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Research Article

Nonself-Adjoint Second-Order Difference Operators in Limit-Circle Cases

Bilender P. Allahverdiev

Department of Mathematics, Suleyman Demirel University, 32260 Isparta, Turkey

Correspondence should be addressed to Bilender P. Allahverdiev, bilender@fef.sdu.edu.tr

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We consider the maximal dissipative second-order difference (or discrete Sturm-Liouville) operators acting in the Hilbert space $\ell_w^2(\mathbb{Z})$ ($\mathbb{Z} := \{0, \pm 1, \pm 2, \ldots\}$), that is, the extensions of a minimal symmetric operator with defect index $(2, 2)$ (in the Weyl-Hamburger limit-circle cases at $\pm\infty$). We investigate two classes of maximal dissipative operators with separated boundary conditions, called "dissipative at $-\infty$" and "dissipative at ∞." In each case, we construct a self-adjoint dilation of the maximal dissipative operator and its incoming and outgoing spectral representations, which make it possible to determine the scattering matrix of the dilation. We also establish a functional model of the maximal dissipative operator and determine its characteristic function through the Titchmarsh-Weyl function of the self-adjoint operator. We prove the completeness of the system of eigenvectors and associated vectors of the maximal dissipative operators.

1. Introduction

Method of contour integration of the resolvent is one of the general methods of the spectral analysis of nonself-adjoint (dissipative) operators. It is related to a fine estimate of the resolvent on expanding contours which separates the spectrum. The feasibility of this method is restricted to weak perturbations of self-adjoint operators and operators having sparse discrete spectrum. Since there are no asymptotics of the solutions for a wide class of singular problems, this method cannot be applied properly.

It is well known [1–4] that the theory of dilations with application of functional models gives an adequate approach to the spectral theory of dissipative (contractive) operators. In this theory, a key role is played by the characteristic function, which carries the full information on the spectral properties of the dissipative operator. Thus, the dissipative operator becomes the model in the incoming spectral representation of the dilation. The completeness problem of the system of eigenvectors and associated vectors is solved by the factorization of the characteristic function. The computation of the characteristic functions of dissipative operators is preceded by the construction and investigation of the self-adjoint dilation and the corresponding scattering problem, in which the characteristic function is realized as the scattering matrix [5]. The adequacy of this approach for dissipative Jacobi operators and second-order difference (or discrete Sturm-Liouville) operators has been indicated in [6–9].

In this paper, we consider the maximal dissipative second-order difference (or discrete Sturm-Liouville) operators acting in the Hilbert space $\ell_w^2(\mathbb{Z})$, that is the extensions of a minimal symmetric operator with defect index $(2,2)$ (in the Weyl-Hamburger limit-circle cases at $\pm\infty$). We investigate two classes of maximal dissipative operators with separated boundary conditions, called "dissipative at $-\infty$" and "dissipative at ∞." In each of these cases we construct a self-adjoint dilation of the maximal dissipative operator and its incoming and outgoing spectral representations, which make it possible to determine the scattering matrix of the dilation according to the scheme of Lax and Phillips [5]. By means of the incoming spectral representation, we establish a functional model of the maximal dissipative operator and construct its characteristic function using the Titchmarsh-Weyl function of the self-adjoint operator. Finally, on the basis of the results obtained for the characteristic functions, we prove the theorems on completeness of the system of eigenvectors and associated vectors (or root vectors) of the maximal dissipative second-order difference operators.

2. Preliminaries

Let $y = \{y_n\}$ be a sequence of complex numbers y_n $(n \in \mathbb{Z})$ and $\ell_1 y$ denote the sequence with components $(\ell_1 y)_n$. We consider the following second-order difference (or discrete Sturm-Liouville) equation on the whole line:

$$(\ell_1 y)_n := -a_{n-1}y_{n-1} + b_n y_n - a_n y_{n+1} = \lambda w_n y_n, \tag{2.1}$$

where λ is a complex spectral parameter, $w_n > 0$, $a_n \neq 0$, and $a_n, b_n \in \mathbb{R} := (-\infty, \infty)$, $n \in \mathbb{Z}$.

If we let $p_n = a_n$, $q_n = b_n - a_n - a_{n-1}$, and $\Delta x_n = x_{n+1} - x_n$, (2.1) can be written in Sturm-Liouville form as follows:

$$-\Delta(p_{n-1}\Delta y_{n-1}) + q_n y_n = \lambda w_n y_n, \quad n \in \mathbb{Z}. \tag{2.2}$$

For arbitrary sequences $y = \{y_n\}$ and $z = \{z_n\}$, $n \in \mathbb{Z}$, we denote by $[y,z]$ the sequence with components $[y,z]_n$ defined as:

$$[y,z]_n := a_n(y_n \overline{z}_{n+1} - y_{n+1}\overline{z}_n), \quad n \in \mathbb{Z}. \tag{2.3}$$

Let $m, n \in \mathbb{Z}$ with $n < m$. Then we have the Green's formula:

$$\sum_{j=n}^{m}\left[(\ell_1 y)_j \overline{z}_j - y_j(\ell_1 \overline{z})_j\right] = [y,z]_m - [y,z]_{n-1}. \tag{2.4}$$

For any sequence $y = \{y_n\}$, let ℓy denote the sequence with components $(\ell y)_n$ given by $(\ell y)_n = (1/w_n)(\ell_1 y)_n$, $n \in \mathbb{Z}$. We denote by $\ell_w^2(\mathbb{Z})$ $(w := \{w_n\}, n \in \mathbb{Z})$ the Hilbert space of all complex sequences $y = \{y_n\}$, $n \in \mathbb{Z}$ such that $\sum_{n=-\infty}^{\infty} w_n |y_n|^2 < \infty$, with the inner product $(y, z) = \sum_{n=-\infty}^{\infty} w_n y_n \overline{z}_n$. Next, we denote by D the set of all vectors $y \in \ell_w^2(\mathbb{Z})$ such that $\ell y \in \ell_w^2(\mathbb{Z})$. We define a *maximal operator* L on D by setting $Ly = \ell y$.

It follows from Green's formula (2.4) that the limits $[y, z]_\infty = \lim_{n\to\infty}[y, z]_n$ and $[y, z]_{-\infty} = \lim_{n\to-\infty}[y, z]_n$ exist and are finite for arbitrary vectors $y, z \in D$. Therefore, taking the limit as $n \to -\infty$ and $m \to \infty$ in (2.4), for all $y, z \in D$, we have

$$(Ly, z) - (y, Lz) = [y, z]_\infty - [y, z]_{-\infty}. \tag{2.5}$$

Denote by L_0 the closure of the symmetric operator L_0' defined by $L_0' y = Ly$ on the linear set D_0' of finite sequences (i.e., vectors having only finitely many nonzero components) $y = \{y_n\}$ $(n \in \mathbb{Z})$. The *minimal operator* L_0 is symmetric and $L_0^* = L$. The computation of the defect index of L_0 can be reduced to the computation of the defect index for the half-line case. In fact, $\ell_w^2(\mathbb{Z})$ is the orthogonal sum of the space $\ell_w^2(\mathbb{N}_-)$, $(\mathbb{N}_- = \{-1, -2, -3, \ldots\})$ and $\ell_w^2(\mathbb{N}_0)$ $(\mathbb{N}_0 = \{0, 1, 2, \ldots\})$ which are imbedded in the natural way in $\ell_w^2(\mathbb{Z})$. Denote by $L_0^-(L_-)$ and $L_0^+(L_+)$ the minimal (maximal) operators generated by ℓ_- and ℓ_+ in the spaces $\ell_w^2(\mathbb{N}_-)$ and $\ell_w^2(\mathbb{N}_0)$, respectively, and $D_0^{\mp}(D_{\mp})$ is a domain of $L_0^{\mp}(L_{\mp})$, where $(\ell_{\mp} y)_n := (\ell y)_n$, $n \in \mathbb{Z} \setminus \{-1, 0\}$, $(\ell_- y)_{-1} := (1/w_{-1})(-a_{-2} y_{-2} + b_{-1} y_{-1})$, $(\ell_+ y)_0 := (1/w_0)(b_0 y_0 - a_0 y_1)$. Then it is easy to see that the equality $\operatorname{def} L_0 = \operatorname{def} L_0^- + \operatorname{def} L_0^+$ is satisfied for the defect number $\operatorname{def} L_0 := \dim\{(L_0 - \lambda I)D(L_0)\}^\perp$, $\operatorname{Im}\lambda \neq 0$, of L_0. This shows that the defect index of L_0 has the form (k, k), where $k = 0, 1$ or 2. For defect index $(0, 0)$ the operator L_0 is self-adjoint, that is, $L_0^* = L_0 = L$.

