$

$$F(t)x = k(t)R(s)x + (a * F)(t)Ax,$$
$$H(t)x = k(t)R(s)x + (a * H)(t)Ax,$$

(3.5)

where we used Definition 2.1, the fact that $A$ is closed and $R(t)Ax = AR(t)x$ for all $x \in D(A)$, $t \geq 0$. It then follows that $W(t) := F(t) - H(t)$ satisfies

$$W(t)x = a * W(t)Ax \quad \forall\, x \in D(A).$$

(3.6)

Note that by (ii) in the Definition 2.1, $W(t)x \in D(A)$ for all $x \in D(A)$ and hence by (iii) we obtain

$$k * W(t)x = (R - a * RA) * W(t)x = R * (W - a * WA)(t)x = 0.$$

(3.7)

Now let $\lambda \in \rho(A)$, $y \in X$ and define $x = (\lambda - A)^{-1}y$. Then $(\lambda - A)k * W(t)x = 0$ implies that $k * W(t)y = 0$ for each $y \in X$. Therefore, by Titchmarsh's Theorem, we obtain that $W(t)x = 0$ for each $x \in X$ which ends the proof. □

*Remark 3.2.* Assume that $R(t)$ is Laplace transformable. We note that an application of the double Laplace transform to (FE) gives the following identity:

$$\widehat{R}(\lambda)\widehat{R}(\mu) = \frac{\widehat{k}(\lambda)}{\widehat{a}(\lambda)} \frac{1}{(1/\widehat{a}(\lambda)) - (1/\widehat{a}(\mu))} \widehat{R}(\mu) - \frac{\widehat{k}(\mu)}{\widehat{a}(\mu)} \frac{1}{(1/\widehat{a}(\lambda)) - (1/\widehat{a}(\mu))} \widehat{R}(\lambda).$$

(3.8)

Let $S, T : \mathbb{R}^+ \to \mathcal{B}(X)$ be strongly continuous functions satisfying $\|S(t)\| \leq Me^{\omega t}$ and $\|T(t)\| \leq Me^{\omega t}$ $(t \geq 0)$ for some $\omega \in \mathbb{R}$, $M \geq 0$ (for simplicity we may assume the same constants). For $h \in \mathbb{R}$ we will denote $S^h$ the translation $S^h(u) := S(u + h)\chi_{[-h,+\infty)}(u)$ for $u \in \mathbb{R}$ and

$$(T * S)(t) := \int_0^t T(t - s)S(s)ds, \quad t > 0,$$

(3.9)

the convolution product between $T$ and $S$. We will need the following result.

**Lemma 3.3** (see Lemma 4.1, [25]). *Let $S, T : [0, \infty) \rightarrow \mathcal{B}(X)$ be strongly continuous functions satisfying the assumptions above. For $\lambda > \mu > \omega$, the following identities are valid:*

$$\widehat{S}(\lambda)\widehat{T}(\mu) = \int_0^\infty e^{-\lambda t} \int_0^\infty e^{-\mu s} S(t)T(s)ds\, dt, \tag{3.10}$$

$$\frac{1}{\mu - \lambda}\left(\widehat{S}(\lambda) - \widehat{S}(\mu)\right) = \int_0^\infty e^{-\lambda t} \int_0^\infty e^{-\mu s} S(t+s)ds\, dt, \tag{3.11}$$

$$\frac{1}{\mu - \lambda}\widehat{T}(\mu)\left[\widehat{S}(\lambda) - \widehat{S}(\mu)\right] = \int_0^\infty e^{-\lambda t} \int_0^\infty e^{-\mu s} (T * S^t)(s)ds\, dt. \tag{3.12}$$

*Defining $S(t) = S(-t)$ for $t < 0$ one has,*

$$\frac{1}{\mu + \lambda}\left(\widehat{S}(\lambda) + \widehat{S}(\mu)\right) = \int_0^\infty e^{-\lambda t} \int_0^\infty e^{-\mu s} S(s-t)ds\, dt, \quad \lambda + \mu > 0, \tag{3.13}$$

$$\frac{1}{\mu + \lambda}\widehat{T}(\mu)\left(\widehat{S}(\lambda) + \widehat{S}(\mu)\right) = \int_0^\infty e^{-\lambda t} \int_0^\infty e^{-\mu s} (T * S^{-t})(s)ds\, dt, \quad \lambda + \mu > 0, \tag{3.14}$$

*and defining $S(t) := -S(-t)$ for $t < 0$ one obtains*

$$\frac{-1}{\mu + \lambda}\widehat{T}(\mu)\left(\widehat{S}(\lambda) - \widehat{S}(\mu)\right) = \int_0^\infty e^{-\lambda t} \int_0^\infty e^{-\mu s} (T * S^{-t})(s)ds\, dt, \quad \lambda + \mu > 0. \tag{3.15}$$

In what follows, we restate and analyze consequences of Theorem 3.1 in several particular cases. They are important because they include different theories of strongly continuous operators and, as consequence, involve the well-posedness of wide classes of abstract evolution equations.

