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Research Article

Inequalities between Power Means and Convex Combinations of the Harmonic and Logarithmic Means

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We prove that $\alpha H(a,b) + (1-\alpha)L(a,b) > M_{(1-4\alpha)/3}(a,b)$ for $\alpha \in (0,1)$ and all $a,b > 0$ with $a \neq b$ if and only if $\alpha \in [1/4, 1)$ and $\alpha H(a,b) + (1-\alpha)L(a,b) < M_{(1-4\alpha)/3}(a,b)$ if and only if $\alpha \in (0, 3\sqrt{345}/80 - 11/16)$, and the parameter $(1-4\alpha)/3$ is the best possible in either case. Here, $H(a,b) = 2ab/(a+b)$, $L(a,b) = (a-b)/(\log a - \log b)$, and $M_p(a,b) = ((a^p + b^p)/2)^{1/p} (p \neq 0)$ and $M_0(a,b) = \sqrt{ab}$ are the harmonic, logarithmic, and pth power means of a and b, respectively.

1. Introduction

The classical logarithmic mean $L(a,b)$ of two positive real numbers a and b with $a \neq b$ is defined by

$$L(a,b) = \frac{a-b}{\log a - \log b}. \tag{1.1}$$

In the recent past, the bivariate means have been the subject of intensive research. In particular, many remarkable inequalities for $L(a,b)$ can be found in the literature [1–21]. It might be surprising that the logarithmic mean has applications in physics, economics, and even in meteorology [22–24]. In [22] the authors study a variant of Jensen's functional equation involving the logarithmic mean, which appears in a heat conduction problem. A representation of $L(a,b)$ as an infinite product and an iterative algorithm for computing it as the common limit of two sequences of special geometric and arithmetic means are given in [4].

In [25, 26] it is shown that $L(a,b)$ can be expressed in terms of Gauss hypergeometric function ${}_2F_1$. And, in [26] the authors prove that the reciprocal of the logarithmic mean is strictly totally positive; that is, every $n \times n$ determinant with elements $1/L(a_i, b_i)$, where $0 < a_1 < a_2 < \cdots < a_n$ and $0 < b_1 < b_2 < \cdots < b_n$, is positive for all $n \geq 1$.

Let $G(a,b) = \sqrt{ab}$, $H(a,b) = 2ab/(a+b)$, $I(a,b) = 1/e(a^a/b^b)^{1/(a-b)}$, $A(a,b) = (a+b)/2$, $M_p(a,b) = ((a^p+b^p)/2)^{1/p}$ $(p \neq 0)$ and $M_0(a,b) = \sqrt{ab}$, and $L_p(a,b) = (a^{p+1}+b^{p+1})/(a^p+b^p)$ be the geometric, harmonic, identric, arithmetic, pth power, and pth Lehmer means of two positive numbers a and b, respectively. Then it is well known that both $M_p(a,b)$ and $L_p(a,b)$ are continuous and strictly increasing with respect to $p \in \mathbb{R}$ for fixed $a, b > 0$ with $a \neq b$, and the inequalities

$$\begin{aligned} \min\{a,b\} &< H(a,b) = M_{-1}(a,b) = L_{-1}(a,b) < G(a,b) = M_0(a,b) = L_{-1/2}(a,b) \\ &< L(a,b) < I(a,b) < A(a,b) = M_1(a,b) = L_0(a,b) < \max\{a,b\} \end{aligned} \tag{1.2}$$

hold for all $a, b > 0$ with $a \neq b$.

In [4], Carlson proves that the double inequality

$$\sqrt{\frac{G(a,b)(A(a,b)+G(a,b))}{2}} < L(a,b) < \frac{1}{2}(A(a,b)+G(a,b)) \tag{1.3}$$

holds for all $a, b > 0$ with $a \neq b$.

In [5], Lin finds the best possible upper and lower power bounds for the logarithmic mean as follows:

$$M_0(a,b) < L(a,b) < M_{1/3}(a,b) \tag{1.4}$$

for all $a, b > 0$ with $a \neq b$.

In [9], Sándor establishes that

$$\sqrt{G(a,b)I(a,b)} < L(a,b) < A(a,b) + G(a,b) - I(a,b) \tag{1.5}$$

for all $a, b > 0$ with $a \neq b$.

In [27], Alzer gives the optimal Lehmer mean bounds for L, $(LI)^{1/2}$, and $(L+I)/2$ as follows:

$$\begin{gathered} L_{-1/3}(a,b) < L(a,b) < L_0(a,b), \\ L_{-1/4}(a,b) < \sqrt{L(a,b)I(a,b)} < L_0(a,b), \\ L_{-1/4}(a,b) < \frac{1}{2}(L(a,b)+I(a,b)) < L_0(a,b) \end{gathered} \tag{1.6}$$

for all $a, b > 0$ with $a \neq b$.

