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Research Article

The Approximate Solution of Fractional Fredholm Integrodifferential Equations by Variational Iteration and Homotopy Perturbation Methods

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Variational iteration method and homotopy perturbation method are used to solve the fractional Fredholm integrodifferential equations with constant coefficients. The obtained results indicate that the method is efficient and also accurate.

1. Introduction

The topic of fractional calculus has attracted many scientists because of its several applications in many areas, such as physics, chemistry, and engineering. For a detail survey with collections of applications in various fields, see, for example, [1–3].

Further, the fractional derivatives technique has been employed for solving linear fractional differential equations including the fractional integrodifferential equations; in this way, much of the efforts is devoted to searching for methods that generate accurate results, see [4, 5]. In this work, we present two different methods, namely, homotopy perturbation method and variational iteration method [6], for solving a fractional Fredholm integro-differential equations with constant coefficients. There is a vast literature, and we only mention the works of Liao which treat a homotopy method in [7, 8].

For the nonlinear equations with derivatives of integer order, many methods are used to derive approximation solution [9–14]. However, for the fractional differential equation, there are some limited approaches, such as Laplace transform method [3], the Fourier transform method [15], the iteration method [16], and the operational calculus method [17].

Recently, there has been considerable researches in fractional differential equations due to their numerous applications in the area of physics and engineering [18], such as phenomena in electromagnetic theory, acoustics, electrochemistry, and material science [3, 16, 18, 19]. Similarly, there is also growing interest in the integrodifferential equations which are combination of differential and Fredholm-Volterra equations. In this work, we study these kind of equations that have the fractional order usually difficult to solve analytically, thus a numerical method is required, for example, the successive approximations, Adomian decomposition, Chebyshev and Taylor collocation, Haar Wavelet, Tau and Walsh series methods.

This note is devoted to the application of variational iteration method (VIM) and homotopy perturbation method (HPM) for solving fractional Fredholm integrodifferential equations with constant coefficients:

$$\sum_{k=0}^{\infty} P_k D_*^{\alpha} y(t) = g(x) + \lambda \int_0^a H(x,t) y(t) dt, \quad a \le x,\ t \le b, \tag{1.1}$$

under the initial-boundary conditions

$$D_*^{\alpha} y(a) = y(0), \tag{1.2}$$

$$D_*^{\alpha} y(0) = y'(a), \tag{1.3}$$

where a is constant and $1 < \alpha < 2$ and D_*^{α} is the fractional derivative operator given in the Caputo sense. For the physical understanding of the fractional integrodifferential equations, see [20]. Further, we also note that fractional integrodifferential equations were associated with a certain class of phase angles and suggested a new way for understanding of Riemann's conjecture, see [21].

Outline of this paper is as follows. Section 2 contains preliminaries on fractional calculus. Section 3 is a short review of the homotopy method and Section 4 variational iteration method. Sections 5 and 6 are devoted to VIM and HPM analysis, respectively. Concluding remarks with suggestions for future work are listed in Section 7.

2. Description of the Fractional Calculus

In the following, we give the necessary notations and basic definitions and properties of fractional calculus theory; for more details, see [3, 13, 16, 22].

Definition 2.1. A real function $f(x),\ x > 0$, is said to be in the space C_{α}, $\alpha \in R$ if there exists a real number $(p > \alpha)$, such that $f(x) = x^p f_1(x)$, where $f_1(x) = C([0,\infty))$. Clearly, $C_{\alpha} \subset C_{\beta}$, if $\beta \le \alpha$.

Definition 2.2. A function $f(x),\ x > 0$, is said to be in space C_{α}^{m}, $m \in N$, if $f^{(m)} \in C_{\alpha}$.

Definition 2.3. The Riemann-Liouville fractional integral of order $\mu \geq 0$ for a function $f \in C_\alpha,\ (\alpha \geq 1)$ is defined as

$$I^\mu f(t) = \frac{1}{\Gamma(\mu)} \int_0^t (t-\tau)^{\mu-1} f(\tau) d\tau, \quad \mu > 0,\ t > 0, \tag{2.1}$$

in particular $I^0 f(t) = f(t)$.