*Example 3.4* (semigroups). $k(t) \equiv a(t) \equiv 1$. In this case we have that $R(t)$ corresponds to a $C_0$ semigroup and the associated functional equation (FE) reads

$$R(s)\int_0^t R(\tau)d\tau - R(t)\int_0^s R(\tau)d\tau = \int_0^t R(\tau)d\tau - \int_0^s R(\tau)d\tau, \quad t, s \geq 0. \tag{3.16}$$

**Corollary 3.5.** *Assume that $R(t)$ is Laplace transformable. Then (3.16) is equivalent to Abel's functional equation.*

*Proof.* By (3.11) we have the identity:

$$\int_0^\infty \int_0^\infty e^{-\lambda t - \mu s} R(t+s)ds\, dt = \frac{\widehat{R}(\lambda) - \widehat{R}(\mu)}{\mu - \lambda}. \tag{3.17}$$

Hence, applying the double Laplace transform to the Abel's functional equation:

$$R(t+s) = R(t)R(s), \quad t, s \geq 0, \tag{3.18}$$

we obtain

$$\widehat{R}(\lambda)\widehat{R}(\mu) = \frac{\widehat{R}(\lambda) - \widehat{R}(\mu)}{\mu - \lambda}, \tag{3.19}$$

which is equivalent to

$$\frac{1}{\mu}\widehat{R}(\lambda)\widehat{R}(\mu) - \frac{1}{\lambda}\widehat{R}(\mu)\widehat{R}(\lambda) = \frac{1}{\lambda\mu}\widehat{R}(\mu) - \frac{1}{\lambda\mu}\widehat{R}(\lambda). \tag{3.20}$$

Hence, inversion of the double Laplace transform to the above identity gives (3.16). The converse is analogue. □

*Example 3.6* (cosine operator families). $k(t) \equiv 1$, $a(t) = t$. In this case we have that $R(t)$ corresponds to a cosine operator family [9, Section 3.14] and the associated functional equation (FE) reads

$$R(s)\int_0^t (t-\tau)R(\tau)d\tau - R(t)\int_0^s (s-\tau)R(\tau)d\tau$$
$$= \int_0^t (t-\tau)R(\tau)d\tau - \int_0^s (s-\tau)R(\tau)d\tau. \tag{3.21}$$

**Corollary 3.7.** *Assume that $R(t)$ is Laplace transformable. Then* (3.21) *is equivalent to the D'Alembert's functional equation.*

*Proof.* By (3.13) we have the identity:

$$\int_0^\infty \int_0^\infty e^{-\lambda t - \mu s} R(t-s)ds\,dt = \frac{\widehat{R}(\lambda) + \widehat{R}(\mu)}{\mu + \lambda} \tag{3.22}$$

valid whenever $R(t)$ is extended as an even function to $\mathbb{R}$, which is indeed the case of cosine operator families.

We apply the double Laplace transform to the D'Alembert's functional equation:

$$R(t+s) + R(t-s) = 2R(t)R(s), \quad t, s \geq 0, \tag{3.23}$$

and we obtain

$$\frac{\widehat{R}(\lambda) - \widehat{R}(\mu)}{\mu - \lambda} + \frac{\widehat{R}(\lambda) + \widehat{R}(\mu)}{\mu + \lambda} = 2\widehat{R}(\lambda)\widehat{R}(\mu), \tag{3.24}$$

which, after an algebraic manipulation, is equivalent to

$$\frac{\lambda}{\lambda^2 - \mu^2}\widehat{R}(\mu) - \frac{\mu}{\lambda^2 - \mu^2}\widehat{R}(\lambda) = \widehat{R}(\lambda)\widehat{R}(\mu), \tag{3.25}$$

(compare with Remark 3.2 in case $a(t) \equiv t$ and $k(t) \equiv 1$). The above identity is equivalent to:

$$\widehat{R}(\lambda)\frac{1}{\mu^2}\widehat{R}(\mu) - \widehat{R}(\mu)\frac{1}{\lambda^2}\widehat{R}(\lambda) = \frac{1}{\mu^2\lambda}\widehat{R}(\mu) - \frac{1}{\lambda^2\mu}\widehat{R}(\lambda). \tag{3.26}$$

Hence, inversion of the double Laplace transform to the above identity gives (3.21). The converse is analogue. $\qquad\square$

*Example 3.8* (sine operator family). $k(t) \equiv t$, $a(t) = t$. In this case we have that $R(t)$ corresponds to a Laplace transformable sine family [9, Section 3.15] and the associated functional equation (FE) reads:

$$R(s)\int_0^t (t-\tau)R(\tau)d\tau - R(t)\int_0^s (s-\tau)R(\tau)d\tau = s\int_0^t (t-\tau)R(\tau)d\tau - t\int_0^s (s-\tau)R(\tau)d\tau.$$

$$\tag{3.27}$$

*Remark 3.9.* The functional equation (3.27) is equivalent to the following:

$$2R(t)R(s) = \int_0^s R(s-\tau)d\tau - \int_0^t R(t+\tau)d\tau + \int_0^t R(\tau)d\tau - \int_0^s R(\tau)d\tau, \quad t,s \geq 0. \tag{3.28}$$

