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Research Article

Multiple-Set Split Feasibility Problems for Asymptotically Strict Pseudocontractions

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In this paper, we introduce an iterative method for solving the multiple-set split feasibility problems for asymptotically strict pseudocontractions in infinite-dimensional Hilbert spaces, and, by using the proposed iterative method, we improve and extend some recent results given by some authors.

1. Introduction

The split feasibility problem (SFP) in finite dimensional spaces was first introduced by Censor and Elfving [1] for modeling inverse problems which arise from phase retrievals and in medical image reconstruction [2]. Recently, it has been found that the SFP can also be used in various disciplines such as image restoration, computer tomograph, and radiation therapy treatment planning [3–5].

The split feasibility problem in an infinite dimensional Hilbert space can be found in [2, 4, 6–8].

Throughout this paper, we always assume that H_1, H_2 are real Hilbert spaces, "$\to$", "$\rightharpoonup$" are denoted by strong and weak convergence, respectively.

The purpose of this paper is to introduce and study the following *multiple-set split feasibility problem* for asymptotically strict pseudocontraction (MSSFP) in the framework of infinite-dimensional Hilbert spaces. Find $x^* \in C$ such that

$$Ax^* \in Q, \tag{1.1}$$

where $A : H_1 \to H_2$ is a bounded linear operator, $\{S_i\}$ and $\{T_i\}$, $i = 1, 2, \ldots, M$, are the families of mappings $S_i : H_1 \to H_1$ and $T_i : H_2 \to H_2$, respectively, $C := \bigcap_{i=1}^{M} F(S_i)$ and $Q := \bigcap_{i=1}^{M} F(T_i)$, where $F(S_i) = \{x_i \in H_1 : S_i x_i = x_i\}$ and $F(T_i) = \{y_i \in H_2 : T_i y_i = y_i\}$ denote the sets of fixed points of S_i and T_i, respectively. In the sequel, we use Γ to denote the set of solutions of the problem (MSSFP), that is,

$$\Gamma = \{x \in C : Ax \in Q\}. \tag{1.2}$$

2. Preliminaries

We first recall some definitions, notations, and conclusions which will be needed in proving our main results.

Let E be a Banach space. A mapping $T : E \to E$ is said to be *demiclosed* at origin if, for any sequence $\{x_n\} \subset E$ with $x_n \rightharpoonup x^*$ and $\|(I - T)x_n\| \to 0$, we have $x^* = Tx^*$. A Banach space E is said to have *Opial's property* if, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x^*$, we have

$$\liminf_{n\to\infty} \|x_n - x^*\| < \liminf_{n\to\infty} \|x_n - y\| \tag{2.1}$$

for all $y \in E$ with $y \neq x^*$.

Remark 2.1. It is well known that each Hilbert space possesses Opial's property.

Definition 2.2. Let H be a real Hilbert space.

(1) A mapping $G : H \to H$ is called *a* $(\gamma, \{k_n\})$*-asymptotically strict pseudocontraction* if there exists a constant $\gamma \in [0, 1)$ and a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ such that

$$\|G^n x - G^n y\|^2 \leq k_n \|x - y\|^2 + \gamma \|(I - G^n)x - (I - G^n)y\|^2, \quad \forall x, y \in H. \tag{2.2}$$

Especially, if $k_n = 1$ for each $n \geq 1$ in (2.2) and there exists $\gamma \in [0, 1)$ such that

$$\|Gx - Gy\|^2 \leq \|x - y\|^2 + \gamma \|(I - G)x - (I - G)y\|^2, \quad \forall x, y \in H, \tag{2.3}$$

then $G : H \to H$ is called a γ*-strict pseudocontraction*.

(2) A mapping $G : H \to H$ is said to be *uniformly L-Lipschitzian* if there exists a constant $L > 0$ such that

$$\|G^n x - G^n y\| \leq L\|x - y\|, \quad \forall x, y \in H, \ n \geq 1. \tag{2.4}$$

(3) A mapping $G : H \to H$ is said to be *semicompact* if, for any bounded sequence $\{x_n\} \subset H$ with $\lim_{n\to\infty}\|x_n - Gx_n\| = 0$, there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that x_{n_i} converges strongly to a point $x^* \in H$.

Now, we give one example of the $(\gamma, \{k_n\})$-asymptotically strict pseudocontraction mapping.