Assume that the symmetric operator L_0 has defect index $(2, 2)$. There are several sufficient conditions that guarantee Weyl-Hamburger limit-circle cases at $\pm\infty$ (i.e., the operator L_0 has defect index $(2, 2)$, see [10–17]). The domain of L_0 consists of precisely those vectors $y \in D$ satisfying the condition

$$[y, z]_\infty - [y, z]_{-\infty} = 0, \quad \forall z \in D. \tag{2.6}$$

Denote by $P^{(1)}(\lambda) = \{P_n^{(1)}(\lambda)\}$ and $P^{(2)}(\lambda) = \{P_n^{(2)}(\lambda)\}$, $n \in \mathbb{Z}$ the solutions of (2.1) satisfying the initial conditions:

$$\begin{gathered} P_{-1}^{(1)}(\lambda) = 0, \qquad P_0^{(1)}(\lambda) = 1, \\ P_{-1}^{(2)}(\lambda) = -\frac{1}{a_{-1}}, \qquad P_0^{(2)}(\lambda) = 0. \end{gathered} \tag{2.7}$$

The *Wronskian* of the two solutions $y = \{y_n\}$ and $z = \{z_n\}$, $n \in \mathbb{N}$ of (2.1) is defined as $W_n(y, z) := a_n(y_n z_{n+1} - y_{n+1} z_n)$, so that $W_n(y, z) = [y, \overline{z}]_n$, $n \in \mathbb{Z}$. The Wronskian of the two solutions of (2.1) is independent of n, and the two solutions of this equation are linearly independent if and only if their Wronskian is nonzero. It follows from the conditions (2.7) and the constancy of the Wronskian that $W_n(P^{(1)}, P^{(2)}) = 1$, $n \in \mathbb{Z}$. Consequently, $P^{(1)}(\lambda)$ and $P^{(2)}(\lambda)$ form a fundamental system of solutions of (2.1), and $P^{(1)}(\lambda), P^{(2)}(\lambda) \in \ell_w^2(\mathbb{Z})$ for all $\lambda \in \mathbb{C}$. The theory of difference equations can be seen in [18, 19].

Let $u = P^{(1)}(0)$ and $v = P^{(2)}(0)$. Since the vectors $u = \{u_n\}$ and $v = \{v_n\}$ $(n \in \mathbb{Z})$ are real valued and $[u, v]_n = 1$ $(n \in \mathbb{Z})$, the following assertion can be verified easily using (2.3).

Lemma 2.1. *For arbitrary vectors* $y = \{y_n\} \in D$ *and* $z = \{z_n\} \in D$*, one has the equality:*

$$[y, z]_n = [y, u]_n [\overline{z}, v]_n - [y, v]_n [\overline{z}, u]_n, \quad (n \in \mathbb{Z} \cup \{-\infty, \infty\}). \tag{2.8}$$

The domain D_0 of the operator L_0 consists of precisely those vectors $y \in D$ satisfying the boundary conditions:

$$[y, u]_{-\infty} = [y, v]_{-\infty} = [y, u]_{\infty} = [y, v]_{\infty} = 0. \tag{2.9}$$

Let us consider the following linear maps from D into $\mathbb{C}^2$

$$\Gamma_1 y = \begin{pmatrix} [y, v]_{-\infty} \\ [y, u]_{\infty} \end{pmatrix}, \qquad \Gamma_2 y = \begin{pmatrix} [y, u]_{-\infty} \\ [y, v]_{\infty} \end{pmatrix}, \quad y \in D. \tag{2.10}$$

Then we have the following result (see [8]).

Theorem 2.2. *For any contraction* K *in* $\mathbb{C}^2$ *the restriction of the operator* L *to the set of vectors* $y \in D$ *satisfying the boundary conditions*

$$(K - I)\Gamma_1 y + i(K + I)\Gamma_2 y = 0, \tag{2.11}$$

or

$$(K - I)\Gamma_1 y - i(K + I)\Gamma_2 y = 0 \tag{2.12}$$

is, respectively, a maximal dissipative or a accretive extension of the operator L_0*. Conversely, every maximal dissipative (accretive) extension of* L_0 *is the restriction of* L *to the set of vectors* $y \in D$ *satisfying* (2.11) (2.12)*, and the contraction* K *is uniquely determined by the extension. These conditions give a self-adjoint extension if and only if* K *is unitary. In the latter case* (2.11) *and* (2.12) *are equivalent to the condition* $(\cos S)\Gamma_1 y - (\sin S)\Gamma_2 y = 0$*, where* S *is a self-adjoint (Hermitian matrix) operator in* $\mathbb{C}^2$*. The general form of dissipative and accretive extensions of the operator* L_0 *is given by the conditions*

$$\begin{aligned} K(\Gamma_1 y + i\Gamma_2 y) &= \Gamma_1 y - i\Gamma_2 y, \quad \Gamma_1 y + i\Gamma_2 y \in D(K), \\ K(\Gamma_1 y - i\Gamma_2 y) &= \Gamma_1 y + i\Gamma_2 y, \quad \Gamma_1 y - i\Gamma_2 y \in D(K), \end{aligned} \tag{2.13}$$

respectively, where K *is a linear operator in* $\mathbb{C}^2$ *with* $\|Kf\| \le \|f\|$, $f \in D\ (K) \subseteq \mathbb{C}^2$*. The general form of symmetric extensions is given by the formulae* (2.13)*, where* K *is an isometric operator.*

In particular, if K *is a diagonal matrix, the boundary conditions*

$$[y, v]_{-\infty} - h_1 [y, u]_{-\infty} = 0, \tag{2.14}$$

$$[y, u]_{\infty} - h_2 [y, v]_{\infty} = 0, \tag{2.15}$$

with $\operatorname{Im} h_1 \geq 0$ *or* $h_1 = \infty$, *and* $\operatorname{Im} h_2 \geq 0$ *or* $h_2 = \infty$ ($\operatorname{Im} h_1 \leq 0$ *or* $h_1 = \infty$, *and* $\operatorname{Im} h_2 \leq 0$ *or* $h_2 = \infty$) *describe all the maximal dissipative (maximal accretive) extensions of* L_0 *with separated boundary conditions. The self-adjoint extensions of* L_0 *are obtained precisely when* $\operatorname{Im} h_1 = 0$ *or* $h_1 = \infty$, *and* $\operatorname{Im} h_2 = 0$ *or* $h_2 = \infty$. *Here for* $h_1 = \infty (h_2 = \infty)$ *condition* (2.14) (2.15) *should be replaced by* $[y,u]_{-\infty} = 0$ ($[y,v]_{\infty} = 0$).

In what follows, we will study the dissipative operators $L^{\mp}_{h_1h_2}$ generated by ℓ and the boundary conditions (2.14) and (2.15) of two types: "dissipative at $-\infty$," that is, when either $\operatorname{Im} h_1 > 0$ and $\operatorname{Im} h_2 = 0$ *or* $h_2 = \infty$; "dissipative at ∞," when $\operatorname{Im} h_1 = 0$ *or* $h_1 = \infty$ and $\operatorname{Im} h_2 > 0$.