The following sharp bounds for $(LI)^{1/2}$ and $(L+I)/2$ in terms of power mean are presented in [28]:

$$M_0(a,b) < \sqrt{L(a,b)I(a,b)} < M_{1/2}(a,b),$$
$$M_{\log 2/(1+\log 2)}(a,b) < \frac{1}{2}(L(a,b)+I(a,b)) < M_{1/2}(a,b) \tag{1.7}$$

for all $a,b>0$ with $a \neq b$.

In [29, 30], the authors obtain the sharp bounds for the products $A^{\alpha}(a,b)L^{1-\alpha}(a,b)$ and $G^{\alpha}(a,b)L^{1-\alpha}(a,b)$ and the sum $\alpha A(a,b)+(1-\alpha)L(a,b)$ in terms of power mean as follows:

$$M_0(a,b) < A^{\alpha}(a,b)L^{1-\alpha}(a,b) < M_{(1+2\alpha)/3}(a,b),$$
$$M_0(a,b) < G^{\alpha}(a,b)L^{1-\alpha}(a,b) < M_{(1-\alpha)/3}(a,b), \tag{1.8}$$
$$M_{\log 2/(\log 2-\log \alpha)}(a,b) < \alpha A(a,b)+(1-\alpha)L(a,b) < M_{(1+2\alpha)/3}(a,b)$$

for any $\alpha \in (0,1)$ and all $a,b>0$ with $a \neq b$.

In [31], Zhu presents some bounds for $I(a,b)$ in terms of $A(a,b)$ and $L(a,b)$ and $L(a,b)$ in terms of $G(a,b)$ and $I(a,b)$.

In [32], Chu et al. prove that the double inequality $\alpha A(a,b)+(1-\alpha)H(a,b) < P(a,b) < \beta A(a,b)+(1-\beta)H(a,b)$ holds for all $a,b>0$ with $a \neq b$ if and only if $\alpha \leq 2/\pi$ and $\beta \geq 5/6$.

It is the aim of this paper to give the optimal power mean bounds for the convex combination of harmonic and logarithmic means. Our main result is the following theorem.

Theorem 1.1. *For $\alpha \in (0,1)$ and all $a,b>0$ with $a \neq b$, one has*

(1) $\alpha H(a,b)+(1-\alpha)L(a,b) > M_{(1-4\alpha)/3}(a,b)$ *if and only if* $\alpha \in [1/4,1)$;

(2) $\alpha H(a,b)+(1-\alpha)L(a,b) < M_{(1-4\alpha)/3}(a,b)$ *if and only if* $\alpha \in (0, 3\sqrt{345}/80-11/16)$.

In particular, the parameter $(1-4\alpha)/3$ is the best possible in either case.

2. Lemmas

In order to establish our main result we need to establish four lemmas, which we present in this section.

Lemma 2.1. *Let $\alpha \in (1/4,1)$, $p=(1-4\alpha)/3 \in (-1,0)$, and $g(t) = -4\alpha p(p+1)^2(p+2)t^{p-1}+2(1-\alpha)p^2(1-p^2)t^{p-2}+2(1-\alpha)p(1-p)^2(2-p)t^{p-3}+12(1-\alpha)(1-p)$. Then $g(t)>0$ for $t \in [1,+\infty)$.*

Proof. Simple computations lead to

$$g(1) = \frac{64}{81}(1-\alpha)^2\left(56\alpha^2+23\alpha+11\right) > 0, \tag{2.1}$$

$$\lim_{t\to+\infty} g(t) = 12(1-\alpha)(1-p) = 8(1-\alpha)(1+2\alpha) > 0, \tag{2.2}$$

$$g'(t) = -2p(1-p)t^{p-4}g_1(t), \tag{2.3}$$

where

$$g_1(t) = -2\alpha(p+1)^2(p+2)t^2 + (1-\alpha)p(p+1)(2-p)t + (1-\alpha)(1-p)(2-p)(3-p),$$
$$g_1(1) = \frac{4}{27}(1-\alpha)\left(148\alpha^2 - 11\alpha + 25\right) > 0, \tag{2.4}$$

$$\lim_{t\to+\infty} g_1(t) = -\infty, \tag{2.5}$$

$$\begin{aligned} g'_1(t) &= -4\alpha(p+1)^2(p+2)t + (1-\alpha)p(p+1)(2-p) \\ &= -\frac{4}{27}(1-\alpha)^2[16\alpha(7-4\alpha)t + (4\alpha-1)(4\alpha+5)] < 0 \end{aligned} \tag{2.6}$$

for $t \in [1, +\infty)$.

Inequality (2.6) implies that $g_1(t)$ is strictly decreasing in $[1, +\infty)$. Then (2.4) and (2.5) lead to the conclusion that there exists $\lambda_1 > 1$ such that $g_1(t) > 0$ for $t \in [1, \lambda_1)$ and $g_1(t) < 0$ for $t \in (\lambda_1, +\infty)$. It follows from (2.3) that $g(t)$ is strictly increasing in $[1, \lambda_1]$ and strictly decreasing in $[\lambda_1, +\infty)$.