Example 2.3. Let B be the unit ball in a Hilbert space l^2, and define a mapping $T : B \to B$ by

$$T = (x_1, x_2, \ldots) = \left(0, x_1^2, a_2x_2, a_3x_3, \ldots\right), \tag{2.5}$$

where $\{a_i\}$ is a sequence in $(0,1)$ such that $\Pi_{i=2}^{\infty} a_i = 1/2$. It is proved in Goebel and Kirk [9] that

(a) $\|Tx = Ty\| \le 2\|x - y\|$ for all $x, y \in B$,

(b) $\|T^n x - T^n y\| \le 2\Pi_{j=2}^{n} a_j$ for all $n \ge 2$ and $x, y \in B$.

Denote by $k_1^{1/2} = 2$, $k_n^{1/2} = 2\Pi_{j=2}^{n} a_j (n \ge 2)$ and $\gamma \in [0,1)$. Then, we have

$$\begin{gathered}\lim_{n\to\infty} k_n = \lim_{n\to\infty}\left(2\prod_{j=2}^{n} a_j\right)^2 = 1, \\ \left\|T^n x - T^n y\right\|^2 \le k_n\left\|x - y\right\|^2 + \gamma\left\|x - y - (T^n x - T^n y)\right\|^2, \quad \forall n \ge 1,\ x, y \in B,\end{gathered} \tag{2.6}$$

and so the mapping T is a $(\gamma, \{k_n\})$-asymptotically strict pseudocontraction.

Remark 2.4. (1) If we put $\gamma = 0$ in (2.2), then the mapping $G : H \to H$ is asymptotically nonexpansive.

(2) If we put $\gamma = 0$ in (2.3), then the mapping $G : H \to H$ is nonexpansive.

(3) Each $(\gamma, \{k_n\})$-asymptotically strict pseudocontraction and each γ-strictly pseudo-contraction both are demiclosed at origin [10].

Proposition 2.5. *Let $G : H \to H$ be a $(\gamma, \{k_n\})$- asymptotically strict pseudocontraction. If $F(G) \neq \emptyset$, then, for any $q \in F(G)$ and $x \in H$, the following inequalities hold and they are equivalent:*

$$\left\|G^n x - q\right\|^2 \le k_n\left\|x - q\right\|^2 + \gamma\|x - G^n x\|^2, \tag{2.7}$$

$$\langle x - G^n x, x - q\rangle \ge \frac{1-\gamma}{2}\|x - G^n x\|^2 - \frac{k_n - 1}{2}\left\|x - q\right\|^2, \tag{2.8}$$

$$\langle x - G^n x, q - G^n x\rangle \le \frac{1+\gamma}{2}\|x - G^n x\|^2 + \frac{k_n - 1}{2}\left\|x - q\right\|^2. \tag{2.9}$$

Lemma 2.6 (see [11]). *Let $\{a_n\}$, $\{b_n\}$, and $\{\delta_n\}$ be sequences of nonnegative real numbers satisfying*

$$a_{n+1} \le (1 + \delta_n)a_n + b_n, \quad \forall n \ge 1. \tag{2.10}$$

If $\sum_{i=1}^{\infty} \delta_n < \infty$ and $\sum_{i=1}^{\infty} b_n < \infty$, then the limit $\lim_{n\to\infty} a_n$ exists.

3. Multiple-Set Split Feasibility Problem

For solving the multiple-set split feasibility problem (1.1), let us assume that the following conditions are satisfied:

(C1) H_1 and H_2 are two real Hilbert spaces, $A : H_1 \to H_2$ is a bounded linear operator;

(C2) $S_i : H_1 \to H_1$, $i = 1, 2, \ldots, M$, is a uniformly L_i-Lipschitzian and $(\beta_i, \{k_{i,n}\})$-asymptotically strict pseudocontraction, and $T_i : H_2 \to H_2$, $i = 1, 2, \ldots, M$, is a uniformly $\widetilde{L}_i$-Lipschitzian and $(\mu_i, \{\widetilde{k}_{i,n}\})$-asymptotically strict pseudocontraction satisfying the following conditions:

(a) $C := \bigcap_{i=1}^{M} F(S_i) \neq \emptyset$ and $Q := \bigcap_{i=1}^{M} F(T_i) \neq \emptyset$,

(b) $\beta = \max_{1 \le i \le M} \beta_i < 1$ and $\mu = \max_{1 \le i \le M} \mu_i < 1$,

(c) $L := \max_{1 \le i \le M} L_i < \infty$ and $\widetilde{L} := \max_{1 \le i \le M} \widetilde{L}_i < \infty$,

(d) $k_n = \max_{1 \le i \le M} \{k_{i,n}, \widetilde{k}_{i,n}\}$ and $\sum_{n=1}^{\infty} (k_n - 1) < \infty$.