3. Self-Adjoint Dilations of the Maximal Dissipative Operators

In order to construct a self-adjoint dilation of the maximal dissipative operator $L^{-}_{h_1h_2}$ in the case of "dissipative at $-\infty$" (i.e., $\operatorname{Im} h_1 > 0$ and $\operatorname{Im} h_2 = 0$ or $h_2 = \infty$), we associate with $H := \ell^2_w(\mathbb{Z})$ the "incoming" and "outgoing" channels $D_- := L^2(-\infty,0)$ and $D_+ := L^2(0,\infty)$, we form the orthogonal sum $\mathcal{H} = D_- \oplus H \oplus D_+$ and we call it the *main Hilbert space of the dilation*. In the space $\mathcal{H}$, we consider the operator $\mathcal{L}^{-}_{h_1h_2}$ generated by the expression

$$\mathcal{L}\langle \varphi_-, y, \varphi_+ \rangle = \left\langle i\frac{d\varphi_-}{d\xi}, \ell y, i\frac{d\varphi_+}{d\varsigma} \right\rangle \tag{3.1}$$

on the set $D(\mathcal{L}^{-}_{h_1h_2})$ of vectors $\langle \varphi_-, y, \varphi_+ \rangle$ satisfying the conditions $\varphi_- \in W^1_2(-\infty,0)$, $\varphi_+ \in W^1_2(0,\infty)$, $y \in D$ and

$$\begin{gathered} [y,v]_{-\infty} - h_1[y,u]_{-\infty} = \alpha\varphi_-(0), \qquad [y,v]_{-\infty} - \overline{h}_1[y,u]_{-\infty} = \alpha\varphi_+(0), \\ [y,u]_{\infty} - h_2[y,v]_{\infty} = 0, \end{gathered} \tag{3.2}$$

where W^1_2 denotes the Sobolev space and $\alpha^2 := 2\operatorname{Im} h_1$, $\alpha > 0$.

Theorem 3.1. *The operator* $\mathcal{L}^{-}_{h_1h_2}$ *is self-adjoint in* $\mathcal{H}$ *and it is a self-adjoint dilation of the maximal dissipative operator* $L^{-}_{h_1h_2}$.

Proof. We assume that $f, g \in D(\mathcal{L}^{-}_{h_1h_2})$ with $f = \langle \varphi_-, y, \varphi_+ \rangle$ and $g = \langle \psi_-, z, \psi_+ \rangle$. Then using integration by parts and (3.1), we obtain

$$\begin{aligned} \left(\mathcal{L}^{-}_{h_1h_2} f, g\right)_{\mathcal{H}} &= \int_0^{\infty} i\varphi'_-\overline{\psi}_- d\xi + (Ly,z)_H + \int_0^{\infty} i\varphi'_+\overline{\psi}_+ d\xi \\ &= i\varphi_-(0)\overline{\psi}_-(0) - i\varphi_+(0)\overline{\psi}_+(0) + [y,z]_{\infty} - [y,z]_{-\infty} + \left(f, \mathcal{L}^{-}_{h_1h_2} g\right)_{\mathcal{H}}. \end{aligned} \tag{3.3}$$

If we use the boundary conditions (3.2) for the components of the vectors f, g and Lemma 2.1, we see that $i\varphi_-(0)\overline{\psi}_-(0) - i\varphi_+(0)\overline{\psi}_+(0) + [y,z]_{\infty} - [y,z]_{-\infty} = 0$. Thus, $\mathcal{L}^{-}_{h_1h_2}$ is symmetric.

Therefore, to prove that $\mathcal{L}^-_{h_1h_2}$ is self-adjoint, it is sufficient to show that $(\mathcal{L}^-_{h_1h_2})^* \subseteq \mathcal{L}^-_{h_1h_2}$. Let us take $g = \langle \psi_-, z, \psi_+ \rangle \in D((\mathcal{L}^-_{h_1h_2})^*)$ and let $(\mathcal{L}^-_{h_1h_2})^* g = g^* = \langle \psi^*_-, z^*, \psi^*_+ \rangle \in \mathcal{H}$ so that

$$(\mathcal{L}^-_{h_1h_2} f, g)_{\mathcal{H}} = (f, g^*)_{\mathcal{H}}, \quad \forall f \in D\left(\mathcal{L}^-_{h_1h_2}\right). \tag{3.4}$$

If we choose the components of $f \in D(\mathcal{L}^-_{h_1h_2})$ properly in (3.4), it becomes easy to show that $\psi_- \in W^1_2(-\infty, 0)$, $\psi_+ \in W^1_2(0, \infty)$, $z \in D$, and $g^* = \mathcal{L}g$, where the operator $\mathcal{L}$ is given by (3.1). As a result, (3.4) takes the form $(\mathcal{L}f, g)_{\mathcal{H}} = (f, \mathcal{L}g)_{\mathcal{H}}$, for all $f \in D(\mathcal{L}^-_{h_1h_2})$. Hence, in the bilinear form $(\mathcal{L}f, g)_{\mathcal{H}}$, the sum of the integral terms must be equal to zero:

$$i\varphi_-(0)\overline{\psi}_-(0) - i\varphi_+(0)\overline{\psi}_+(0) + [y, z]_{\infty} - [y, z]_{-\infty} = 0, \tag{3.5}$$

for all $f = \langle \varphi_-, y, \varphi_+ \rangle \in D(\mathcal{L}^-_{h_1h_2})$. In addition, if we solve the boundary conditions (3.2) for $[y, u]_{-\infty}$ and $[y, v]_{-\infty}$, we get

$$[y, u]_{-\infty} = -\frac{i}{\alpha}(\varphi_+(0) - \varphi_-(0)), \qquad [y, v]_{-\infty} = \alpha\varphi_-(0) - \frac{ih_1}{\alpha}(\varphi_+(0) - \varphi_-(0)). \tag{3.6}$$

It follows from Lemma 2.1 and (3.6) that (3.5) is equivalent to the following equality:

$$\begin{aligned} i\varphi_-(0)\overline{\psi}_-(0) - i\varphi_+(0)\overline{\psi}_+(0) &= [y, z]_{-\infty} - [y, z]_{\infty} \\ &= -\frac{i}{\alpha}(\varphi_+(0) - \varphi_-(0))[\overline{z}, v]_{-\infty} \\ &\quad - \alpha\left[\varphi_-(0) - \frac{ih_1}{\alpha^2}(\varphi_+(0) - \varphi_-(0))\right][\overline{z}, u]_{-\infty} \\ &\quad - [y, u]_{\infty}[\overline{z}, v]_{\infty} + [y, v]_{\infty}[\overline{z}, u]_{\infty} \\ &= -\frac{i}{\alpha}(\varphi_+(0) - \varphi_-(0))[\overline{z}, v]_{-\infty} \\ &\quad - \alpha\left[\varphi_-(0) - \frac{ih_1}{\alpha^2}(\varphi_+(0) - \varphi_-(0))\right][\overline{z}, u]_{-\infty} \\ &\quad + ([\overline{z}, u]_{\infty} - h_2[\overline{z}, v]_{\infty})[y, v]_{\infty}. \end{aligned} \tag{3.7}$$

Note that the values $\varphi_\pm(0)$ can be any complex numbers. Therefore, when we compare the coefficients of $\varphi_\pm(0)$ on the left and right of the last equality we see that the vector $g = \langle \psi_-, z, \psi_+ \rangle$ satisfies the boundary conditions $[z, v]_{-\infty} - h_1[z, u]_{-\infty} = \alpha\psi_-(0)$, $[z, v]_{-\infty} - h_1[z, u]_{-\infty} = \alpha\psi_+(0)$, $[z, u]_{\infty} - h_2[z, v]_{\infty} = 0$. Consequently, we obtain $(\mathcal{L}^-_{h_1h_2})^* \subseteq \mathcal{L}^-_{h_1h_2}$, and hence $\mathcal{L}^-_{h_1h_2} = (\mathcal{L}^-_{h_1h_2})^*$.