To show this, we apply the double Laplace transform to (3.28) and then Lemma 3.3, to conclude that:

$$2\widehat{R}(t)\widehat{R}(s) = -\frac{\widehat{R}(\lambda)-\widehat{R}(\mu)}{\mu(\mu+\lambda)} - \frac{\widehat{R}(\lambda)-\widehat{R}(\mu)}{\lambda(\mu-\lambda)} + \frac{\widehat{R}(\lambda)}{\lambda\mu} - \frac{\widehat{R}(\mu)}{\lambda\mu}$$

$$= 2\frac{\widehat{R}(\lambda)-\widehat{R}(\mu)}{\mu^2-\lambda^2}. \tag{3.29}$$

Finally, using Remark 3.2 in case $a(t) \equiv t$ and $k(t) \equiv t$ we see from the last identity that (3.27) is equivalent to (3.28). We notice that the functional equations (3.27) and (3.28) seem to be new for the theory of sine operator functions [9].

*Example 3.10* (Laplace transformable $(\alpha,\beta)$-resolvent operators). This example recovers Theorem 3.11 in [7]. Let $k(t) = t^\beta/\Gamma(\beta+1)$; $a(t) \equiv g_\alpha(t) = t^{\alpha-1}/\Gamma(\alpha)$; $\alpha > 0$, $\beta \geq 0$. In this case we have that $R(t)$ corresponds to a Laplace transformable $(g_\alpha, g_{\beta+1})$-regularized resolvent family [7, Theorem 3.12] and the associated functional equation (FE) reads:

$$R(s)\int_0^t g_\alpha(t-\tau)R(\tau)d\tau - R(t)\int_0^s g_\alpha(s-\tau)R(\tau)d\tau$$

$$= g_{\beta+1}(s)\int_0^t g_\alpha(t-\tau)R(\tau)d\tau - g_{\beta+1}(t)\int_0^s g_\alpha(s-\tau)R(\tau)d\tau. \tag{3.30}$$

Note that, taking $a(t) = k(t) = g_\alpha(t)$, $\alpha > 0$ (i.e., $\beta+1 = \alpha$) this example also includes the concept of $\alpha$-resolvent families introduced in [26], which characterizes the well-posedness of the fractional Cauchy problem:

$$D_t^\alpha u(t) = Au(t), \quad t \geq 0, \tag{3.31}$$

where $D_t^\alpha$ denotes the Riemann-Liouville fractional derivative. On the other hand, the special case $k(t) \equiv 1$ and $a(t) = g_\alpha(t)$, $\alpha > 0$ gives the theory of solution operators introduced by Bazhlekova [27] for the fractional Cauchy problem (3.31) where now $D_t^\alpha$ denotes Caputo's fractional derivative instead of Riemann-Liouville fractional derivative. Note that the functional equation has now the following form:

$$R(s) \int_0^t (t - \tau)^{\alpha-1} R(\tau) d\tau - R(t) \int_0^s (s - \tau)^{\alpha-1} R(\tau) d\tau$$
$$= \int_0^t (t - \tau)^{\alpha-1} R(\tau) d\tau - \int_0^s (s - \tau)^{\alpha-1} R(\tau) d\tau, \tag{3.32}$$

for $\alpha > 0$.

Observe the remarkable fact that in the scalar case, that is, $X = \mathbb{R}$ and $A = \rho \in \mathbb{R}$, we have that the Mittag-Leffler function $E_\alpha(\rho t^\alpha)$ satisfies the functional equation (3.32), because it is the unique solution of (3.31) with the fractional derivative considered in the sense of Caputo. In other words, we have

$$E_\alpha(\rho s^\alpha) \int_0^t (t - \tau)^{\alpha-1} E_\alpha(\rho \tau^\alpha) d\tau - E_\alpha(\rho t^\alpha) \int_0^s (s - \tau)^{\alpha-1} E_\alpha(\rho \tau^\alpha) d\tau$$
$$= \int_0^t (t - \tau)^{\alpha-1} E_\alpha(\rho \tau^\alpha) d\tau - \int_0^s (s - \tau)^{\alpha-1} E_\alpha(\rho \tau^\alpha) d\tau, \quad t, s \geq 0, \tag{3.33}$$

for $\alpha > 0$. In particular, it shows that the functional equation (3.32) is a proper generalization of Abel's functional equation (corresponding to the case $\alpha = 1$) and D'Alembert functional equation (corresponding to the case $\alpha = 2$) since in case $\alpha = 1$ we have $E_1(\rho t) = e^{\rho t}$ and in case $\alpha = 2$ we have $E_2(\rho t^2) = \cosh(\sqrt{\rho} t)$.