We are now in a position to give the following result.

Theorem 3.1. *Let H_1, H_2, A, $\{S_i\}$, $\{T_i\}$, C, Q, β, μ, L, $\widetilde{L}$, and $\{k_n\}$ be the same as above. Let $\{x_n\}$ be the sequence generated by*

$$\begin{gathered} x_1 \in H_1 \text{ chosen arbitrarily}, \\ x_{n+1} = (1 - \alpha_n) u_n + \alpha_n S_n^n (u_n), \\ u_n = x_n + \gamma A^* (T_n^n - I) A x_n, \quad \forall n \ge 1, \end{gathered} \tag{3.1}$$

where $S_n^n = S_{n(\bmod M)}^n$, $T_n^n = T_{n(\bmod M)}^n$ for all $n \ge 1$, $\{\alpha_n\}$ is a sequence in $[0,1]$, and $\gamma > 0$ is a constant satisfying the following conditions.

(e) *$\alpha_n \in (\delta, 1 - \beta)$ for all $n \ge 1$ and $\gamma \in (0, (1 - \mu)/\|A\|^2)$, where $\delta \in (0, 1 - \beta)$ is a positive constant.*

(1) *If $\Gamma \neq \emptyset$, then the sequence $\{x_n\}$ converges weakly to a point $x^* \in \Gamma$.*

(2) *In addition, if there exists a positive integer j such that S_j is semicompact, then the sequences $\{x_n\}$ and $\{u_n\}$ both converge strongly to a point $x^* \in \Gamma$.*

Proof. (1) The proof is divided into 5 steps as follows.

Step 1. We first prove that, for any $p \in \Gamma$, the limit

$$\lim_{n \to \infty} \|x_n - p\| \tag{3.2}$$

exists. In fact, since $p \in \Gamma$, $p \in C := \bigcap_{i=1}^{M} F(S_i)$, and $Ap \in Q := \bigcap_{i=1}^{M} F(T_i)$. From (3.1) and (2.8), it follows that

$$\begin{aligned}\|x_{n+1} - p\|^2 &= \|u_n - p - \alpha_n(u_n - S_n^n u_n)\|^2 \\ &= \|u_n - p\|^2 - 2\alpha_n\langle u_n - p, u_n - S_n^n u_n\rangle + \alpha_n^2\|u_n - S_n^n u_n\|^2 \\ &\le \|u_n - p\|^2 - \alpha_n\left\{(1-\beta)\|u_n - S_n^n u_n\|^2 - (k_n - 1)\|u_n - p\|^2\right\} \\ &\quad + \alpha_n^2\|u_n - S_n^n u_n\|^2 \\ &= (1 + \alpha_n(k_n - 1))\|u_n - p\|^2 - \alpha_n(1 - \beta - \alpha_n)\|u_n - S_n^n u_n\|^2.\end{aligned} \tag{3.3}$$

On the other hand, since

$$\begin{aligned}\|u_n - p\|^2 &= \|x_n - p + \gamma A^*(T_n^n - I)Ax_n\|^2 \\ &= \|x_n - p\|^2 + \gamma^2\|A^*(T_n^n - I)Ax_n\|^2 + 2\gamma\langle x_n - p, A^*(T_n^n - I)Ax_n\rangle,\end{aligned} \tag{3.4}$$

$$\begin{aligned}\gamma^2\|A^*(T_n^n - I)Ax_n\|^2 &= \gamma^2\langle A^*(T_n^n - I)Ax_n,\ A^*(T_n^n - I)Ax_n\rangle \\ &= \gamma^2\langle AA^*(T_n^n - I)Ax_n,\ (T_n^n - I)Ax_n\rangle \\ &\le \gamma^2\|A\|^2\|T_n^n Ax_n - Ax_n\|^2,\end{aligned} \tag{3.5}$$

$$\begin{aligned}2\gamma\langle x_n - p, A^*(T_n^n - I)Ax_n\rangle &= 2\gamma\langle Ax_n - Ap, (T_n^n - I)Ax_n\rangle \\ &= 2\gamma\langle (Ax_n - Ap) + (T_n^n - I)Ax_n - (T_n^n - I)Ax_n, (T_n^n - I)Ax_n\rangle \\ &= 2\gamma\left\{\langle T_n^n Ax_n - Ap, T_n^n Ax_n - Ax_n\rangle - \|(T_n^n - I)Ax_n\|^2\right\}.\end{aligned} \tag{3.6}$$