The self-adjoint operator $\mathcal{L}^-_{h_1h_2}$ generates in $\mathcal{H}$ a unitary group $U^-_t = \exp[i\mathcal{L}^-_{h_1h_2}t]$, $t \in \mathbb{R}$. Let $P : \mathcal{H} \to H$ and $P_1 : H \to \mathcal{H}$ denote the mappings acting according to the formulas $P : \langle \varphi_-, y, \varphi_+ \rangle \to y$ and $P_1 : y \to \langle 0, y, 0 \rangle$. Let $Z^-_t = PU^-_t P_1$ $t \geq 0$. The family $\{Z^-_t\}$, $t \geq 0$, of operators is a strongly continuous semigroup of completely nonunitary contractions on H. (We recall that the linear bounded operator A acting in the Hilbert space H is called *completely nonunitary* if invariant subspace $M \subseteq H$ ($M \neq \{0\}$ of operator A whose restriction to M is unitary, does not exist). Let us denote by $A_{h_1h_2}$ the generator of this semigroup: $A_{h_1h_2}y = \lim_{t\to+0}(it)^{-1}(Z_t^- y - y)$. The domain of $A_{h_1h_2}$ consists of all the vectors for which the limit exists. $A_{h_1h_2}$ is a maximal dissipative operator. The operator $\mathcal{L}_{h_1h_2}^-$ is called the *self-adjoint dilation* of $A_{h_1h_2}$ [1–4]. We show that $A_{h_1h_2} = L_{h_1h_2}^-$, and thus $\mathcal{L}_{h_1h_2}^-$ is a self-adjoint dilation of $L_{h_1h_2}^-$. To do this, we first verify the equality [1–4]:

$$P\left(\mathcal{L}_{h_1h_2}^- - \lambda I\right)^{-1} P_1 y = \left(L_{h_1h_2}^- - \lambda I\right)^{-1} y, \quad y \in H, \ \operatorname{Im}\lambda < 0. \tag{3.8}$$

Denote $(\mathcal{L}_{h_1h_2}^- - \lambda I)^{-1} P_1 y = g = \langle \psi_-, z, \psi_+ \rangle$. Then $(\mathcal{L}_{h_1h_2}^- - \lambda I) g = P_1 y$, and hence $Lz - \lambda z = y$, $\psi_-(\xi) = \psi_-(0)e^{-i\lambda\xi}$ and $\psi_+(\varsigma) = \psi_+(0)e^{-i\lambda\varsigma}$. Since $g \in D(\mathcal{L}_{h_1h_2}^-)$, and hence, $\psi_- \in L^2(-\infty, 0)$; it follows that $\psi_-(0) = 0$, and, consequently, z satisfies the boundary conditions $[z,v]_{-\infty} - h_1[z,u]_{-\infty} = 0$, $[z,u]_{\infty} - h_2[z,v]_{\infty} = 0$. Therefore, $z \in D(L_{h_1h_2}^-)$ and since a point λ with $\operatorname{Im}\lambda < 0$ cannot be an eigenvalue of a dissipative operator, it follows that $z = (L_{h_1h_2}^- - \lambda I)^{-1} y$. Note that $\psi_+(0)$ is obtained from the formula $\psi_+(0) = \alpha^{-1}([z,v]_{-\infty} - \overline{h}_1[z,u]_{-\infty})$. Then

$$\left(\mathcal{L}_{h_1h_2}^- - \lambda I\right)^{-1} P_1 y = \left\langle 0,\ \left(L_{h_1h_2}^- - \lambda I\right)^{-1} y,\ \alpha^{-1}\left([z,v]_{-\infty} - \overline{h}_1[z,u]_{-\infty}\right)e^{-i\lambda\varsigma} \right\rangle, \tag{3.9}$$

for $y \in H$ and $\operatorname{Im}\lambda < 0$. By applying P, one can obtain (3.8).

Now, it is not difficult to show that $A_h = L_h^-$. In fact, it follows from (3.8) that

$$\begin{aligned}\left(L_{h_1h_2}^- - \lambda I\right)^{-1} &= P\left(\mathcal{L}_{h_1h_2}^- - \lambda I\right)^{-1} P_1 = -iP\int_0^{\infty} U_t^- e^{-i\lambda t}\, dt P_1 \\ &= -i\int_0^{\infty} Z_t^- e^{-i\lambda t}\, dt = (A_{h_1h_2} - \lambda I)^{-1}, \quad \operatorname{Im}\lambda < 0,\end{aligned} \tag{3.10}$$

and thus $L_{h_1h_2}^- = A_{h_1h_2}$. Theorem 3.1. is proved. □

In order to construct a self-adjoint dilation of the maximal dissipative operator $L_{h_1h_2}^+$ in the case "dissipative at ∞" (i.e., $\operatorname{Im} h_1 = 0$ or $h_1 = \infty$ and $\operatorname{Im} h_2 > 0$) in $\mathcal{H}$, we consider the operator $\mathcal{L}_{h_1h_2}^+$ generated by the expression (3.1) on the set $D(\mathcal{L}_{h_1h_2}^+)$ of vectors $\langle \varphi_-, y, \varphi_+ \rangle$ satisfying the conditions $\varphi_- \in W_2^1(-\infty, 0)$, $\varphi_+ \in W_2^1(0,\infty)$, $y \in D$ and

$$\begin{aligned}&[y,v]_{-\infty} - h_1[y,u]_{-\infty} = 0, \qquad [y,u]_{\infty} - h_2[y,v]_{\infty} = \alpha\varphi_-(0), \\ &[y,u]_{\infty} - \overline{h}_2[y,v]_{\infty} = \alpha\varphi_+(0),\end{aligned} \tag{3.11}$$

where $\alpha^2 := 2\operatorname{Im} h_2$, $\alpha > 0$.

The proof of the next theorem is similar to that of Theorem 3.1.

Theorem 3.2. *The operator $\mathcal{L}_{h_1h_2}^+$ is self-adjoint in $\mathcal{H}$ and it is a self-adjoint dilation on the maximal dissipative operator $L_{h_1h_2}^+$.*

4. Scattering Theory of the Dilations and Functional Models of the Maximal Dissipative Operators

The unitary group $U_t^{\pm} = \exp[i\mathcal{L}_{h_1h_2}^{\pm} t]$ $(t \in \mathbb{R})$ has a crucial property which enables us to apply the Lax-Phillips scheme [5]. In other words, it has incoming and outgoing subspaces $D_- = \langle L^2(-\infty,0),0,0\rangle$ and $D_+ = \langle 0,0,L^2(0,\infty)\rangle$ satisfying the following properties:

(1) $U_t^{\pm}D_- \subset D_-$, $t \le 0$ and $U_t^{\pm}D_+ \subset D_+$, $t \ge 0$;

(2) $\bigcap_{t\le 0} U_t^{\pm}D_- = \bigcap_{t\ge 0} U_t^{\pm}D_+ = \{0\}$;

(3) $\overline{\bigcup_{t\ge 0} U_t^{\pm}D_-} = \overline{\bigcup_{t\le 0} U_t^{\pm}D_+} = \mathcal{H}$;

(4) $D_- \perp D_+$.

Property (4) is obvious. To verify property (1) for D_+ (the proof for D_- is similar), we set $R_\lambda^{\pm} = (\mathcal{L}_{h_1h_2}^{\pm} - \lambda I)^{-1}$, for all λ with $\operatorname{Im}\lambda < 0$. Then, for any $f = \langle 0,0,\varphi_+\rangle \in D_+$, we have

$$R_\lambda^{\pm} f = \left\langle 0,0,-ie^{-i\lambda\varsigma}\int_0^{\varsigma} e^{-i\lambda s}\varphi_+(s)ds\right\rangle. \tag{4.1}$$

Hence, we find $R_\lambda f \in D_+$. Therefore, if $g \perp D_+$, then it follows that

$$0 = \left(R_\lambda^{\pm} f, g\right)_{\mathcal{H}} = -i\int_0^{\infty} e^{-i\lambda t}\left(U_t^{\pm} f, g\right)_{\mathcal{H}} dt, \quad \operatorname{Im}\lambda < 0. \tag{4.2}$$

From this, we conclude that $(U_t^{\pm} f, g)_{\mathcal{H}} = 0$ for all $t \ge 0$. Hence $U_t^{\pm}D_+ \subset D_+$, for $t \ge 0$, which completes the proof of property (1).