*Remark 3.11.* Suppose the family $\{R(t)\}_{t \geq 0}$ of bounded linear operators of $X$ is Laplace transformable. Then for $0 < \alpha < 1$ the functional equation (3.32) is equivalent to the functional equation:

$$\int_0^{t+s} \frac{R(\tau)}{(t + s - \tau)^\alpha} d\tau - \int_0^t \frac{R(\tau)}{(t + s - \tau)^\alpha} d\tau - \int_0^s \frac{R(\tau)}{(t + s - \tau)^\alpha} d\tau$$
$$= \alpha \int_0^t \int_0^s \frac{R(\tau_1) R(\tau_2)}{(t + s - \tau_1 - \tau_2)^{1+\alpha}} d\tau_1 d\tau_2 \tag{3.34}$$

proposed by Peng and Li in [5, 6]. Indeed, we apply the double Laplace transform to (3.34) and use Lemma 3.3 to conclude that the following identity holds for $0 < \alpha < 1$:

$$
\frac{\Gamma(1-\alpha)}{\mu-\lambda}\left[\frac{\widehat{R}(\lambda)}{\lambda^{1-\alpha}} - \frac{\widehat{R}(\mu)}{\mu^{1-\alpha}}\right] - \frac{\Gamma(1-\alpha)\widehat{R}(\lambda)}{\lambda-\mu}\left[\frac{1}{\mu^{1-\alpha}} - \frac{1}{\lambda^{1-\alpha}}\right] - \frac{\Gamma(1-\alpha)\widehat{R}(\mu)}{\mu-\lambda}\left[\frac{1}{\lambda^{1-\alpha}} - \frac{1}{\mu^{1-\alpha}}\right]
$$
$$
= \alpha\Gamma(1-\alpha)\frac{\mu^{\alpha}-\lambda^{\alpha}}{\lambda-\mu}\widehat{R}(\mu)\widehat{R}(\lambda).
$$

$$(3.35)$$

Then, algebraic manipulation shows that it is equivalent to the identity:

$$
\widehat{R}(\mu)\widehat{R}(\lambda) = \frac{1/\lambda}{1/\lambda^{\alpha}}\frac{\widehat{R}(\mu)}{\lambda^{\alpha}-\mu^{\alpha}} - \frac{1/\mu}{1/\mu^{\alpha}}\frac{\widehat{R}(\lambda)}{\lambda^{\alpha}-\mu^{\alpha}}.
$$

$$(3.36)$$

Finally, using Remark 3.2 in case $a(t) = t^{\alpha-1}/\Gamma(\alpha)$ and $k(t) \equiv 1$ we see from the last identity and inversion of the double Laplace transform that (3.34) is equivalent to the functional equation (3.32).

*Example 3.12* (Laplace transformable $k$-time integrated semigroups). We take in this example $k(t) = t^k/\Gamma(k+1), k = 0, 1, \ldots$ and $a(t) \equiv 1$. We have that $R(t)$ is a $k$-time integrated semigroup and the functional equation has the form:

$$
R(s)\int_0^t R(\tau)d\tau - R(t)\int_0^s R(\tau)d\tau = \frac{s^k}{k!}\int_0^t R(\tau)d\tau - \frac{t^k}{k!}\int_0^s R(\tau)d\tau.
$$

$$(3.37)$$

Following the same type of arguments as in Corollaries 3.5 and 3.7 (see also the following example), we note that the above equation is equivalent to the following well-known formula that originally define $k$-time integrated semigroups (see [9, Section 3.2, Proposition 2.3.4]):

$$
R(t)R(s) = \frac{1}{(k-1)!}\left[\int_t^{t+s}(s+t-r)^{k-1}R(r)dr - \int_0^s(s+t-r)^{k-1}R(r)dr\right].
$$

$$(3.38)$$

*Example 3.13* ($K$-convoluted semigroups). Let $K$ be a complex valued, locally integrable function on $[0, \infty)$. We take in this example $k(t) = \int_0^t K(\sigma)d\sigma$ and $a(t) \equiv 1$. We have that $R(t)$ is an $K$-convoluted semigroup (see [28, Definition 2.1] and references therein) and the functional equation has the form:

$$
R(s)\int_0^t R(\tau)d\tau - R(t)\int_0^s R(\tau)d\tau = \int_0^s K(\sigma)d\sigma\int_0^t R(\tau)d\tau - \int_0^t K(\sigma)d\sigma\int_0^s R(\tau)d\tau, \quad (3.39)
$$

which is equivalent with the standard definition:

$$
R(t)R(s)x = \left[\int_0^{t+s} - \int_0^t - \int_0^s\right]K(t+s-r)R(r)x\,dr.
$$

$$(3.40)$$

Indeed, if $R$ is Laplace transformable then we can apply the double Laplace transform to (3.40) and use Lemma 3.3 to conclude that

$$
\begin{aligned}
2\widehat{R}(\lambda)\widehat{R}(\mu) &= \frac{\widehat{K}(\lambda)\widehat{R}(\lambda) - \widehat{K}(\mu)\widehat{R}(\mu)}{\mu - \lambda} - \widehat{R}(\lambda)\frac{\widehat{K}(\mu) - \widehat{K}(\lambda)}{\lambda - \mu} - \widehat{R}(\mu)\frac{\widehat{K}(\lambda) - \widehat{R}(\mu)}{\mu - \lambda} \\
&= \frac{\widehat{K}(\lambda)\widehat{R}(\mu) - \widehat{K}(\mu)\widehat{R}(\lambda)}{\lambda - \mu}.
\end{aligned}
$$

(3.41)

Then observing the formula in Remark 3.2 we note that (3.39) is equivalent to the functional equation (3.40).