Further, letting $x = Ax_n$, $G^n = T_n^n$, $q = Ap$, $\gamma = \mu$ in (2.9) and noting $Ap \in F(T_n)$, it follows that

$$\begin{aligned}\langle T_n^n Ax_n - Ap, T_n^n Ax_n - Ax_n\rangle &\le \frac{1+\mu}{2}\|(T_n^n - I)Ax_n\|^2 + \frac{k_n - 1}{2}\|Ax_n - Ap\|^2 \\ &\le \frac{1+\mu}{2}\|(T_n^n - I)Ax_n\|^2 + \frac{(k_n - 1)\|A\|^2}{2}\|x_n - p\|^2.\end{aligned} \tag{3.7}$$

Substituting (3.7) into (3.6) and simplifying it, we have

$$2\gamma\langle x_n - p, A^*(T_n^n - I)Ax_n\rangle \le \gamma(\mu - 1)\|(T_n^n - I)Ax_n\|^2 + (k_n - 1)\gamma\|A\|^2\|x_n - p\|^2. \tag{3.8}$$

Substituting (3.5) and (3.8) into (3.4) and simplifying it, we have

$$\begin{aligned}\|u_n - p\|^2 &\le \|x_n - p\|^2 + \gamma^2\|A\|^2\|T_n^n Ax_n - Ax_n\|^2 \\ &\quad + \gamma(\mu - 1)\|(T_n^n - I)Ax_n\|^2 + (k_n - 1)\gamma\|A\|^2\|x_n - p\|^2\end{aligned}$$

$$\begin{aligned}&= \|x_n - p\|^2 - \gamma\left(1-\mu-\gamma\|A\|^2\right)\|T_n^n A x_n - A x_n\|^2 \\ &\quad + (k_n-1)\gamma\|A\|^2\|x_n-p\|^2.\end{aligned} \tag{3.9}$$

Again, substituting (3.9) into (3.3) and simplifying it, we have

$$\begin{aligned}\|x_{n+1}-p\|^2 \le{}& (1+\alpha_n(k_n-1)) \\ &\times\left\{\|x_n-p\|^2 - \gamma\left(1-\mu-\gamma\|A\|^2\right)\|T_n^n A x_n - A x_n\|^2 + (k_n-1)\gamma\|A\|^2\|x_n-p\|^2\right\} \\ &- \alpha_n(1-\beta-\alpha_n)\|u_n - S_n^n u_n\|^2 \\ \le{}& (1+\alpha_n(k_n-1))\|x_n-p\|^2 - \gamma\left(1-\mu-\gamma\|A\|^2\right)\|T_n^n A x_n - A x_n\|^2 \\ &+ (1+\alpha_n(k_n-1))(k_n-1)\gamma\|A\|^2\|x_n-p\|^2 - \alpha_n(1-\beta-\alpha_n)\|u_n - S_n^n u_n\|^2.\end{aligned} \tag{3.10}$$

By the condition (e), we have

$$\begin{aligned}\|x_{n+1}-p\|^2 &\le (1+\alpha_n(k_n-1))\|x_n-p\|^2 + (1+\alpha_n(k_n-1))(k_n-1)\gamma\|A\|^2\|x_n-p\|^2 \\ &\le (1+K(k_n-1))\|x_n-p\|^2,\end{aligned} \tag{3.11}$$

where

$$K = \sup_{n\ge1}\left(\alpha_n + (1+\alpha_n(k_n-1))\gamma\|A\|^2\right) < \infty. \tag{3.12}$$