To prove property (2), we denote by $P^+ : \mathcal{H} \to L^2(0,\infty)$ and $P_1^+ : L^2(0,\infty) \to D_+$ the mappings acting according to the formulas $P^+ : \langle \varphi_-, u, \varphi_+\rangle \to \varphi_+$ and $P_1^+ : \varphi \to \langle 0,0,\varphi\rangle$, respectively. Note that the semigroup of isometries $V_t^{\pm} = P^+U_t^{\pm}P_1^+$, $t \ge 0$ is a one-sided shift in $L^2(0,\infty)$. Indeed, the generator of the semigroup of the one-sided shift V_t in $L^2(0,\infty)$ is the differential operator $i(d/d\xi)$ satisfying the boundary condition $\varphi(0) = 0$. On the other hand, the generator $A^{\pm}$ of the semigroup of isometries $V_t^{\pm}$, $t \ge 0$, is the operator $A^{\pm}\varphi = P^+\mathcal{L}_{h_1h_2}^{\pm}P_1^+ f = P^+\mathcal{L}_{h_1h_2}^{\pm}\langle 0,0,\varphi\rangle = P^+\langle 0,0,i(d\varphi/d\xi)\rangle = i(d\varphi/d\xi)$, where $\varphi \in W_2^1(0,\infty)$ and $\varphi(0) = 0$. As a semigroup is uniquely determined by its generator, it follows that $V_t^{\pm} = V_t$, and thus, $\bigcap_{t\ge 0} U_t^{\pm}D_+ = \langle 0,0,\bigcap_{t\ge 0} V_t L^2(0,\infty)\rangle = \{0\}$, which verifies the property (2).

The scattering matrix is defined in terms of the spectral representations theory in this scheme of the Lax-Phillips scattering theory. We will continue with their construction and prove property (3) of the incoming and outgoing subspaces along the way.

We recall that the linear operator A (with domain $D(A)$) acting in the Hilbert space H is called *completely nonself-adjoint* (or *simple*) if the invariant subspace $M \subseteq D(A)$ $(M \ne \{0\})$ of the operator A whose restriction to M is self-adjoint, does not exist.

Lemma 4.1. *The operator $L_{h_1h_2}^{\pm}$ is completely nonself-adjoint (simple).*

Proof. Let $H' \subset H$ be a nontrivial subspace where $L_{h_1h_2}^{-}$ (the proof for $L_{h_1h_2}^{+}$ is similar) induces a self-adjoint operator L' with domain $D(L') = H' \cap D(L_{h_1h_2}^{-})$. If $f \in D(L')$, then we get $f \in D(L'^{*})$ and $[y,u]_{-\infty} - h_1[y,v]_{-\infty} = 0$, $[y,u]_{-\infty} - \overline{h}_1[y,v]_{-\infty} = 0$. It follows that $[y,u]_{-\infty} = 0$, $[y,v]_{-\infty} = 0$ and $y(\lambda) = 0$ for the eigenvectors $y(\lambda)$ of the operator $L_{h_1h_2}^{-}$ that lie in H' and are eigenvectors of L'. Since all solutions of (2.1) belong to $\ell_w^2(\mathbb{Z})$, we conclude that the resolvent $R_\lambda(L_{h_1h_2}^-)$ of the operator $L_{h_1h_2}^-$ is a Hilbert-Schmidt operator, and hence the spectrum of $L_{h_1h_2}^-$ is purely discrete. Using the theorem on expansion in eigenvectors of the self-adjoint operator L', we see that $H' = \{0\}$, that is, the operator $L_{h_1h_2}^-$ is simple. The lemma is proved. □

To prove property (3) we first set

$$\mathscr{H}_-^{\pm} = \overline{\bigcup_{t\geq 0} U_t^{\pm} D_-}, \qquad \mathscr{H}_+^{\pm} = \overline{\bigcup_{t\leq 0} U_t^{\pm} D_+}, \tag{4.3}$$

and prove the following lemma.

Lemma 4.2. *The equality $\mathscr{H}_-^{\pm} + \mathscr{H}_+^{\pm} = \mathscr{H}$ holds.*

Proof. Using property (1) of the subspace $D_\pm$, we can easily show that the subspace $\mathscr{H}'_\pm = \mathscr{H} \ominus (\mathscr{H}_-^{\pm} + \mathscr{H}_+^{\pm})$ is invariant with respect to the group $\{U_t^{\pm}\}$ and has the form $\mathscr{H}'_\pm = \langle 0, H'_\pm, 0\rangle$, where $H'_\pm$ is a subspace in H. Accordingly, if the subspace $\mathscr{H}'_\pm$ (and thus, $H'_\pm$ as well) were nontrivial, then the unitary group $\{U_t^{\pm\prime}\}$, restricted to this subspace, would be a unitary part of the group $\{U_t^{\pm}\}$, and thus the restriction $L_{h_1h_2}^{\pm\prime}$ of $L_{h_1h_2}^{\pm}$ to $H'_\pm$ would be a self-adjoint operator in $H'_\pm$. It follows from the simplicity of the operator $L_{h_1h_2}^{\pm}$ that $H'_\pm = \{0\}$, that is, $\mathscr{H}'_\pm = \{0\}$. The proof is completed. □

Let $\varphi(\lambda)$ and $\psi(\lambda)$ be the solutions of (2.1) satisfying the conditions:

$$\begin{aligned} [\varphi, u]_{-\infty} &= -1, \qquad [\varphi, v]_{-\infty} = 0, \\ [\psi, u]_{-\infty} &= 0, \qquad [\psi, v]_{-\infty} = 1. \end{aligned} \tag{4.4}$$

The *Titchmarsh-Weyl function* $m_{\infty h_2}(\lambda)$ of the self-adjoint operator $L_{\infty h_2}^-$ is determined by the condition $[\psi + m_{\infty h_2}\varphi, u]_\infty - h_2[\psi + m_{\infty h_2}\varphi, v]_\infty = 0$. Then, we have

$$m_{\infty h_2}(\lambda) = -\frac{[\psi, u]_\infty - h_2[\psi, v]_\infty}{[\varphi, u]_\infty - h_2[\varphi, v]_\infty}. \tag{4.5}$$

The last equality implies that $m_{\infty h_2}(\lambda)$ is a meromorphic function on the complex plane $\mathbb{C}$ with a countable number of poles on the real axis, which coincide with the eigenvalues of the self-adjoint operator $L_{\infty h_2}$. One can also show that the function $m_{\infty h_2}(\lambda)$ has the following properties: $\operatorname{Im}\lambda \operatorname{Im} m_{\infty h_2}(\lambda) > 0$ for $\operatorname{Im}\lambda \neq 0$ and $m_{\infty h_2}(\overline{\lambda}) = \overline{m_{\infty h_2}(\lambda)}$ for complex λ with the exception of the real poles of $m_{\infty h_2}(\lambda)$.

We adopt the following notations: $\theta(\lambda) = \psi(\lambda) + m_{\infty h_2}(\lambda)\varphi(\lambda)$,

$$S_{h_1h_2}^-(\lambda) = \frac{m_{\infty h_2}(\lambda) - h_1}{m_{\infty h_2}(\lambda) - \overline{h}_1}. \tag{4.6}$$

Let

$$U_\lambda^-(\xi, \varsigma) = \left\langle e^{-i\lambda\xi},\ (m_{\infty h_2}(\lambda) - h_1)^{-1}\alpha\theta(\lambda),\ \overline{S_{h_1h_2}^-}(\lambda)e^{-i\lambda\varsigma} \right\rangle. \tag{4.7}$$

For real values of λ, the vectors $U_\lambda^-(\xi,\varsigma)$ do not belong to the space $\mathcal{H}$, but they satisfy the equation $\mathcal{L}U = \lambda U$ and the boundary conditions (3.2). Using $U_\lambda^-(\xi,\varsigma)$, we define the transformation $F_- : f \to \widetilde{f}_-(\lambda)$ by $(F_-f)(\lambda) := \widetilde{f}_-(\lambda) := (1/\sqrt{2\pi})(f,U_\lambda^-)_{\mathcal{H}}$ on the vector $f = \langle \varphi_-, y, \varphi_+\rangle$, where φ_-, φ_+ are smooth, compactly supported functions, and $y = \{y_n\}$, $n \in \mathbb{Z}$, is a finite sequence.

Lemma 4.3. *The transformation F_- isometrically maps $\mathcal{H}_-$ onto $L^2(\mathbb{R})$. For all vectors $f, g \in \mathcal{H}_-^-$, the Parseval equality and the inversion formula hold:*

$$(f,g)_{\mathcal{H}} = \left(\widetilde{f}_-,\widetilde{g}_-\right)_{L^2} = \int_{-\infty}^{\infty} \widetilde{f}_-(\lambda)\overline{\widetilde{g}_-(\lambda)}d\lambda, \qquad f = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \widetilde{f}_-(\lambda)U_\lambda^- d\lambda, \tag{4.8}$$

where $\widetilde{f}_-(\lambda) = (F_-f)(\lambda)$ and $\widetilde{g}_-(\lambda) = (F_-g)(\lambda)$.