Therefore, Lemma 2.1 follows from (2.1) and (2.2) together with the piecewise monotonicity of $g(t)$. □

Lemma 2.2. *Let* $\alpha \in (1/4, 1), p = (1-4\alpha)/3 \in (-1, 0)$, *and* $h(t) = -(1-\alpha)(p+1)(p+2)^2(p+3)t^p + (p+1)(p^3 - \alpha p^3 - 19\alpha p^2 + 3p^2 - 34\alpha p + 2p - 8\alpha)t^{p-1} + (1-\alpha)p(p^3 - 8p^2 - p + 4)t^{p-2} + (1-\alpha)(1-p)(p^3 + 5p^2 - 14p + 4)t^{p-3} + 4(1-\alpha)(7-4p) - 4p(1-\alpha)t^{-1} + 4\alpha(1+p)t^{-2}$. *Then* $h(t) > 0$ *for* $t \in [1, +\infty)$.

Proof. Let

$$h_1(t) = t^{3-p}h(t). \tag{2.7}$$

Then simple computations lead to

$$h_1(1) = \frac{16}{27}(1-\alpha)\left(80\alpha^2 + 110\alpha - 1\right) > 0, \tag{2.8}$$

$$\begin{aligned} h'_1(t) &= -3(1-\alpha)(p+1)(p+2)^2(p+3)t^2 + 2(p+1) \\ &\quad \times \left(p^3 - \alpha p^3 - 19\alpha p^2 + 3p^2 - 34\alpha p + 2p - 8\alpha\right)t + (1-\alpha)p\left(p^3 - 8p^2 - p + 4\right) \\ &\quad + 4(1-\alpha)(7-4p)(3-p)t^{2-p} - 4p(1-\alpha)(2-p)t^{1-p} + 4\alpha\left(1-p^2\right)t^{-p}, \\ h'_1(1) &= \frac{32}{27}(1-\alpha)\left(-16\alpha^3 + 38\alpha^2 + 176\alpha - 9\right) > 0, \end{aligned} \tag{2.9}$$

$$
\begin{aligned}
h_1''(t) &= -6(1-\alpha)(p+1)(p+2)^2(p+3)t + 2(p+1) \\
&\quad \times \left(p^3 - \alpha p^3 - 19\alpha p^2 + 3p^2 - 34\alpha p + 2p - 8\alpha\right) \\
&\quad + 4(1-\alpha)(7-4p)(3-p)(2-p)t^{1-p} \\
&\quad - 4p(1-\alpha)(2-p)(1-p)t^{-p} - 4\alpha p\left(1-p^2\right)t^{-p-1},
\end{aligned}
\tag{2.10}
$$

$$
\begin{aligned}
h_1''(1) &= \frac{8}{81}(1-\alpha)\left(-128\alpha^4 + 896\alpha^3 + 288\alpha^2 + 5294\alpha - 437\right) > 0, \\
h_1'''(t) &= -6(1-\alpha)(p+1)(p+2)^2(p+3) + 4(1-\alpha)(7-4p)(3-p)(2-p)(1-p)t^{-p} \\
&\quad + 4p^2(1-\alpha)(2-p)(1-p)t^{-p-1} + 4\alpha p(1+p)^2(1-p)t^{-p-2}, \\
h_1'''(1) &= \frac{8}{81}(1-\alpha)\left(576\alpha^4 + 3872\alpha^3 + 660\alpha^2 + 6612\alpha - 785\right) > 0,
\end{aligned}
\tag{2.11}
$$

$$
h_1^{(4)}(t) = -4p(1-p)t^{-p-3}h_2(t),
\tag{2.12}
$$

where

$$
\begin{aligned}
h_2(t) &= (1-\alpha)(7-4p)(3-p)(2-p)t^2 + (1-\alpha)p(2-p)(p+1)t + \alpha(p+1)^2(p+2), \\
h_2(1) &= \frac{4}{27}(1-\alpha)\left(96\alpha^3 + 232\alpha^2 + 388\alpha + 175\right) > 0, \\
h_2'(t) &= 2(1-\alpha)(7-4p)(3-p)(2-p)t + (1-\alpha)p(2-p)(p+1) \\
&\geq h_2'(1) = \frac{4}{9}(1-\alpha)(5+4\alpha)\left(12\alpha^2 + 31\alpha + 23\right) > 0
\end{aligned}
\tag{2.13}
$$

for $t \in [1, +\infty)$.

From inequalities (2.13) we clearly see that $h_2(t) > 0$ for $t \in [1, +\infty)$. Then (2.12) leads to the conclusion that $h_1'''(t)$ is strictly increasing in $[1, +\infty)$.