By the condition (d), $\sum_{n=1}(k_n-1) < \infty$; hence, from Lemma 2.6, we know that the following limit exists:

$$\lim_{n\to\infty}\|x_n-p\|. \tag{3.13}$$

Step 2. We will now prove that, for each $p \in \Gamma$, the limit

$$\lim_{n\to\infty}\|u_n-p\| \tag{3.14}$$

exists. In fact, from (3.10) and (3.13), it follows that

$$\begin{aligned}&\gamma\left(1-\mu-\gamma\|A\|^2\right)\|(T_n^n - I)Ax_n\|^2 + \alpha_n(1-\beta-\alpha_n)\|u_n - S_n^n u_n\|^2 \\ &\quad\le \|x_n-p\|^2 - \|x_{n+1}-p\|^2 + K(k_n-1)\|x_n-p\|^2 \longrightarrow 0 \quad (n\longrightarrow\infty).\end{aligned} \tag{3.15}$$

This together with the condition (e) implies that

$$\lim_{n\to\infty}\|u_n - S_n^n u_n\| = 0, \tag{3.16}$$

$$\lim_{n\to\infty}\|(T_n^n - I)Ax_n\| = 0. \tag{3.17}$$

Therefore, it follows from (3.4), (3.13), and (3.17) that the limit $\lim_{n\to\infty}\|u_n - p\|$ exists.

Step 3. Now, we prove that

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0, \qquad \lim_{n\to\infty}\|u_{n+1} - u_n\| = 0. \tag{3.18}$$

In fact, it follows from (3.1) that

$$\begin{aligned}
\|x_{n+1} - x_n\| &= \|(1-\alpha_n)u_n + \alpha_n S_n^n(u_n) - x_n\| \\
&= \left\|(1-\alpha_n)\left(x_n + \gamma A^*(T_n^n - I)Ax_n\right) + \alpha_n S_n^n(u_n) - x_n\right\| \\
&= \left\|(1-\alpha_n)\gamma A^*(T_n^n - I)Ax_n + \alpha_n(S_n^n(u_n) - x_n)\right\| \\
&= \left\|(1-\alpha_n)\gamma A^*(T_n^n - I)Ax_n + \alpha_n(S_n^n(u_n) - u_n) + \alpha_n(u_n - x_n)\right\| \\
&= \left\|(1-\alpha_n)\gamma A^*(T_n^n - I)Ax_n + \alpha_n(S_n^n(u_n) - u_n) + \alpha_n\gamma \mathrm{A}^*(T_n^n - I)Ax_n\right\| \\
&= \left\|\gamma A^*(T_n^n - I)Ax_n + \alpha_n(S_n^n(u_n) - u_n)\right\|.
\end{aligned} \tag{3.19}$$

In view of (3.16) and (3.17), we have

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{3.20}$$

Similarly, it follows from (3.1), (3.17), and (3.20) that

$$\begin{aligned}
\|u_{n+1} - u_n\| &= \left\|x_{n+1} + \gamma A^*\left(T_{n+1}^{n+1} - I\right)Ax_{n+1} - \left(x_n + \gamma A^*(T_n^n - I)Ax_n\right)\right\| \\
&\le \|x_{n+1} - x_n\| + \gamma\left\|A^*\left(T_{n+1}^{n+1} - I\right)Ax_{n+1}\right\| \\
&\quad + \gamma\|A^*(T_n^n - I)Ax_n\| \longrightarrow 0 \quad (n \longrightarrow \infty).
\end{aligned} \tag{3.21}$$

The conclusion (3.18) is proved.

Step 4. Next, we prove that, for each $j = 1, 2, \ldots, M$,

$$\left\|u_{iM+j} - S_j u_{iM+j}\right\| \longrightarrow 0, \quad \left\|Ax_{iM+j} - T_j Ax_{iM+j}\right\| \longrightarrow 0 \quad (i \longrightarrow \infty). \tag{3.22}$$

In fact, from (3.16), it follows that

$$\zeta_{iM+j} := \left\|u_{iM+j} - S_j^{iM+j} u_{iM+j}\right\| \longrightarrow 0 \quad (i \longrightarrow \infty). \tag{3.23}$$