*Example 3.14* ($K$-convoluted cosine functions). Let $K$ be a complex valued, locally integrable function on $[0, \infty)$. We take in this example $k(t) = \int_0^t K(\sigma)d\sigma$ and $a(t) = t$. We have that $R(t)$ is an $K$-convoluted semigroup and the functional equation has the form:

$$
\begin{aligned}
R(s)\int_0^t (t - \tau)R(\tau)d\tau &- R(t)\int_0^s (s - \tau)R(\tau)d\tau \\
&= \int_0^s K(\sigma)d\sigma \int_0^t (t - \tau)R(\tau)d\tau - \int_0^t K(\sigma)d\sigma \int_0^s (s - \tau)R(\tau)d\tau,
\end{aligned}
$$

(3.42)

in contrast with the (equivalent) recently discovered expression [25]:

$$
\begin{aligned}
2R(t)R(s) &= \int_t^{s+t} K(s + t - r)R(r)dr - \int_0^s K(s + t - r)R(r)dr \\
&+ \int_{t-s}^t K(r - t + s)R(r)dr + \int_0^s K(r + t - s)R(r)dr,
\end{aligned}
$$

(3.43)

where $t \geq s \geq 0$.

*Example 3.15* (resolvent families). Taking $k(t) \equiv 1$, $a \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$ we obtain the following functional equation, which seems to be the first, for the theory of integral equations of convolution type [4]:

$$
R(s)\int_0^t a(t - \tau)R(\tau)d\tau - R(t)\int_0^s a(s - \tau)R(\tau)d\tau = \int_0^t a(t - \tau)R(\tau)d\tau - \int_0^s a(s - \tau)R(\tau)d\tau.
$$

(3.44)

*Example 3.16* (integral resolvents). Taking $a(t) = k(t)$ we obtain, to our knowledge, the first functional equation for the theory of integral resolvents. Of course, it includes the scalar case [29]:

$$R(s) \int_0^t a(t - \tau) R(\tau) d\tau - R(t) \int_0^s a(s - \tau) R(\tau) d\tau$$
$$= a(s) \int_0^t a(t - \tau) R(\tau) d\tau - a(t) \int_0^s a(s - \tau) R(\tau) d\tau. \tag{3.45}$$

*Example 3.17* (a special case). Taking $k(t) = 1$ and $a(t) = (t^{\alpha-1} / \Gamma(\alpha)) \gamma e^{-\beta t}$, $\gamma \neq 0$, we obtain

$$R(s) \int_0^t (t - \tau)^{\alpha-1} e^{-\beta(t-\tau)} R(\tau) d\tau - R(t) \int_0^s (s - \tau)^{\alpha-1} e^{-\beta(s-\tau)} R(\tau) d\tau$$
$$= \int_0^t (t - \tau)^{\alpha-1} e^{-\beta(t-\tau)} R(\tau) d\tau - \int_0^s (s - \tau)^{\alpha-1} e^{-\beta(s-\tau)} R(\tau) d\tau. \tag{3.46}$$

In this example, the kernel $a(t)$ is important in viscoelasticity theory [4].

## 4. Sufficient Conditions

Let $k, a \in L^1_{\text{loc}}(\mathbb{R}_+)$ be given and $R : \mathbb{R}_+ \to B(X)$ be a strongly continuous family such that

$$\text{(FE)} \quad R(s)(a * R)(t) - (a * R)(s)R(t) = k(s)(a * R)(t) - k(t)(a * R)(s) \tag{4.1}$$

holds for all $s, t \geq 0$. In this section, we study in what extent the functional equation (FE) is sufficient to imply that $R(t)$ is an $(a, k)$ resolvent family. In passing, we are going to unify and clarify from a general perspective a basic property of the theories of semigroups and cosine operator families: automatic denseness of the domain of the generator.

**Theorem 4.1.** *Let $k \in C(\mathbb{R}_+)$, $a \in L^1_{\text{loc}}(\mathbb{R}_+)$ be given and let $R(t) \subset \mathcal{B}(X)$ be a commutative and strongly continuous family such that $R(0) = k(0)I$ and satisfies* (FE). *Define*

$$D(B) := \left\{ x \in X : \lim_{t \to 0^+} \frac{R(t)x - k(t)x}{(a * k)(t)} \ exist \right\}, \tag{4.2}$$

$$Bx := \lim_{t \to 0^+} \frac{R(t)x - k(t)x}{(a * k)(t)} \quad x \in D(B) \tag{4.3}$$

*and suppose the following condition:*

$$\sup_{t > 0} \frac{\int_0^t |a(s)| ds}{|a * k(t)|} < +\infty. \tag{4.4}$$

*Then $R(t)$ is an $(a, k)$-regularized resolvent family with generator $B$. Moreover, $B$ is closed and $D(B)$ is dense in $X$.*

*Proof.* Since $R(0) = k(0)I$, we have to prove conditions (ii) and (iii) from Definition 2.1. Fix $x \in D(B)$ and $t \geq 0$. For $s \geq 0$ we have from the commutativity of the family $R(t)$ that

$$\frac{R(s)R(t)x - k(s)R(t)x}{a * k(s)} = \frac{R(t)(R(s)x - k(s)x)}{a * k(s)}. \tag{4.5}$$