Therefore, Lemma 2.2 follows from (2.7)–(2.11) and the monotonicity of $h_1'''(t)$. □

Lemma 2.3. *Let* $\alpha \in (0,1), p = (1-4\alpha)/3, \lambda_0 = 3\sqrt{345}/80 - 11/16 = 0.00903\ldots$, *and* $f(t) = 2\alpha(1-t^{p+1})t\log^2 t + (1-\alpha)(1+t^{p-1})(1+t)^2 t\log t + (1-\alpha)(1+t)^2(1-t)(t^p+1)$. *Then the following two statements are true:*

(1) *if* $\alpha \in (1/4, 1)$, *then* $f(t) > 0$ *for* $t \in (1, +\infty)$;

(2) *if* $\alpha \in (0, \lambda_0]$, *then* $f(t) < 0$ *for* $t \in (1, +\infty)$.

Proof. Let $f_1(t) = t^{-p}f''(t)$, $f_2(t) = t^{p+2}f_1'(t)$, $f_3(t) = t^{4-p}f_2'''(t)$, $f_4(t) = t^{p+2}f_3'''(t)$ and $f_5(t) = t^{4-p}f_4'''(t)$. Then simple computations lead to

$$f(1) = 0, \tag{2.14}$$

$$\begin{aligned}
f'(t) = {} & 2\alpha\left[1-(p+2)t^{p+1}\right]\log^2 t \\
& + \Big[(p+2-\alpha p-6\alpha)t^{p+1} + 2(1-\alpha)(p+1)t^p \\
& \quad +(1-\alpha)pt^{p-1} + 3(1-\alpha)t^2 + 4(1-\alpha)t + 3\alpha + 1\Big]\log t \\
& - (1-\alpha)\left[(p+3)t^{p+2} + (p+1)t^{p+1} - (p+3)t^p - (p+1)t^{p-1} + 2t^2 - 2\right], \\
f'(1) = {} & 0,
\end{aligned} \tag{2.15}$$

$$\begin{aligned}
f_1(t) = {} & -2\alpha(p+1)(p+2)\log^2 t \\
& + \Big[\left(p^2 - \alpha p^2 + 3p - 11\alpha p - 14\alpha + 2\right) + 2(1-\alpha)p(p+1)t^{-1} \\
& \quad -(1-\alpha)p(1-p)t^{-2} + 6(1-\alpha)t^{1-p} + 4(1-\alpha)t^{-p} + 4\alpha p t^{-1-p}\Big]\log t \\
& - (1-\alpha)(p+2)(p+3)t + (1-\alpha)\left(p^2+5p+2\right)t^{-1} \\
& + (1-\alpha)\left(p^2+p-1\right)t^{-2} - (1-\alpha)t^{1-p} + 4(1-\alpha)t^{-p} + (1+3\alpha)t^{-1-p} \\
& - (1-\alpha)p^2 - (1-\alpha)p - 5\alpha + 1, \\
f_1(1) = {} & 0,
\end{aligned} \tag{2.16}$$

$$\begin{aligned}
f_2(t) = {} & -\Big[4\alpha(p+1)(p+2)t^{p+1} + 2(1-\alpha)p(p+1)t^p - 2(1-\alpha)p(1-p)t^{p-1} \\
& \quad -6(1-\alpha)(1-p)t^2 + 4(1-\alpha)pt + 4\alpha(1+p)\Big]\log t \\
& - (1-\alpha)(p+2)(p+3)t^{p+2} + \left(p^2 - \alpha p^2 + 3p - 11\alpha p - 14\alpha + 2\right)t^{p+1} \\
& + (1-\alpha)\left(p^2-3p-2\right)t^p - (1-\alpha)\left(p^2+3p-2\right)t^{p-1} + (1-\alpha)(p+5)t^2 \\
& + 4(1-\alpha)(1-p)t + \alpha - 3\alpha p - p - 1, \\
f_2(1) = {} & 0,
\end{aligned} \tag{2.17}$$

$$\begin{aligned}
f_2'(t) = {} & -\Big[4\alpha(p+1)^2(p+2)t^p + 2(1-\alpha)p^2(p+1)t^{p-1} + 2(1-\alpha)p(1-p)^2t^{p-2} \\
& \quad -12(1-\alpha)(1-p)t + 4(1-\alpha)p\Big]\log t - (1-\alpha)(p+2)^2(p+3)t^{p+1} \\
& + (p+1)\left(p^2 - \alpha p^2 - 15\alpha p + 3p - 22\alpha + 2\right)t^p + (1-\alpha)p\left(p^2-5p-4\right)t^{p-1} \\
& + (1-\alpha)(1-p)\left(p^2+5p-2\right)t^{p-2} + 4(1-\alpha)(4-p)t - 4\alpha(1+p)t^{-1} \\
& + 4(1-\alpha)(1-2p), \\
f_2'(1) = {} & 0,
\end{aligned} \tag{2.18}$$