Since S_j is uniformly L_j-Lipschitzian continuous, it follows from (3.18) and (3.23) that

$$\begin{aligned}
&\|u_{iM+j} - S_j u_{iM+j}\| \\
&\quad\le \left\|u_{iM+j} - S_j^{iM+j} u_{iM+j}\right\| + \left\|S_j^{iM+j} u_{iM+j} - S_j u_{iM+j}\right\| \\
&\quad\le \zeta_{iM+j} + L_j \left\|S_j^{iM+j-1} u_{iM+j} - u_{iM+j}\right\| \\
&\quad\le \zeta_{iM+j} + L_j \left\{\left\|S_j^{iM+j-1} u_{iM+j} - S_j^{iM+j-1} u_{iM+j-1}\right\| + \left\|S_j^{iM+j-1} u_{iM+j-1} - u_{iM+j}\right\|\right\} \\
&\quad\le \zeta_{iM+j} + L_j^2 \|u_{iM+j} - u_{iM+j-1}\| \\
&\qquad + L_j \left\|S_j^{iM+j-1} u_{iM+j-1} - u_{iM+j-1} + u_{iM+j-1} - u_{iM+j}\right\| \\
&\quad\le \zeta_{iM+j} + L_j(1 + L_j)\|u_{iM+j} - u_{iM+j-1}\| + L_j \zeta_{iM+j-1} \longrightarrow 0 \quad (i \longrightarrow \infty).
\end{aligned} \tag{3.24}$$

Similarly, for each $j = 1, 2, \ldots, M$, it follows from (3.17) that

$$\xi_{iM+j} := \left\|Ax_{iM+j} - T_j^{iM+j} Ax_{iM+j}\right\| \longrightarrow 0 \quad (i \longrightarrow \infty). \tag{3.25}$$

Since T_j is uniformly $\widetilde{L}_j$-Lipschitzian continuous, by the same way as above, from (3.18) and (3.25), we can also prove that

$$\|Ax_{iM+j} - T_j Ax_{iM+j}\| \longrightarrow 0 \quad (i \longrightarrow \infty). \tag{3.26}$$

Step 5. Finally, we prove that $x_n \rightharpoonup x^*$ and $u_n \rightharpoonup x^*$, which is a solution of the problem (MSSFP). In fact, since $\{u_n\}$ is bounded, there exists a subsequence $\{u_{n_i}\} \subset \{u_n\}$ such that $u_{n_i} \rightharpoonup x^* \in H_1$. Hence, for any positive integer $j = 1, 2, \ldots, M$, there exists a subsequence $\{n_i(j)\} \subset \{n_i\}$ with $n_i(j) (\bmod M) = j$ such that $u_{n_i(j)} \rightharpoonup x^*$. Again, from (3.22), it follows that

$$\|u_{n_i(j)} - S_j u_{n_i(j)}\| \longrightarrow 0 \quad (n_i(j) \longrightarrow \infty). \tag{3.27}$$

Since S_j is demiclosed at zero (see Remark 2.4), it follows that $x^* \in F(S_j)$. By the arbitrariness of $j = 1, 2, \ldots, M$, we have $x^* \in C := \bigcap_{j=1}^M F(S_j)$.

Moreover, it follows from (3.1) and (3.17) that

$$x_{n_i} = u_{n_i} - \gamma A^*(T_{n_i}^{n_i} - I)Ax_{n_i} \rightharpoonup x^*. \tag{3.28}$$

Since A is a linear bounded operator, it follows that $Ax_{n_i} \rightharpoonup Ax^*$. For any positive integer $k = 1, 2, \ldots, M$, there exists a subsequence $\{n_i(k)\} \subset \{n_i\}$ with $n_i(k) (\bmod M) = k$ such that $Ax_{n_i(k)} \rightharpoonup Ax^*$. In view of (3.22), we have

$$\|Ax_{n_i(k)} - T_k Ax_{n_i(k)}\| \longrightarrow 0 \quad (n_i(k) \longrightarrow \infty). \tag{3.29}$$

Since T_k is demiclosed at zero, we have $Ax^* \in F(T_k)$. By the arbitrariness of $k = 1, 2, \ldots, M$, it follows that $Ax^* \in Q := \bigcap_{k=1}^M F(T_k)$. This together with $x^* \in C$ shows that $x^* \in \Gamma$, that is, x^* is a solution to the problem (MSSFP).