Since $R(t)$ is bounded, by definition of $B$ we have that

$$\lim_{s \to 0^+} \frac{R(t)(R(s)x - k(s)x)}{a * k(s)} \tag{4.6}$$

exists and equals to $R(t)Bx$. Then (4.5) implies that $R(t)x \in D(B)$ and $BR(t)x = R(t)Bx$ for all $t \geq 0$. It proves condition (ii) in Definition 2.1. In order to show condition (iii), let $x \in X$ be given and note that by (4.4) we have that

$$\left\| \frac{a * R(s)x}{a * k(s)} - x \right\| \leq \frac{1}{|a * k(s)|} \int_0^s |a(s - \mu)| \|R(\mu)x - k(\mu)x\| d\mu$$

$$\leq \frac{1}{|a * k(s)|} \int_0^s |a(s - \mu)| d\mu \sup_{\mu \in [0,s]} \|R(\mu)x - k(\mu)x\| \longrightarrow 0 \quad \text{as } s \longrightarrow 0^+. \tag{4.7}$$

It follows from (4.1) and (4.7) that

$$\frac{R(s)(a * R)(t)x - k(s)(a * R)(t)x}{a * k(s)} = \frac{(a * R(s))(R(t)x - k(t)x)}{a * k(s)} \longrightarrow R(t)x - k(t)x, \tag{4.8}$$

as $s \to 0^+$ for all $x \in X$. Then, for all $x \in X$ and $t \geq 0$, $(a * R)(t)x \in D(B)$ and

$$B(a * R)(t)x = R(t)x - k(t)x. \tag{4.9}$$

Now let $x \in D(B)$. Note that $(a * R)(t)x \in D(B)$ and hence by (4.3) and the commutativity of $R(t)$, we have

$$B(a * R)(t)x = \lim_{s \to 0^+} \frac{R(s)(a * R)(t)x - k(s)(a * R)(t)x}{a * k(s)}$$

$$= \lim_{s \to 0^+} (a * R)(t) \frac{[R(s)x - k(s)x]}{a * k(s)} \tag{4.10}$$

$$= (a * R)(t)Bx.$$

It then follows from (4.9) and (4.10) that for all $x \in D(B)$,

$$R(t)x = k(t)x + B(a * R)(t)x = k(t)x + (a * R)(t)Bx. \tag{4.11}$$

It shows (iii) in Definition 2.1 and hence that $R(t)$ is an $(a, k)$-regularized resolvent family generated by $B$.

We now show closedness. Let $(x_n) \in D(B)$ be a sequence such that $x_n \to x$ and $Ax_n \to y$ as $n \to \infty$. It follows from the second equality in (4.11) that

$$R(t)x = k(t)x + (a * R)(t)y. \tag{4.12}$$

Then by (4.7) we have

$$\frac{R(t)x - k(t)x}{a * k(t)} = \frac{(a * R)(t)y}{a * k(t)} \longrightarrow y, \tag{4.13}$$

as $t \to 0^+$. It shows $x \in D(B)$ and $Bx = y$.

Finally, we show that $D(B)$ is dense in $X$. Let $x \in X$ be given. Then it was proved that $a * R(t)x \in D(B)$ for all $t \geq 0$. Hence, defining

$$x_n = \frac{a * R(1/n)}{a * k(1/n)} x, \quad n \in \mathbb{N}, \tag{4.14}$$

it follows from (4.7) that $x_n \in D(B)$ and $\lim_{n \to \infty} x_n = x$, proving the claim and the theorem.
□

*Remark 4.2.* We notice that the denseness of $D(A)$ is always present under the conditions of the above theorem. In particular, we recover a well-known result in the theories of semigroups and cosine operator functions and, more important, give new results for other theories of strongly continuous functions of operators. An immediate application of this result is that the Laplacian operator with Dirichlet boundary conditions cannot be the generator on $L^p$ spaces for $p \neq 2$, of those classes of $(a, k)$-regularized resolvent families that have $a(t)$ and $k(t)$ satisfying (4.4).

Our next result studies a complementary case of Theorem 4.1. Here we deal with integrated versions of the operator families where, as we known, the density of the domain is not present, in general.

**Theorem 4.3.** *Let $k \in C(\mathbb{R}_+)$ and $a \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$. Let $R(t) \subset \mathcal{B}(X)$ be a commutative strongly continuous family such that $R(0) = k(0)I$ and satisfies* (FE). *Define $D(B)$ and $B$ as in the above theorem. Suppose the following condition:*

$$\lim_{s \to 0+} \frac{(a * (a * k))(s)}{a * k(s)} = 0. \tag{4.15}$$

*Then $R(t)$ is an $(a, k)$-regularized resolvent family with generator $B$.*

*Proof.* Since $R(0) = k(0)$, we have to prove conditions (ii)-(iii) in Definition 2.1. The proof of (ii) is the same as in Theorem 4.1. To show (iii), fix $x \in D(B)$ and note that

$$\left\| \frac{a * R(s)x}{a * k(s)} - x \right\| \leq \frac{1}{|a * k(s)|} \int_0^s |a(s - \mu)| \|(R(\mu)x - k(\mu)x)\| d\mu. \tag{4.16}$$