$$f_2''(t) = -\Big[4\alpha p(p+1)^2(p+2)t^{p-1} - 2(1-\alpha)p^2\left(1-p^2\right)t^{p-2} - 2(1-\alpha)p(1-p)^2 \times(2-p)t^{p-3} - 12(1-\alpha)(1-p)\Big]\log t - (1-\alpha)(p+1)(p+2)^2 \times (p+3)t^p + (p+1)\left(p^3 - \alpha p^3 - 19\alpha p^2 + 3p^2 - 34\alpha p + 2p - 8\alpha\right)t^{p-1} + (1-\alpha)p\left(p^3 - 8p^2 - p + 4\right)t^{p-2} + (1-\alpha)(1-p)\left(p^3 + 5p^2 - 14p + 4\right)t^{p-3} - 4(1-\alpha)pt^{-1} + 4\alpha(1+p)t^{-2} + 4(1-\alpha)(7-4p). \tag{2.19}$$

(1) If $\alpha \in (1/4, 1)$, then from (2.19) we note that

$$f_2''(t) = g(t)\log t + h(t), \tag{2.20}$$

where $g(t)$ and $h(t)$ are defined as in Lemmas 2.1 and 2.2, respectively.

Lemmas 2.1 and 2.2 together with (2.20) imply that $f_2'(t)$ is strictly increasing in $[1, +\infty)$. Therefore, $f(t) > 0$ for $t \in (1, +\infty)$ follows from (2.14)–(2.18) and the monotonicity of $f_2'(t)$.

(2) If $\alpha \in (0, \lambda_0]$, then from (2.19) we have

$$f_2''(1) = \frac{16}{27}(1-\alpha)\left(80\alpha^2 + 110\alpha - 1\right) = \frac{1280}{27}(1-\alpha)(\alpha - \lambda_0)\left(\alpha + \frac{3\sqrt{345}}{80} + \frac{11}{16}\right) \le 0, \tag{2.21}$$

$$f_3(t) = \Big[4\alpha p(1-p)(p+1)^2(p+2)t^2 - 2(1-\alpha)p^2\left(1-p^2\right)(2-p)t - 2(1-\alpha)p(1-p)^2(2-p)(3-p)\Big]\log t - (1-\alpha)p(p+1)(p+2)^2(p+3)t^3 + (p+1)\left(p^4 - \alpha p^4 - 22\alpha p^3 + 2p^3 - 27\alpha p^2 - p^2 - 2p + 18\alpha p + 8\alpha\right)t^2 + (1-\alpha)p\left(p^4 - 12p^3 + 15p^2 + 8p - 8\right)t + (1-\alpha)(1-p)\left(p^4 + 4p^3 - 35p^2 + 50p - 12\right) + 12(1-\alpha)(1-p)t^{3-p} + 4p(1-\alpha)t^{2-p} - 8\alpha(1+p)t^{1-p},$$
$$f_3(1) = \frac{32}{81}(1-\alpha)(1-4\alpha)\left(80\alpha^2 + 110\alpha - 1\right) = \frac{10240}{81}(1-\alpha)\left(\frac{1}{4} - \alpha\right)(\alpha - \lambda_0)\left(\alpha + \frac{3\sqrt{345}}{80} + \frac{11}{16}\right) \le 0, \tag{2.22}$$

$$\begin{aligned}
f_3'(t) &= \Big[8\alpha p(1-p)(p+1)^2(p+2)t - 2(1-\alpha)p^2\left(1-p^2\right)(2-p)\Big]\log t \\
&\quad - 3(1-\alpha)p(p+1)(p+2)^2(p+3)t - 2(p+1) \\
&\quad \times \left(3\alpha p^4 - p^4 + 26\alpha p^3 - 2p^3 + 25\alpha p^2 + p^2 - 22\alpha p + 2p - 8\alpha\right)t \\
&\quad - (1-\alpha)p\left(p^4 + 8p^3 - 17p^2 - 4p + 8\right) - 2(1-\alpha)p(1-p)^2(2-p)(3-p)t^{-1} \\
&\quad + 12(1-\alpha)(1-p)(3-p)t^{2-p} + 4(1-\alpha)p(2-p)t^{1-p} - 8\alpha\left(1-p^2\right)t^{-p}, \\
f_3'(1) &= \frac{8}{243}(1-\alpha)\left(3328\alpha^4 + 128\alpha^3 - 7248\alpha^2 + 7118\alpha - 167\right) \\
&< \frac{8}{243}(1-\alpha)\left(3328\lambda_0^4 + 128\lambda_0^3 + 7118\lambda_0 - 167\right) \\
&< \frac{8}{243}(1-\alpha)\Big[3328\times(0.01)^4 + 128\times(0.01)^3 + 7118\times 0.01 - 167\Big] < 0,
\end{aligned} \tag{2.23}$$