Proof. For $f, g \in D_-$, $f = \langle \varphi_-, 0, 0\rangle$, $g = \langle \psi_-, 0, 0\rangle$, we have

$$\widetilde{f}_-(\lambda) := \frac{1}{\sqrt{2\pi}}(f,U_\lambda^-)_{\mathcal{H}} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0} \varphi_-(\xi)e^{i\lambda\xi}d\xi \in H_-^2, \tag{4.9}$$

and, by the usual Parseval equality for Fourier integrals,

$$(f,g)_{\mathcal{H}} = \int_{-\infty}^{0} \varphi_-(\xi)\overline{\psi_-(\xi)}d\xi = \int_{-\infty}^{\infty} \widetilde{f}_-(\lambda)\overline{\widetilde{g}_-(\lambda)}d\lambda = (F_-f, F_-g)_{L^2}. \tag{4.10}$$

From now on, let $H_\pm^2$ denote the Hardy classes in $L^2(\mathbb{R})$ consisting of the functions which are analytically extendable to the upper and lower half-planes, respectively.

Let us extend the Parseval equality to the whole $\mathcal{H}_-^-$. To this end, we consider in $\mathcal{H}_-^-$ the dense set $\mathcal{H}'_-$ of vectors obtained from the smooth, compactly supported functions in D_- : $f \in \mathcal{H}'_-$ if $f = U_T^- f_0$, $f_0 = \langle \varphi_-, 0, 0\rangle$, $\varphi_- \in C_0^\infty(-\infty,0)$, where $T = T_f$ is a nonnegative number (depending on f). In this case, if $f, g \in \mathcal{H}'_-$, then $U_{-T}^- f$, $U_{-T}^- g \in D_-$ for $T > T_f$ and $T > T_g$. Furthermore, the first components of these vectors belong to $C_0^\infty(-\infty,0)$. Since the operators U_t^-, $t \in \mathbb{R}$, are unitary, the equality $F_-U_{-T}^- f = (U_{-T}^- f, U_\lambda^-)_{\mathcal{H}} = e^{-i\lambda T}(f,U_\lambda^-)_{\mathcal{H}} = e^{-i\lambda T}F_-f$ gives us that

$$\begin{aligned}(f,g)_{\mathcal{H}} &= (U_{-T}^- f, U_{-T}^- g)_{\mathcal{H}} = (F_-U_{-T}^- f, F_-U_{-T}^- g)_{L^2} \\ &= \left(e^{-i\lambda T}F_-f, e^{-i\lambda T}F_-g\right)_{L^2} = (F_-f, F_-g)_{L^2}.\end{aligned} \tag{4.11}$$

If we take the closure in (4.11), we get the Parseval equality for the whole space $\mathcal{H}_-^-$. The inversion formula follows from the Parseval equality if all integrals in it are considered as limits in the mean of integrals over finite intervals. In conclusion, we have $F_-\mathcal{H}_-^- = \overline{\bigcup_{t\geq 0} F_-U_t^- D_-} = \overline{\bigcup_{t\geq 0} e^{-i\lambda t}H_-^2} = L^2(\mathbb{R})$ which implies that F_- maps $\mathcal{H}_-^-$ onto the whole of $L^2(\mathbb{R})$. The lemma is proved. □

Now, we let

$$U_\lambda^+(\xi,\varsigma) = \left\langle S_{h_1h_2}^-(\lambda)e^{-i\lambda\xi},\ \left(m_{\infty h_2}^-(\lambda) - \overline{h}_1\right)^{-1}\alpha\theta(\lambda),\ e^{-i\lambda\varsigma}\right\rangle. \tag{4.12}$$

Note as in the previous case that the vectors $U_\lambda^+(\xi,\varsigma)$, for real values of λ, do not belong to the space $\mathscr{H}$. But, $U_\lambda^+(\xi,\varsigma)$ satisfies the equation $\mathcal{L}U = \lambda U$, $\lambda \in \mathbb{R}$, and the boundary conditions (3.2). By means of $U_\lambda^+(\xi,\varsigma)$, we consider the transformation $F_+ : f \to \widetilde{f}_+(\lambda)$ by setting $(F_+f)(\lambda) := \widetilde{f}_+(\lambda) := (1/\sqrt{2\pi})(f, U_\lambda^+)_{\mathscr{H}}$ on vectors $f = \langle \varphi_-, y, \varphi_+\rangle$, where φ_-, φ_+ are smooth, compactly supported functions, and $y = \{y_n\}$, $n \in \mathbb{Z}$, is a finite sequence. The proof of the next result is similar to that of Lemma 4.3.

Lemma 4.4. *The transformation F_+ isometrically maps $\mathscr{H}_+^-$ onto $L^2(\mathbb{R})$, and for all vectors $f, g \in \mathscr{H}_+^-$, the Parseval equality and the inversion formula hold:*

$$(f,g)_{\mathscr{H}} = \left(\widetilde{f}_+, \widetilde{g}_+\right)_{L^2} = \int_{-\infty}^{\infty} \widetilde{f}_+(\lambda)\overline{\widetilde{g}_+(\lambda)}d\lambda, \qquad f = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \widetilde{f}_+(\lambda)U_\lambda^+ d\lambda, \tag{4.13}$$

where $\widetilde{f}_+(\lambda) = (F_+f)(\lambda)$ and $\widetilde{g}_+(\lambda) = (F_+g)(\lambda)$.

From (4.6), we see that $|S_{h_1h_2}^-(\lambda)| = 1$ for all $\lambda \in \mathbb{R}$. Therefore, it follows from the explicit formula for the vectors U_λ^+ and U_λ^- that

$$U_\lambda^- = \overline{S_{h_1h_2}^-}(\lambda)U_\lambda^+, \quad (\lambda \in \mathbb{R}). \tag{4.14}$$

Lemmas 4.3 and 4.4 imply that $\mathscr{H}_-^- = \mathscr{H}_+^-$. Together with Lemma 4.2, this results in $\mathscr{H} = \mathscr{H}_-^- = \mathscr{H}_+^-$ and the property (3) of the incoming and outgoing subspaces for U_t^-.

Therefore, the transformation F_- maps isometrically onto $L^2(\mathbb{R})$ with the subspace D_- mapped onto H_-^2 and the operators U_t^- are transformed into the operators of multiplication by $e^{i\lambda t}$, that is, F_- is the incoming spectral representation for the group $\{U_t^-\}$. Similarly F_+ is the outgoing spectral representation for $\{U_t^-\}$. It is seen from (4.14) that we can realize the passage from the F_+-representation of a vector $f \in \mathscr{H}$ to its F_--representation multiplying by the function $S_{h_1h_2}^-(\lambda) : \widetilde{f}_-(\lambda) = S_{h_1h_2}^-(\lambda)\widetilde{f}_+(\lambda)$. According to [5], the scattering function (matrix) of the group $\{U_t^-\}$ with respect to the subspaces D_- and D_+, is the coefficient by which the F_--representation of a vector $f \in \mathscr{H}$ must be multiplied in order to get the corresponding F_+-representation: $\widetilde{f}_+(\lambda) = \overline{S_{h_1h_2}^-}(\lambda)\widetilde{f}_-(\lambda)$ and, thus, we have proved the following theorem.