Since $x \in D(B)$, there exists $\delta > 0$ such that for all $|\mu| \leq \delta$

$$\left\| \frac{R(\mu)x - k(\mu)x}{(a * k)(\mu)} - Bx \right\| < 1. \tag{4.17}$$

Then for all $|\mu| \leq \delta$,

$$\|R(\mu)x - k(\mu)x\| \leq (1 + \|Bx\|)|(a * k)(\mu)| =: C_x |(a * k)(\mu)|. \tag{4.18}$$

It follows from (4.16), (4.18), and the condition (4.15) that

$$\left\| \frac{a * R(s)x}{a * k(s)} - x \right\| \leq \frac{1}{|a * k(s)|} \int_0^s |a(s - \mu)C_x(a * k)(\mu)|d\mu$$

$$= \frac{|a * (a * k)(s)|}{|a * k(s)|} C_x \longrightarrow 0 \quad \text{as } s \longrightarrow 0^+. \tag{4.19}$$

Since $R(t)x - k(t)x \in D(B)$, it follows from (4.1) and (4.19) that

$$\frac{R(s)(a * R)(t)x - k(s)(a * R)(t)x}{a * k(s)} = \frac{(a * R(s))(R(t)x - k(t)x)}{a * k(s)} \longrightarrow R(t)x - k(t)x, \tag{4.20}$$

as $s \to 0^+$. Then for all $x \in D(B)$, we have $(a * R)(t)x \in D(B)$ and

$$B(a * R)(t)x = R(t)x - k(t)x. \tag{4.21}$$

Finally, note that $(a * R)(t)x \in D(B)$ and

$$B(a * R)(t)x = (a * R)(t)Bx. \tag{4.22}$$

Hence

$$R(t)x = k(t)x + B(a * R)(t)x = k(t)x + (a * R)(t)Bx, \tag{4.23}$$

proving the theorem. □

*Example 4.4.* Suppose that $a(t)$ and $k(t)$ are positive kernels, then the condition (4.15) holds. In fact, denote $c(t) = (a * k)(t)$. Then $c(t)$ is positive nondecreasing and therefore

$$(a * c)(t) = \int_0^t a(t - s)c(s)ds \leq \int_0^t a(t - s)c(t)ds = c(t) \int_0^t a(s)ds, \quad t \geq 0. \tag{4.24}$$

Hence

$$\frac{(a * a * k)(t)}{(a * k)(t)} \leq \int_0^t a(s)ds \longrightarrow 0, \tag{4.25}$$

as $t \to 0$.

*Remark 4.5.* In view of Theorems 3.1, 4.1, and 4.3, the Definition-regularized family is equivalent, under certain conditions on the kernels $a(t)$ and $k(t)$, to the functional equation (FE). This fact can be used to define $(a, k)$-regularized families in a local way and avoid the use of Laplace transform in the development of the theory. We note that more general classes of families of bounded operators like, for example, $(a, k)$-regularized-$C$-resolvent families can be understood using (FE) (see, e.g., [30, Definition 1.2]). We left the details to the interested reader.

## 5. Application

In this section, we give one example to show how this class of regularized families defined by functional equations appears in new concrete problems, while other methods do not apply directly.

Consider the following nonlinear third order differential equation:

$$u'''(t) + \alpha u''(t) + c^2 Au(t) + bAu'(t) = f\left(t, u(t), u'(t), u''(t)\right), \tag{5.1}$$

with given initial conditions $u(0)$, $u'(0)$, $u''(0)$, and where $\alpha$, $b$, $c$ are positive real numbers and $f$ is a vector-valued function.

Equation (1.5) has recently attracted the attention of a number of authors because their applications in different fields as, for example, high-intensity ultrasound and vibrations of flexible materials. See [31–33] and references therein. In such cases, usually the operator $A$ is the negative Laplacian and $f(t, u, u_t, u_{tt}) = (K(u_t)^2 + |\nabla u|^2)_t$ for a suitable constant $K > 0$. For example, in high-intensity ultrasound, $u$ is the velocity potential of the acoustic phenomenon described on some bounded $\mathbb{R}^3$ domain. In the abstract case, $A$ is a nonnegative, self-adjoint operator (possibly with compact resolvent) defined on a Hilbert space $H$.

Mathematical understanding of the linearized equation:

$$u'''(t) + \alpha u''(t) + c^2 Au(t) + bAu'(t) = 0 \tag{5.2}$$

is meant as a preliminary critical step for the subsequent analysis of the full nonlinear model (5.1).