$$\begin{aligned}
f_3''(t) &= 8\alpha p(1-p)(p+1)^2(p+2)\log t - 6(1-\alpha)p(p+1)(p+2)^2(p+3)t \\
&\quad - 2(1-\alpha)p^2\left(1-p^2\right)(2-p)t^{-1} + 2(1-\alpha)p(1-p)^2(2-p)(3-p)t^{-2} \\
&\quad + 12(1-\alpha)(1-p)(2-p)(3-p)t^{1-p} + 4(1-\alpha)p(1-p)(2-p)t^{-p} \\
&\quad + 8\alpha p\left(1-p^2\right)t^{-1-p} - 2(p+1) \\
&\quad \times \left(7\alpha p^4 - p^4 + 34\alpha p^3 - 2p^3 + 21\alpha p^2 + p^2 - 30\alpha p + 2p - 8\alpha\right), \\
f_3''(1) &= \frac{8}{243}(1-\alpha)(7-4\alpha)\left(256\alpha^4 - 64\alpha^3 - 1152\alpha^2 + 2066\alpha - 53\right) \\
&< \frac{8}{243}(1-\alpha)(7-4\alpha)\left(256\lambda_0^4 + 2066\lambda_0 - 53\right) \\
&< \frac{8}{243}(1-\alpha)[7-4\alpha)\left(256\times(0.01)^4 + 2066\times 0.01 - 53\right] < 0,
\end{aligned} \tag{2.24}$$

$$\begin{aligned}
f_4(t) &= -6(1-\alpha)p(p+1)(p+2)^2(p+3)t^{p+2} + 8\alpha p(1-p)(p+1)^2(p+2)t^{p+1} \\
&\quad + 2(1-\alpha)p^2\left(1-p^2\right)(2-p)t^{p} - 4(1-\alpha)p(1-p)^2(2-p)(3-p)t^{p-1} \\
&\quad + 12(1-\alpha)(1-p)^2(2-p)(3-p)t^2 - 4(1-\alpha)p^2(1-p)(2-p)t \\
&\quad - 8\alpha p(1-p)(1+p)^2, \\
f_4(1) &= \frac{8}{243}(1-\alpha)\left(-1024\alpha^4 + 21952\alpha^3 - 10968\alpha^2 + 13474\alpha - 835\right) \\
&< \frac{8}{243}(1-\alpha)\left(21952\lambda_0^3 + 13474\lambda_0 - 835\right) \\
&< \frac{8}{243}(1-\alpha)\Big[21952\times(0.01)^3 + 13474\times 0.01 - 835\Big] < 0,
\end{aligned} \tag{2.25}$$

$$\begin{aligned}
f_4'(t) &= -6(1-\alpha)p(p+1)(p+2)^3(p+3)t^{p+1} + 8\alpha p(1-p)(p+1)^3(p+2)t^{p} \\
&\quad + 2(1-\alpha)p^3\left(1-p^2\right)(2-p)t^{p-1} + 4(1-\alpha)p(1-p)^3(2-p)(3-p)t^{p-2} \\
&\quad + 24(1-\alpha)(1-p)^2(2-p)(3-p)t - 4(1-\alpha)p^2(1-p)(2-p),
\end{aligned}$$

$$\begin{aligned}
f_4'(1) &= \frac{8}{729}(1-\alpha)(7-4\alpha)\left(-1024\alpha^4+21952\alpha^3-10968\alpha^2+13474\alpha-835\right)\\
&< \frac{8}{729}(1-\alpha)(7-4\alpha)\left(21952\lambda_0^3+13474\lambda_0-835\right)\\
&< \frac{8}{729}(1-\alpha)(7-4\alpha)\left[21952\times(0.01)^3+13474\times 0.01-835\right]<0,
\end{aligned} \tag{2.26}$$

$$\begin{aligned}
f_4''(t) &= -6(1-\alpha)p(p+1)^2(p+2)^3(p+3)t^p+8\alpha p^2(1-p)(p+1)^3(p+2)t^{p-1}\\
&\quad -2(1-\alpha)p^3(1+p)(1-p)^2(2-p)t^{p-2}-4(1-\alpha)p(1-p)^3(2-p)^2(3-p)t^{p-3}\\
&\quad +24(1-\alpha)(1-p)^2(2-p)(3-p),\\
f_4''(1) &= \frac{32}{2187}(1-\alpha)\Big(-4096\alpha^6+136320\alpha^5-241440\alpha^4+383672\alpha^3\\
&\qquad\qquad -209850\alpha^2+100113\alpha-7255\Big)\\
&< \frac{32}{2187}(1-\alpha)\left(136320\lambda_0^5+383672\lambda_0^3+100113\lambda_0-7255\right)\\
&< \frac{32}{2187}(1-\alpha)\left[136320\times(0.01)^5+383672\times(0.01)^3+100113\times 0.01-7255\right]\\
&< 0,
\end{aligned} \tag{2.27}$$