Definition 2.4. The Caputo fractional derivative of $f \in C_{-1}^m$, $m \in N$, is defined as

$$D_c^\mu f(t) = \begin{cases} \left[I^{m-\mu} f^{(m)}(t)\right], & m-1 < \mu \leq m,\ m \in N, \\ \dfrac{d^m}{dt^m} f(t), & \mu = m. \end{cases} \tag{2.2}$$

Note that

(i) $I^\mu t^\gamma = (\Gamma(\gamma+1)/\Gamma(\gamma+\mu+1)) t^{\gamma+\mu}$, $\mu > 0$, $\gamma > -1$, $t > 0$,

(ii) $I^\mu\ {}^C D_{0^+}^\mu f(t) = f(t) - \sum_{k=0}^{m-1} f^{(k)}(0_+)(t^k/k!)$, $m-1 < \mu \leq m$, $m \in N$,

(iii) ${}^C D_{0^+}^\mu f(t) = D^\mu(f(t) - \sum_{k=0}^{m-1} f^{(k)}(0_+)(t^k/k!))$, $m-1 < \mu \leq m$, $m \in N$,

(iv)

$$D^\beta I^\alpha f(t) = \begin{cases} I^{\alpha-\beta} f(t), & \text{if } \alpha > \beta, \\ f(t), & \text{if } \alpha = \beta, \\ D^{\beta-\alpha} f(t), & \text{if } \alpha < \beta, \end{cases} \tag{2.3}$$

(v) ${}^C D_{0^+}^\beta D^m f(t) = D^{\beta+m} f(t)$, $m = 0, 1, 2, \ldots, n-1 < \beta < n$.

Definition 2.5 (see [3, 16]). The Riemann-Liouville fractional integral operator of order $\rho \geq 0$ for a function $f \in C_\mu,\ (\mu \geq -1)$ is defined as

$$K^\rho f(x) = \frac{1}{\Gamma(\rho)} \int_0^x (x-t)^{\rho-1} f(t) dt, \quad \rho > 0,\ x > 0, \qquad K^0 f(x) = f(x), \tag{2.4}$$

having the properties

$$\begin{gathered} K^\rho K^\beta f(x) = K^{\rho+\beta} f(x), \\ K^\rho x^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\rho+\beta+1)} x^{\alpha+\beta}. \end{gathered} \tag{2.5}$$

According to the Caputo's derivatives, we obtain the following expressions:

$$
\begin{gathered}
{}^{C}D^{\mu}C = 0, \quad C = \text{constant}, \\
{}^{C}D^{\mu}t^{\beta} = \begin{cases} \dfrac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}t^{\beta-\alpha}, & \beta > \alpha - 1, \\ 0, & \beta \leq \alpha - 1. \end{cases}
\end{gathered} \tag{2.6}
$$

Lemma 2.6. *If $m-1<\alpha \leq m$, $m \in N$, $f \in C_{\mu}^{m}$, $\mu \geq -1$, then the following two properties hold:*

$$
(1)\ D^{\alpha}K^{\alpha}f(t) = f(t), \qquad (2)\ (D^{\alpha}K^{\alpha})f(t) = f(t) - \sum_{k=0}^{m-1} f^{(k)}(0^{+})\frac{t^{k}}{k!}. \tag{2.7}
$$

In fact, Kılıçman and Zhour introduced the Kronecker convolution product and expanded to the Riemann-Liouville fractional integrals of matrices by using the Block Pulse operational matrix as follows:

$$
\frac{1}{\Gamma(\alpha)}\int_{0}^{t}(t-t_1)^{\alpha-1}\phi_m(t_1)dt_1 \simeq F_{\alpha}\phi_m(t), \tag{2.8}
$$

where

$$
F_{\alpha} = \left(\frac{b}{m}\right)^{\alpha}\frac{1}{\Gamma(\alpha+2)}\begin{bmatrix} 1 & \xi_2 & \xi_3 & \cdots & \xi_m \\ 0 & 1 & \xi_2 & \cdots & \xi_{m-1} \\ 0 & 0 & 1 & \cdots & \xi_{m-2} \\ 0 & 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \tag{2.9}
$$

see [23].