Theorem 4.5. *The function $\overline{S_{h_1h_2}^-}(\lambda)$ is the scattering matrix of the group $\{U_t^-\}$ (of the self-adjoint operator $\mathcal{L}_{h_1h_2}^-$).*

Let $S(\lambda)$ be an arbitrary nonconstant inner function [1–4] on the upper half-plane (the analytic function $S(\lambda)$ on the upper half-plane $\mathbb{C}_+$ is called *inner function* on $\mathbb{C}_+$ if $|S(\lambda)| \le 1$ for $\lambda \in \mathbb{C}_+$ and $|S(\lambda)| = 1$ for almost all $\lambda \in \mathbb{R}$). Let $\mathcal{K} = H_+^2 \ominus SH_+^2$. We can see that $\mathcal{K} \neq \{0\}$ is a subspace of the Hilbert space H_+^2. Now, let us consider the semigroup of the operators Z_t, $t \ge 0$, acting in $\mathcal{K}$ according to the formula $Z_t\varphi = P[e^{i\lambda t}\varphi]$, $\varphi := \varphi(\lambda) \in \mathcal{K}$, where P

denotes the orthogonal projection from H_+^2 onto $\mathcal{K}$. The generator of the semigroup $\{Z_t\}$ is denoted by $T : T\varphi = \lim_{t\to+0}(it)^{-1}(Z_t\varphi - \varphi)$, which is a maximal dissipative operator acting in $\mathcal{K}$ with the domain $D(T)$ consisting of all vectors $\varphi \in \mathcal{K}$, so that the limit exists. The operator T is called a *model dissipative operator* (note that this model dissipative operator, which is associated with the names of Lax and Phillips [5], is a special case of a more general model dissipative operator constructed by Sz-Nagy and Foiaş [1, 2]). The basic assertion is that $S(\lambda)$ is the characteristic function of the operator T.

Let $K = \langle 0, H, 0\rangle$ so that $\mathcal{H} = D_- \oplus K \oplus D_+$. It can be concluded from the explicit form of the unitary transformation F_- that

$$\begin{gathered} \mathcal{H} \longrightarrow L^2(\mathbb{R}), \qquad f \longrightarrow \tilde{f}_-(\lambda) = (F_- f)(\lambda), \\ D_- \longrightarrow H_-^2, \qquad D_+ \longrightarrow S_{h_1h_2}^- H_+^2, \\ K \longrightarrow H_+^2 \ominus S_{h_1h_2}^- H_+^2, \qquad U_t^- f \longrightarrow \left(F_- U_t^- F_-^{-1}\tilde{f}_-\right)(\lambda) = e^{i\lambda t}\tilde{f}_-(\lambda). \end{gathered} \tag{4.15}$$

The formulas (4.15) show that the operator $L_{h_1h_2}^-$ is unitarily equivalent to the model dissipative operator with the characteristic function $S_{h_1h_2}^-(\lambda)$. Since the characteristic functions of unitarily equivalent dissipative operators coincide [1–4], we have proved the theorem below.

Theorem 4.6. *The characteristic function of the maximal dissipative operator $L_{h_1h_2}^-$ coincides with the function $S_{h_1h_2}^-(\lambda)$ defined in* (4.6).

If $m_{h_1\infty}(\lambda)$ is the Titchmarsh-Weyl function of the self-adjoint operator $L_{h_1\infty}$, then it can be expressed in terms of the Wronskian of the solutions as follows:

$$m_{h_1\infty}(\lambda) = -\frac{[\chi, v]_\infty}{[\phi, v]_\infty}. \tag{4.16}$$

Here $\phi(\lambda)$ and $\chi(\lambda)$ are solutions of (2.1) and normalized by

$$\begin{gathered} [\phi, u]_{-\infty} = -\frac{1}{\sqrt{1+h_1^2}}, \qquad [\phi, v]_{-\infty} = -\frac{h_1}{\sqrt{1+h_1^2}}, \\ [\chi, u]_{-\infty} = \frac{h_1}{\sqrt{1+h_1^2}}, \qquad [\chi, v]_{-\infty} = \frac{1}{\sqrt{1+h_1^2}}. \end{gathered} \tag{4.17}$$

Let us adopt the following notations:

$$\begin{gathered} n(\lambda) := \frac{[\phi, u]_\infty}{[\chi, v]_\infty}, \qquad m(\lambda) := m_{h_1\infty}(\lambda), \\ S^+(\lambda) := S_{h_1h_2}^+(\lambda) := \frac{m(\lambda)n(\lambda) - h_2}{m(\lambda)n(\lambda) - \overline{h}_2}. \end{gathered} \tag{4.18}$$

Let

$$V_\lambda^-(\xi,\varsigma)=\left\langle e^{-i\lambda\xi},\ \alpha m(\lambda)\left[(m(\lambda)n(\lambda)-h_2)\left[\chi,v\right]_\infty\right]^{-1}\phi(\lambda),\ \overline{S}^+(\lambda)e^{-i\lambda\varsigma}\right\rangle. \tag{4.19}$$

One can see that the vector $V_\lambda^-(\xi,\varsigma)$ does not belong to $\mathscr{H}$ for $\lambda\in\mathbb{R}$, but V_λ^- satisfies the equation $\mathcal{L}V=\lambda V$, $\lambda\in\mathbb{R}$, and the boundary conditions (3.11). By means of V_λ^-, we define the transformation $F_-:f\to\widetilde{f}_-(\lambda)$ by $(F_-f)(\lambda):=\widetilde{f}_-(\lambda):=(1/\sqrt{2\pi})(f,V_\lambda^-)_{\mathscr{H}}$ on the vector $f=\langle\varphi_-,y,\varphi_+\rangle$, where φ_-, φ_+ are smooth, compactly supported functions, and $y=\{y_n\}$, $n\in\mathbb{Z}$, is a finite sequence. The next result can be proved following the steps similar to the proof of Lemma 4.3.

Lemma 4.7. *The transformation F_- isometrically maps $\mathscr{H}_-^+$ onto $L^2(\mathbb{R})$. For all vectors $f,g\in\mathscr{H}_-^+$, the Parseval equality and the inversion formula hold:*

$$(f,g)_{\mathscr{H}}=\left(\widetilde{f}_-,\widetilde{g}_-\right)_{L^2}=\int_{-\infty}^{\infty}\widetilde{f}_-(\lambda)\overline{\widetilde{g}_-(\lambda)}d\lambda,\qquad f=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\widetilde{f}_-(\lambda)U_\lambda^-d\lambda, \tag{4.20}$$

where $\widetilde{f}_-(\lambda)=(F_-f)(\lambda)$ and $\widetilde{g}_-(\lambda)=(F_-g)(\lambda)$.

Let

$$V_\lambda^+(\xi,\varsigma)=\left\langle S^+(\lambda)e^{-i\lambda\xi},\ \alpha m(\lambda)\left[\left(m(\lambda)n(\lambda)-\overline{h}_2\right)\left[\chi,v\right]_\infty\right]^{-1}\phi(\lambda),\ e^{-i\lambda\varsigma}\right\rangle. \tag{4.21}$$

The vector $V_\lambda^+(\xi,\varsigma)$ does not belong to $\mathscr{H}$ for $\lambda\in\mathbb{R}$. However, V_λ^+ satisfies the equation $\mathcal{L}V=\lambda V$, $\lambda\in\mathbb{R}$, and the boundary conditions (3.11). Using $V_\lambda^+(\xi,\varsigma)$, let us consider the transformation $F_+:f\to\widetilde{f}_+(\lambda)$ on vectors $f=\langle\varphi_-,y,\varphi_+\rangle$, in which φ_-, φ_+ are smooth, compactly supported functions, and $y=\{y_n\}$, $n\in\mathbb{Z}$, is a finite sequence, by setting $(F_+f)(\lambda):=\widetilde{f}_+(\lambda):=(1/\sqrt{2\pi})(f,U_\lambda^+)_{\mathscr{H}}$.

Lemma 4.8. *The transformation F_+ isometrically maps $\mathscr{H}_+^+$ onto $L^2(\mathbb{R})$. For all vectors $f,g\in\mathscr{H}_+^+$, the Parseval equality and the inversion formula hold:*

$$(f,g)_{\mathscr{H}}=\left(\widetilde{f}_+,\widetilde{g}_+\right)_{L^2}=\int_{-\infty}^{\infty}\widetilde{f}_-(\lambda)\overline{\widetilde{g}_-(\lambda)}d\lambda,\qquad f=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\widetilde{f}_+(\lambda)U_\lambda^+d\lambda, \tag{4.22}$$

where $\widetilde{f}_+(\lambda)=(F_+f)(\lambda)$ and $\widetilde{g}_+(\lambda)=(F_+g)(\lambda)$.