$$\begin{aligned}
f_5(t) &= -6(1-\alpha)p^2(p+1)^2(p+2)^3(p+3)t^3-8\alpha p^2(1-p)^2(p+1)^3(p+2)t^2\\
&\quad +2(1-\alpha)p^3(1+p)(1-p)^2(2-p)^2t+4(1-\alpha)p(1-p)^3(2-p)^2(3-p)^2,\\
f_5(1) &= \frac{32}{6561}(1-\alpha)(1-4\alpha)\Big(-4096\alpha^6+173568\alpha^5-190368\alpha^4+439136\alpha^3\\
&\qquad\qquad -191370\alpha^2+96723\alpha-8665\Big)\\
&< \frac{32}{6561}(1-\alpha)(1-4\alpha)\left(173568\lambda_0^5+439136\lambda_0^3+96723\lambda_0-8665\right)\\
&< \frac{32}{6561}(1-\alpha)(1-4\alpha)\left[173568\times(0.01)^5+439136\times(0.01)^3+96723\times 0.01-8665\right]\\
&< 0,
\end{aligned} \tag{2.28}$$

$$\begin{aligned}
f_5'(t) &= -18(1-\alpha)p^2(p+1)^2(p+2)^3(p+3)t^2-16\alpha p^2(1-p)^2(p+1)^3(p+2)t\\
&\quad +2(1-\alpha)p^3(1+p)(1-p)^2(2-p)^2,\\
f_5'(1) &= \frac{32}{6561}(1-\alpha)^2(1-4\alpha)^2\left(-17408\alpha^4+69920\alpha^3-119136\alpha^2+95282\alpha-30845\right)\\
&< \frac{32}{6561}(1-\alpha)(1-4\alpha)^2\left(69920\lambda_0^3+95282\lambda_0-30845\right)\\
&< \frac{32}{6561}(1-\alpha)(1-4\alpha)^2\left(69920\times(0.01)^3+95282\times 0.01-30845\right)\\
&< 0,
\end{aligned} \tag{2.29}$$

$$
\begin{aligned}
f_5''(t) &= -36(1-\alpha)p^2(p+1)^2(p+2)^3(p+3)t - 16\alpha p^2(1-p)^2(p+1)^3(p+2),\\
f_5''(1) &= \frac{128}{6561}(1-\alpha)^3(1-4\alpha)^2(7-4\alpha)\left(160\alpha^3 - 1856\alpha^2 + 3370\alpha - 2205\right)\\
&< \frac{128}{6561}(1-\alpha)^3(1-4\alpha)^2(7-4\alpha)\left(160\lambda_0^3 + 3370\lambda_0 - 2205\right)\\
&< \frac{128}{6561}(1-\alpha)^3(1-4\alpha)^2(7-4\alpha)\left(160\times(0.01)^3 + 3370\times 0.01 - 2205\right)\\
&< 0,\\
f_5'''(t) &= -36(1-\alpha)p^2(p+1)^2(p+2)^3(p+3)\\
&= -\frac{128}{729}(5-2\alpha)(1-4\alpha)^2(1-\alpha)^3(7-4\alpha)^3 < 0.
\end{aligned}
\tag{2.30}
$$

Inequalities (2.30) imply that $f_5'(t)$ is strictly decreasing in $[1,+\infty)$. Then (2.29) leads to the conclusion that $f_5(t)$ is strictly decreasing in $[1,+\infty)$.

It follows from (2.28) and the monotonicity of $f_5(t)$ that $f_4''(t)$ is strictly decreasing in $[1,+\infty)$. Then inequalities (2.25)–(2.27) lead to the conclusion that $f_4(t) < 0$ for $t \in [1,+\infty)$. Thus, $f_3''(t)$ is strictly decreasing in $[1,+\infty)$.

From inequalities (2.22)–(2.24) and the monotonicity of $f_3''(t)$ we clearly see that $f_3(t) < 0$ for $t \in (1,+\infty)$. Thus, $f_2''(t)$ is strictly decreasing in $[1,+\infty)$.

It follows from (2.17) and (2.18) and inequality (2.21) together with the monotonicity of $f_2''(t)$ that $f_2(t) < 0$ for $t \in (1,+\infty)$, which implies that $f_1(t)$ is strictly decreasing in $[1,+\infty)$.

Therefore, $f(t) < 0$ for $t \in (1,+\infty)$ follows from (2.14)–(2.16) and the monotonicity of $f_1(t)$. □

Lemma 2.4. $3t^4 - 4t(2t^2 - t + 2)\log t - 3 > 0$ *for* $t > 1$.