In our work, we consider Caputo fractional derivatives and apply the homotopy method in order to derive an approximate solutions of the fractional integrodifferential equations.

3. Homotopy Method

To illustrate the basic ideas of this method, we consider the following nonlinear differential equation:

$$
A(u) + f(\mathbf{r}) = 0, \quad \mathbf{r} \in \boldsymbol{\Omega}, \tag{3.1}
$$

with boundary conditions

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0, \quad \mathbf{r} \in \boldsymbol{\Gamma}, \tag{3.2}$$

where A is a general differential operator, B is a boundary operator, $f(\mathbf{r})$ is a known analytical function, and Γ is the boundary of the domain Ω, see [24].

In general, the operator A can be divided into two parts L and N, where L is linear, while N is nonlinear. Equation (3.1), therefore, can be rewritten as follows:

$$L(u) + N(u) - f(\mathbf{r}) = 0. \tag{3.3}$$

By using the homotopy technique that was proposed by Liao in [7, 8], we construct a homotopy of (3.1) $v(\mathbf{r}, p) : \Omega \times [0, 1] \to \mathcal{R}$ which satisfies

$$\mathscr{H}(v, p) = (1 - p)[L(v) - L(u_0)] + p[A(v) + f(\mathbf{r})] = 0, \quad p \in [0, 1], \ \mathbf{r} \in \boldsymbol{\Omega}, \tag{3.4}$$

or

$$\mathscr{H}(v, p) = L(v) - L(u_0) + pL(u_0) + p[N(v) - f(\mathbf{r})] = 0, \tag{3.5}$$

where $p \in [0, 1]$ is an embedding parameter and u_0 is an initial approximation which satisfies the boundary conditions. By using (3.4) and (3.5), we have

$$\begin{aligned} \mathscr{H}(v, 0) &= L(v) - L(u_0) = 0, \\ \mathscr{H}(v, 1) &= A(v) - f(\mathbf{r}) = 0. \end{aligned} \tag{3.6}$$

The changing in the process of p from zero to unity is just that of $v(\mathbf{r}, p)$ from u_0 to $u(\mathbf{r})$. In a topology, this is also

known deformation, further $L(v) - L(u_0)$ and $A(v) - f(\mathbf{r})$ are homotopic.

Now, assume that the solution of (3.4) and (3.5) can be expressed as

$$v = v + pv_1 + p^2 v_2 + \cdots. \tag{3.7}$$

The approximate solution of (3.1), therefore, can be readily obtained:

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + \cdots. \tag{3.8}$$

The convergence of the series of (3.8) has been proved in the [25, 26].

4. The Variational Iteration Method

To illustrate the basic concepts of the VIM, we consider the following differential equation:

$$Lu + Nu = g(x), \tag{4.1}$$

where L is a linear operator, N is a nonlinear operator, and $g(x)$ is an nonhomogenous term; for more details, see [19].

According to the VIM, one construct a correction functional as follows:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda\left[Lu_n(s) + N\widetilde{u}_n(s) - g(s)\right]ds, \tag{4.2}$$

where λ is a general Lagrange multiplier, which can be identified optimally via the variational theory, and the subscript n denotes the order of approximation, $\widetilde{u}_u$ is considered variation [6, 27], that is, $\delta\widetilde{u}_u = 0$.