Now, we prove that $x_n \rightharpoonup x^*$ and $u_n \rightharpoonup x^*$. In fact, assume that there exists another subsequence $\{u_{n_j}\} \subset \{u_n\}$ such that $u_{n_j} \rightharpoonup y^* \in \Gamma$ with $y^* \neq x^*$. Consequently, by virtue of (3.2) and Opial's property of Hilbert space, we have

$$\begin{aligned}
\liminf_{n_i \to \infty} \|u_{n_i} - x^*\| &< \liminf_{n_i \to \infty} \|u_{n_i} - y^*\| \\
&= \lim_{n \to \infty} \|u_n - y^*\| \\
&= \lim_{n_j \to \infty} \left\|u_{n_j} - y^*\right\| \\
&< \liminf_{n_j \to \infty} \left\|u_{n_j} - x^*\right\| \\
&= \lim_{n \to \infty} \|u_n - x^*\| \\
&= \liminf_{n_i \to \infty} \|u_{n_i} - x^*\|.
\end{aligned} \tag{3.30}$$

This is a contradiction. Therefore, $u_n \rightharpoonup x^*$. By using (3.1) and (3.17), we have

$$x_n = u_n - \gamma A^*(T_n^n - I)Ax_n \rightharpoonup x^*. \tag{3.31}$$

Therefore, the conclusion (I) follows.

(2) Without loss of generality, we can assume that S_1 is semicompact. It follows from (3.27) that

$$\left\|u_{n_i(1)} - S_1 u_{n_i(1)}\right\| \longrightarrow 0 \quad (n_i(1) \longrightarrow \infty). \tag{3.32}$$

Therefore, there exists a subsequence of $\{u_{n_i(1)}\}$ (for the sake of convenience, we still denote it by $\{u_{n_i(1)}\}$) such that $u_{n_i(1)} \to u^* \in H$. Since $u_{n_i(1)} \rightharpoonup x^*$, $x^* = u^*$ and so $u_{n_i(1)} \to x^* \in \Gamma$. By virtue of (3.2), we know that

$$\lim_{n \to \infty} \|u_n - x^*\| = 0, \qquad \lim_{n \to \infty} \|x_n - x^*\| = 0, \tag{3.33}$$

that is, $\{u_n\}$ and $\{x_n\}$ both converge strongly to the point $x^* \in \Gamma$. This completes the proof. □

If we put $\gamma = 0$ in Theorem 3.1, we can get the following.

Corollary 3.2. *Let H, C, L and $\{k_n\}$ be the same as above and $\{S_i\}$ a family of asymptotically nonexpansive mappings. Let $\{x_n\}$ be the sequence generated by*

$$\begin{gathered}
x_1 \in H_1 \text{ chosen arbitrarily}, \\
x_{n+1} = (1 - \alpha_n)x_n + \alpha_n S_n^n(x_n), \quad \forall n \geq 1,
\end{gathered} \tag{3.34}$$

where $S_n^n = S_{n(\bmod M)}^n$ for all $n \geq 1$ and $\{\alpha_n\}$ is a sequence in $[0,1]$ satisfying the following conditions.

(e) *$\alpha_n \in (\delta, 1 - \beta)$ for all $n \geq 1$, where $\delta \in (0, 1 - \beta)$ is a positive constant.*

(1) *If* $\Gamma \neq \emptyset$, *then the sequence* $\{x_n\}$ *converges weakly to a point* $x^* \in \Gamma$.

(2) *In addition, if there exists a positive integer* j *such that* S_j *is semicompact, then the sequence* $\{x_n\}$ *converges strongly to a point* $x^* \in \Gamma$.

The following theorem can be obtained from Theorem 3.1 immediately.

Theorem 3.3. *Let* H_1 *and* H_2 *be two real Hilbert spaces,* $A : H_1 \to H_2$ *a bounded linear operator,* $S_i : H_1 \to H_1$, $i = 1, 2, \ldots, M$, *a uniformly* L_i*-Lipschitzian and* β_i*-strict pseudocontraction, and* $T_i : H_2 \to H_2$, $i = 1, 2 \ldots, M$, *a uniformly* $\widetilde{L}_i$*-Lipschitzian and* μ_i*-strict pseudocontraction satisfying the following conditions:*

(a) $C := \bigcap_{i=1}^{M} F(S_i) \neq \emptyset$ *and* $Q := \bigcap_{i=1}^{M} F(T_i) \neq \emptyset$,

(b) $\beta = \max_{1 \leq i \leq M} \beta_i < 1$ *and* $\mu = \sup_{1 \leq i \leq M} \mu_i < 1$.