It is seen from (4.18) that the function $S_{h_1h_2}^+(\lambda)$ satisfies $|S_{h_1h_2}^+(\lambda)|=1$ for $\lambda\in\mathbb{R}$. Therefore, the explicit formula for the vectors U_λ^+ and U_λ^- gives us that

$$V_\lambda^-=\overline{S}_{h_1h_2}^+(\lambda)V_\lambda^+,\quad \lambda\in\mathbb{R}. \tag{4.23}$$

Hence, we conclude the equality $\mathcal{H}_-^+ = \mathcal{H}_+^+$ from Lemmas 4.7 and 4.8. Together with Lemma 4.2, we get $\mathcal{H} = \mathcal{H}_-^+ = \mathcal{H}_+^+$. We can see from (4.23) that the passage from the F_--representation of a vector $f \in \mathcal{H}$ to its F_+-representation is realized as follows: $\widetilde{f}_+(\lambda) = \overline{S}^+_{h_1h_2}(\lambda)\widetilde{f}_-(\lambda)$. Thus, we have proved the following assertion.

Theorem 4.9. *The function $\overline{S}^+_{h_1h_2}(\lambda)$ is the scattering matrix of the group $\{U_t^+\}$ (of the self-adjoint operator $\mathcal{L}^+_{h_1h_2}$).*

Using the explicit form of the unitary transformation F_-, we obtain

$$\begin{gathered}\mathcal{H} \longrightarrow L^2(\mathbb{R}), \qquad f \longrightarrow \widetilde{f}_-(\lambda) = (F_- f)(\lambda), \\ D_- \longrightarrow H_-^2, \qquad D_+ \longrightarrow S^+_{h_1h_2} H_+^2, \\ K \longrightarrow H_+^2 \ominus S^+_{h_1h_2} H_+^2, \qquad U_t^+ f \longrightarrow \left(F_- U_t^+ F_-^{-1} \widetilde{f}_-\right)(\lambda) = e^{i\lambda t}\widetilde{f}_-(\lambda).\end{gathered} \tag{4.24}$$

We conclude from (4.24) that the operator $L^+_{h_1h_2}$ is a unitary equivalent to the model dissipative operator with characteristic function $S^+_{h_1h_2}(\lambda)$, which in turn proves the next theorem.

Theorem 4.10. *The characteristic function of the maximal dissipative operator $L^+_{h_1h_2}$ coincides with the function $S^+_{h_1h_2}(\lambda)$ defined by* (4.18).

5. Completeness Theorems for the System of Eigenvectors and Associated Vectors of the Maximal Dissipative Operators

We know that the characteristic function of a maximal dissipative operator $L^\pm_{h_1h_2}$ carries complete information about the spectral properties of this operator [1–4]. For example, completeness of the system of eigenvectors and associated vectors of the maximal dissipative operators $L^\pm_{h_1h_2}$ is guaranteed by the absence of a singular factor of the characteristic function $S^\pm_{h_1h_2}(\lambda)$ in the factorization $S^\pm_{h_1h_2}(\lambda) = S^\pm(\lambda)B^\pm(\lambda)$ (where $B^\pm(\lambda)$ is a Blaschke product).

Let A be a linear operator in the Hilbert space $\mathbf{H}$ with the domain $D(A)$. The complex number λ_0 is called an *eigenvalue* of the operator A if there exists a nonzero element $y_0 \in D(A)$ satisfying $Ay_0 = \lambda_0 y_0$. Such an element y_0 is called the *eigenvector* of the operator A corresponding to the eigenvalue λ_0. The elements $y_1, y_2, \ldots, y_k$ are called the *associated vectors* of the eigenvector y_0 if they belong to $D(A)$ and satisfy $Ay_j = \lambda_0 y_j + y_{j-1}$, $j = 1, 2, \ldots, k$. The element $y \in D(A)$, $y \neq 0$ is called a *root vector* of the operator A corresponding to the eigenvalue λ_0, if all powers of A are defined on this element and $(A - \lambda_0 I)^m y = 0$ for some integer m. The set of all root vectors of A corresponding to the same eigenvalue λ_0 with the vector $y = 0$ forms a linear set N_{λ_0} and is called the root lineal. The dimension of the lineal N_{λ_0} is called the *algebraic multiplicity* of the eigenvalue λ_0. The root lineal N_{λ_0} coincides with the linear span of all eigenvectors and associated vectors of A corresponding to the eigenvalue λ_0. Therefore, the completeness of the system of all eigenvectors and associated vectors of A is equivalent to the completeness of the system of all root vectors of this operator.

Theorem 5.1. *For all values of h_1 with* $\operatorname{Im} h_1 > 0$, *except possibly for a single value $h_1 = h_1^0$, and for fixed h_2* ($\operatorname{Im} h_2 = 0$ *or* $h_2 = 0$), *the characteristic function $S^-_{h_1h_2}(\lambda)$ of the maximal dissipative* *operator* $L^-_{h_1 h_2}$ *is a Blaschke product and the spectrum of* $L^-_{h_1 h_2}$ *is purely discrete and belongs to the open upper half plane. The operator* $L^-_{h_1 h_2}(h_1 \neq h_1^0)$ *has a countable number of isolated eigenvalues with finite algebraic multiplicity and limit points at infinity, and the system of all eigenvectors and associated vectors (or root vectors) of this operator is complete in the space* $\ell^2_w(\mathbb{Z})$.

Proof. It can be easily seen from (4.6) that $S^-_{h_1 h_2}(\lambda)$ is an inner function in the upper half-plane and, moreover, it is meromorphic in the whole λ-plane. Then, it can be factorized as

$$S^-_{h_1 h_2}(\lambda) = e^{i\lambda c} B_{h_1 h_2}(\lambda), \quad c = c(h_1) \geq 0, \tag{5.1}$$

where $B_{h_1 h_2}(\lambda)$ is a Blaschke product. It can be inferred from (5.1) that

$$\left| S^-_{h_1 h_2}(\lambda) \right| \leq e^{-c(h_1)\,\mathrm{Im}\,\lambda}, \quad \mathrm{Im}\,\lambda \geq 0. \tag{5.2}$$

Further, if we express $m_{\infty h_2}(\lambda)$ in terms of $S^-_{h_1 h_2}(\lambda)$ and then use (4.6), we find

$$m_{\infty h_2}(\lambda) = \frac{\overline{h}_1 S^-_{h_1 h_2}(\lambda) - h_1}{S^-_{h_1 h_2}(\lambda) - 1}. \tag{5.3}$$

If $c(h_1) > 0$ for a given value h_1 ($\mathrm{Im}\,h_1 > 0$), then $\lim_{t\to+\infty} S^-_{h_1 h_2}(it) = 0$ follows from (5.2). Hence, we obtain $\lim_{t\to+\infty} m_{\infty h_2}(it) = h_1$ in the light of (5.3). Since $m_{\infty h_2}(\lambda)$ is independent of h_1, $c(h_1)$ can be nonzero at not more than a single point $h_1 = h_1^0$ (and, further, $h_1^0 = \lim_{t\to+\infty} m_{\infty h_2}(it)$). Hence, the theorem is proved. □

The proof of the next result is similar to that of Theorem 5.1.

Theorem 5.2. *For all values of* h_2 *with* $\mathrm{Im}\,h_2 > 0$, *except possibly for a single value* $h_2 = h_2^0$, *and for fixed* h_1 ($\mathrm{Im}\,h_1 = 0$ *or* $h_1 = \infty$), *the characteristic function* $S^+_{h_1 h_2}(\lambda)$ *of the maximal dissipative operator* $L^+_{h_1 h_2}$ *is a Blaschke product and the spectrum of* $L^+_{h_1 h_2}$ *is purely discrete and belongs to the open upper half-plane. The operator* $L^+_{h_1 h_2}(h_2 \neq h_2^0)$ *has a countable number of isolated eigenvalues with finite algebraic multiplicity and limit points at infinity, and the system of all eigenvectors and associated vectors of this operator is complete in the space* $\ell^2_w(\mathbb{Z})$.

Since a linear operator **S** acting in a Hilbert space **H** is maximal accretive if and only if −**S** is maximal dissipative, all results obtained for maximal dissipative operators can be immediately transferred to maximal accretive operators.

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