5. Analysis of VIM

To solve the fractional integrodifferential equation (1.1) by using the variational iteration method, with boundary conditions (1.2), one can construct the following correction functional:

$$y_{k+1}(x) = y_k(x) + \int_0^t \mu\sum_{k=0}^{\infty} P_k D_*^{\alpha} y(s)ds - \mu\widetilde{g}_k(x) - \lambda\int_a^b \mu H(x,s)\widetilde{y}_k(s)ds, \tag{5.1}$$

where μ is a general Lagrange multiplier and $\widetilde{g}_k(x)$ and $\widetilde{y}_k(x)$ are considered as restricted variations, that is, $\delta\widetilde{g}_k(x) = 0$ and $\delta\widetilde{y}_k(x) = 0$.

Making the above correction functional stationary, the following conditions can be obtained:

$$\delta y_{k+1}(x) = \delta y_k(x) + \int_0^t \left[\sum_{k=0}^{\infty} P_k \mu(s)\delta D_*^{\alpha} y(s) - \delta\widetilde{g}_k(x) - \lambda\int_a^b H(x,s)\mu(s)\delta\widetilde{y}_k(s)ds\right], \tag{5.2}$$

having the boundary conditions as follows:

$$1 - \mu'(s)\big|_{x=s} = 0, \qquad \mu(s)\big|_{x=s} = 1. \tag{5.3}$$

The Lagrange multipliers can be identified as follows:

$$\mu(s) = \frac{1}{2}(s - x). \tag{5.4}$$

Substituting the value of μ from (5.4) into correction functional of (5.1) leads to the following iteration formulae:

$$\begin{aligned} y_{k+1}(x) &= y_k(x) + \frac{\mu}{2\Gamma(\alpha-1)} \int_0^x (x-s)^{\alpha-2}(s-x) \\ &\quad \times \left[\int_0^t \sum_{k=0}^{\infty} P_k D_*^{\alpha} y(s)ds - \tilde{g}_k(x) - \lambda \int_a^b H(x,s)\tilde{y}_k(s) \right] ds, \\ y_{k+1}(x) &= y_k(x) - \frac{\mu(\alpha-1)}{2\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} \\ &\quad \times \left[\int_0^t \sum_{k=0}^{\infty} P_k D_*^{\alpha} y(s)ds - \tilde{g}_k(x) - \lambda \int_a^b H(x,s)\tilde{y}_k(s) \right] ds, \end{aligned} \tag{5.5}$$

by applying formulae (2.4), we get

$$y_{k+1}(x) = y_k(x) - \frac{(\alpha-1)K^{\alpha}}{2\Gamma(\alpha)} \left[\int_0^t \mu \sum_{k=0}^{\infty} P_k D_*^{\alpha} y(v)dv - \mu\tilde{g}_k(x) - \lambda \int_a^b \mu H(x,v)\tilde{y}_k(v) \right] dv. \tag{5.6}$$

The initial approximation can be chosen in the following manner which satisfies initial boundary conditions (1.2)-(1.3):

$$y_0(x) = v_0 + v_1 x, \quad \text{where } v_1 = D_*^{\alpha} y(0),\ v_0 = D_*^{\alpha} y(a). \tag{5.7}$$

We can obtain the following first-order approximation by substitution of (5.7) into (5.6)

$$y_1(x) = y_0(x) - \frac{(\alpha-1)K^{\alpha}}{2\Gamma(\alpha)} \left[\int_0^t \mu \sum_{k=0}^{N} P_k D_*^{\alpha} y(v)dv - \mu\tilde{g}_0(x) - \lambda \int_a^b \mu H(x,v)\tilde{y}_k(v) \right] dv. \tag{5.8}$$

Substituting the constant value of v_0 and v_1 in the expression (5.8) results in the approximation solution of (1.1)–(1.3).

6. Analysis of HPM

This section illustrates the basic of HPM for fractional Fredholm integrodifferential equations with constant coefficients (1.1) with initial-boundary conditions (1.2).