Let $\{x_n\}$ *be the sequence generated by*

$$
\begin{gathered}
x_1 \in H_1 \text{ chosen arbitrarily}, \\
x_{n+1} = (1 - \alpha_n) u_n + \alpha_n S_n(u_n), \\
u_n = x_n + \gamma A^*(T_n - I) A x_n, \quad \forall n \geq 1,
\end{gathered}
\tag{3.35}
$$

where $S_n = S_{n(\bmod M)}$, $T_n = T_{n(\bmod M)}$, $\{\alpha_n\}$ *is a sequence in* $[0, 1]$, *and* $0 < \gamma < 1$ *is a constant. If* $\Gamma \neq \emptyset$ *and the following condition is satisfied:*

(c) $\alpha_n \in (\delta, 1 - \beta)$ *for all* $n \geq 1$ *and* $\gamma \in (0, (1 - \mu)/\|A\|^2)$, *where* $\delta \in (0, 1 - \beta)$ *is a constant,*

then the sequence $\{x_n\}$ *converges weakly to a point* $x^* \in \Gamma$. *In addition, if there exists a positive integer* j *such that* S_j *is semicompact, then the sequences* $\{x_n\}$ *and* $\{u_n\}$ *both converge strongly to the point* x^*.

Proof. By the same way as given in the proof of Theorem 3.1 and using the case of strict pseudocontraction with the sequence $\{k_n = 1\}$, we can prove that, for each $p \in \Gamma$, the limits $\lim_{n \to \infty} \|x_n - p\|$ and $\lim_{n \to \infty} \|u_n - p\|$ exist,

$$
\begin{gathered}
\|u_n - S_n u_n\| \to 0, \qquad \|A x_n - T_n A x_n\| \to 0, \qquad \|u_n - u_{n+1}\| \to 0, \qquad \|x_n - x_{n+1}\| \to 0, \\
x_n \rightharpoonup x^*, \qquad u_n \rightharpoonup x^* \in \Gamma.
\end{gathered}
\tag{3.36}
$$

In addition, if there exists a positive integer j such that S_j is semicompact, we can also prove that $\{x_n\}$ and $\{u_n\}$ both converge strongly to the point x^*. This completes the proof. □

If you put $S_i = T_i$ or $T_i = I$ (: the identity mapping) for each $i = 1, 2 \ldots, M$ in Theorem 3.3, then we have the following.

Corollary 3.4. *Let* H *be a real Hilbert space and* $S_i : H \to H$, $i = 1, 2, \ldots, M$, *a uniformly* L_i*-Lipschitzian and* β_i*-strict pseudocontraction satisfying the following conditions:*

(a) $C := \bigcap_{i=1}^{M} F(S_i) \neq \emptyset$,

(b) $\beta = \max_{1 \leq i \leq M} \beta_i < 1$.

Let $\{x_n\}$ *be the sequence generated by*

$$\begin{gathered} x_1 \in H_1 \text{ chosen arbitrarily}, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n S_n(x_n), \quad \forall n \geq 1, \end{gathered} \tag{3.37}$$

where $S_n = S_{n(\bmod M)}$ *and* $\{\alpha_n\}$ *is a sequence in* $[0,1]$. *If* $\Gamma \neq \emptyset$ *and the following condition is satisfied:*

(c) $\alpha_n \in (\delta, 1-\beta)$ *for all* $n \geq 1$, *where* $\delta \in (0, 1-\beta)$ *is a constant,*

then the sequence $\{x_n\}$ *converges weakly to a point* $x^* \in \Gamma$. *In addition, if there exists a positive integer* j *such that* S_j *is semicompact, then the sequences* $\{x_n\}$ *converges strongly to the point* x^*.

Remark 3.5. Theorems 3.1 and 3.3 improve and extend the corresponding results of Censor et al. [1, 4, 5], Byrne [2], Yang [7], Moudafi [12], Xu [13], Censor and Segal [14], Masad and Reich [15], and others in the following aspects:

(1) for the framework of spaces, we extend the space from finite dimension Hilbert space to infinite dimension Hilbert space;

(2) for the mappings, we extend the mappings from nonexpansive mappings, quasi-nonexpansive mapping or demicontractive mappings to finite families of asymptotically strictly pseudocontractions;

(3) for the algorithms, we propose some new hybrid iterative algorithms which are different from ones given in [1, 2, 4, 5, 7, 14, 15]. And, under suitable conditions, some weak and strong convergences for the algorithms are proved.

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