The usual operator-theoretic method to solve (5.2) is to rewrite it as a first-order abstract Cauchy problem of the form

$$U'(t) = \mathcal{A}U(t), \quad t \geq 0, \ U(0) = U_0, \tag{5.3}$$

on the Banach space $\mathcal{X} := X \times X \times X$ with the usual norm, and where

$$\mathcal{A} = \begin{pmatrix} 0 & I & 0 \\ 0 & 0 & I \\ -c^2 A & -bA & -\alpha I \end{pmatrix} \tag{5.4}$$

is defined on $D(\mathcal{A}) := D(A) \times D(A) \times X$. It is known that if $A$ is unbounded then, for $b = 0$, the matrix of operators $\mathcal{A}$ cannot be the generator of a $C_0$ semigroup on $\mathcal{X}$ or even in some

subspaces of it (see [31, Theorem 1.1]). It can be seen directly observing the entries of the resolvent operator

$$(\lambda I - \mathcal{A})^{-1} = \begin{pmatrix} (\lambda^2 + \alpha\lambda + bA)H(\lambda) & (\lambda + \alpha)H(\lambda) & H(\lambda) \\ -c^2 AH(\lambda) & (\lambda^2 + \alpha\lambda)H(\lambda) & \lambda H(\lambda) \\ -\lambda c^2 AH(\lambda) & -(\lambda b + c^2)H(\lambda) & \lambda^2 H(\lambda) \end{pmatrix}, \tag{5.5}$$

where $H(\lambda) = (\lambda^3 + \alpha\lambda^2 + b\lambda A + c^2 A)^{-1}$. Indeed, for $b = 0$ we replace the identity:

$$\lambda^3 H(\lambda) + \alpha\lambda^2 H(\lambda) + c^2 AH(\lambda) = I \tag{5.6}$$

and obtain

$$(\lambda I - \mathcal{A})^{-1} = \begin{pmatrix} (\lambda^2 + \alpha\lambda)H(\lambda) & (\lambda + \alpha)H(\lambda) & H(\lambda) \\ \lambda^3 H(\lambda) + \alpha\lambda^2 H(\lambda) - I & (\lambda^2 + \alpha\lambda)H(\lambda) & \lambda H(\lambda) \\ \lambda^4 H(\lambda) + \alpha\lambda^3 H(\lambda) - \lambda I & -(\lambda b + c^2)H(\lambda) & \lambda^2 H(\lambda) \end{pmatrix}, \tag{5.7}$$

and now we observe that the entries $\lambda^3 H(\lambda) + \alpha\lambda^2 H(\lambda) - I$ and $\lambda^4 H(\lambda) + \alpha\lambda^3 H(\lambda) - \lambda I$ cannot be a Laplace transform. Therefore, $\mathcal{A}$ cannot generate a $C_0$ semigroup on $\mathcal{X}$.

However, for $\alpha \neq 0$, we can directly associate to (5.2) an $(a, k)$-regularized family on $X$, with $k(t) = e^{-\alpha t}$ and $a(t) = ((b\alpha - c^2)/\alpha^2)(1 - e^{-\alpha t}) + (c^2/\alpha)t$. Indeed, for such choice of the pair $(a, k)$ we have

$$\widehat{k}(\lambda) = \frac{1}{\lambda + \alpha}, \qquad \widehat{a}(\lambda) = \frac{c^2 + \lambda b}{\lambda^3 + \alpha\lambda^2}. \tag{5.8}$$

Hence

$$\lambda^2 H(\lambda) = \frac{\widehat{k}(\lambda)}{\widehat{a}(\lambda)} \left( \frac{1}{\widehat{a}(\lambda)} - A \right)^{-1}. \tag{5.9}$$

Now take the (formal) Laplace transform of the left-hand side of (5.2) with initial conditions $u(0) = x$, $u'(0) = y$, $u''(0) = z$. We obtain in case $b = 0$

$$\widehat{u}(\lambda) = \left( \lambda^2 + \alpha\lambda \right) H(\lambda)x + (\lambda + \alpha)H(\lambda)y + H(\lambda)z. \tag{5.10}$$

Therefore, in case $b = 0$ we can still have that (5.2) is well posed in the sense that $A$ is the generator of an $(a, k)$-regularized family $\{S(t)\}_{t \geq 0}$ and, in this case, a (mild) solution of the problem can be explicitly given by the formula:

$$u(t) = [S(t) + \alpha(1 * S)(t)]x + [(1 * S)(t) + \alpha(t * S)(t)]y + (t * S)(t)z, \tag{5.11}$$

where $x, y, z \in X$. Finally, note that

$$k(t) = e^{-\alpha t}, \qquad a(t) = c^2 \int_0^t (t-s)k(s)\,ds + b \int_0^t k(s)\,ds \qquad (5.12)$$

are positive kernels and therefore, by Example 4.4, the condition (4.15) in Theorem 4.3 is satisfied. In this way, a method based on a direct approach, defined by an specific functional equation, is now available. Note that in case $b \neq 0$ the matrix operator $\mathcal{A}$ is the generator of a $C_0$ semigroup $\mathcal{T}(t)$ on $\mathcal{X}$, but it can never be compact. It has nothing to do with the corresponding properties of $S(t)$. Therefore maximal regularity results for the corresponding nonhomogeneous version of (5.2) cannot be proved by reduction to a first-order problem. The same remark applies to stability questions; even if $S(t)$ is integrable, the type of $\mathcal{T}(t)$ cannot be negative unless the type of $S(t)$ already is, this implies that it is not possible to obtain sharp integrability results for the solution $u(t)$ of (5.2) by means of an indirect approach.

*Remark 5.1.* The argument given above for the justification of the introduction of $(a, k)$-regularized families for a third-order abstract differential equation is analog to those given in the origins of the theory of cosine families, see for example, Fattorini [34], which has proved along the years to be very efficient to handle directly incomplete second-order abstract Cauchy problems.

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