In view of HPM [25, 26], construct the following homotopy for (1.1):

$$\sum_{k=0}^{\infty} P_k D_*^{\alpha} y(x) = p \left[\sum_{k=0}^{\infty} P_k D_*^{\alpha} y(x) + \left(g(t) - \lambda \int_a^b H(x,t)y(x)dx \right) \right]. \tag{6.1}$$

In view of basic assumption of HPM, solution of (1.1) can be expressed as a power series in p:

$$y(x) = D_*^\alpha y_0(x) + pD_*^\alpha y_1(x) + p^2 D_*^\alpha y_2(x) + p^3 D_*^\alpha y_3(x) + \cdots. \tag{6.2}$$

If we put $p \to 1$ in (6.2), we get the approximate solution of (1.1):

$$y(x) = D_*^\alpha y_0(x) + D_*^\alpha y_1(x) + D_*^\alpha y_2(x) + D_*^\alpha y_3(x) + \cdots. \tag{6.3}$$

The convergence of series (6.3) has been proved in [28].

Now, we substitute (6.2) into (6.1); then equating the terms with identical power of p, we obtain the following series of linear equations:

$$\begin{aligned}
p^0 &: \sum_{k=0}^{\infty} P_k D_*^\alpha y_0(t) = 0,\\
p^1 &: \sum_{k=0}^{\infty} P_k D_*^\alpha y_1(t) = \sum_{k=0}^{\infty} P_k D_*^\alpha y_0(t) - \lambda \int_a^b H(x,t) y_0(x)\,dx,\\
p^2 &: \sum_{k=0}^{\infty} P_k D_*^\alpha y_2(t) = \sum_{k=0}^{\infty} P_k D_*^\alpha y_1(t) + g(x) - \lambda \int_a^b H(x,t) y_1(x)\,dx,\\
p^3 &: \sum_{k=0}^{\infty} P_k D_*^\alpha y_3(t) = \sum_{k=0}^{\infty} P_k D_*^\alpha y_2(t) - \lambda \int_a^b H(x,t) y_2(x)\,dx,\\
p^4 &: \sum_{k=0}^{\infty} P_k D_*^\alpha y_4(t) = \sum_{k=0}^{\infty} P_k D_*^\alpha y_3(t) - \lambda \int_a^b H(x,t) y_3(x)\,dx,
\end{aligned} \tag{6.4}$$

with the initial-boundary conditions

$$D_*^\alpha y(a) = y(0), \quad D_*^\alpha y(0) = y'(a). \tag{6.5}$$

We can also take the initial approximation in the following manner which satisfies initial-boundary conditions (1.2)-(1.3):

$$y_0(x) = v_0 + v_1 x, \quad \text{where } v_1 = D_*^\alpha y(0), \quad v_0 = D_*^\alpha y(a). \tag{6.6}$$

Note that (6.4) can be solved by applying the operator K^β, which is the inverse of operator D^α we approximate the series solution of HPM by the following n-term truncated series [29]:

$$\chi_n(x) = D_*^\alpha y_0(x) + D_*^\alpha y_1(x) + D_*^\alpha y_2(x) + D_*^\alpha y_3(x) + \cdots + D_*^\alpha y_{n-1}(x), \tag{6.7}$$

which results, the approximate solutions of (1.2)-(1.3). For further analysis, the variational iteration method, see [30] and the algorithm by the homotopy perturbation method, see [31].

7. Conclusion

The proposed methods are used to solve fractional Fredholm integrodifferential equations with constant coefficients. Comparison of the results obtained by the present method with that obtained by other method reveals that the present method is very effective and convenient. Unfortunately, the disadvantage of the second method is that the embedding parameter p is quite casual, and often enough the approximations obtained by this method will not be uniform. So, in our future work we expect to study this kind of equation by using a combination of the variational iteration method and the homotopy perturbation method which has shown reliable results in supplying analytical approximation that converges very rapidly. However, we note that the papers [32, 33] suggest alternative ways for similar problems.

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