Proof. Let

$$
J(t) = 3t^4 - 4t\left(2t^2 - t + 2\right)\log t - 3. \tag{2.31}
$$

Then simple computations lead to

$$
\begin{aligned}
J(1) &= 0,\\
J'(t) &= 4\left(3t^3 - 2t^2 + t - 2\right) - 8\left(3t^2 - t + 1\right)\log t,\\
J'(1) &= 0,\\
J''(t) &= \frac{4}{t}J_1(t),
\end{aligned}
\tag{2.32}
$$

where $J_1(t) = 9t^3 - 10t^2 + 3t - 2 - 2(6t-1)t\log t$,

$$
J_1(1) = 0, \tag{2.33}
$$

$$
\begin{aligned}
J_1'(t) &= 27t^2 - 32t + 5 - 2(12t-1)\log t,\\
J_1'(1) &= 0,
\end{aligned}
\tag{2.34}
$$

$$
J_1''(t) = \frac{2}{t}J_2(t), \tag{2.35}
$$

where $J_2(t) = 27t^2 - 12t\log t - 28t + 1$,

$$J_2(1) = 0,$$
$$J_2'(t) = 54t - 12\log t - 40 > 0 \tag{2.36}$$

for $t > 1$. □

Therefore, Lemma 2.4 follows from (2.31)–(2.36).

3. Proof of Theorem 1.1

Proof of Theorem 1.1. For all $a, b > 0$ with $a \neq b$, we first prove that

$$\alpha H(a,b) + (1-\alpha)L(a,b) > M_{(1-4\alpha)/3}(a,b) \tag{3.1}$$

for $\alpha \in [1/4, 1)$,

$$\alpha H(a,b) + (1-\alpha)L(a,b) < M_{(1-4\alpha)/3}(a,b) \tag{3.2}$$

for $\alpha \in (0, 3\sqrt{345}/80 - 11/16)$.

Without loss of generality, we assume that $a > b, t = a/b > 1$ and $p = (1-4\alpha)/3$. We divide the proof into two cases.

Case 1 ($\alpha = 1/4$). Let $x = \sqrt{t} = \sqrt{a/b} > 1$. Then we clearly see that

$$\begin{aligned}\alpha H(a,b) + (1-\alpha)L(a,b) - M_{(1-4\alpha)/3}(a,b) &= \frac{1}{4}[H(a,b) + 3L(a,b)] - \sqrt{ab}\\ &= \frac{b[3x^4 - 4x(2x^2 - x + 2)\log x - 3]}{8(x^2+1)\log x}.\end{aligned} \tag{3.3}$$

Therefore, inequality (3.1) follows from (3.3) and Lemma 2.4.

Case 2 ($\alpha \in (0, 3\sqrt{345}/80 - 11/16) \cup (1/4, 1)$). Then we have

$$\begin{aligned}\alpha H(a,b) + (1-\alpha)L(a,b) - M_{(1-4\alpha)/3}(a,b) &= \alpha H(a,b) + (1-\alpha)L(a,b) - M_p(a,b)\\ &= b\left[\frac{2\alpha t}{t+1} + \frac{(1-\alpha)(t-1)}{\log t} - \left(\frac{t^p+1}{2}\right)^{1/p}\right].\end{aligned} \tag{3.4}$$

Let

$$F(t) = \log\left[\frac{2\alpha t}{t+1} + \frac{(1-\alpha)(t-1)}{\log t}\right] - \frac{1}{p}\log\left(\frac{t^p+1}{2}\right). \tag{3.5}$$

Then simple computations lead to

$$\lim_{t\to 1} F(t) = 0,$$
$$F'(t) = \frac{f(t)}{t(t+1)(t^p+1)\left[2\alpha t\log t + (1-\alpha)(t^2-1)\right]\log t}, \tag{3.6}$$

where $f(t)$ is defined as in Lemma 2.3.

If $\alpha \in (1/4, 1)$, then inequality (3.1) follows from (3.4)–(3.6) and Lemma 2.3(1). If $\alpha \in (0, 3\sqrt{345}/80 - 11/16)$, then inequality (3.2) follows from (3.4)–(3.6) and Lemma 2.3(2).

Next, we prove that the parameter $(1-4\alpha)/3$ in inequalities (3.1) and (3.2) is the best possible.

For any $\alpha \in (0, 3\sqrt{345}/80 - 11/16) \cup (1/4, 1)$, $p \neq 0$, and $x > 0$, one has

$$\alpha H(1,1+x) + (1-\alpha)L(1,1+x) - M_p(1,1+x) = \frac{Q(x)}{2^{1/p}(1+x/2)\log(1+x)}, \tag{3.7}$$

where $Q(x) = 2^{1/p}\alpha(1+x)\log(1+x) + 2^{1/p}(1-\alpha)x(1+x/2) - (1+x/2)\log(1+x)[1+(1+x)^p]^{1/p}$.

Letting $x \to 0$ and making use of Taylor expansion, we get

$$Q(x) = \frac{2^{1/p}}{8}\left(\frac{1-4\alpha}{3} - p\right)x^3 + o\left(x^3\right). \tag{3.8}$$

If $\alpha \in [1/4, 1)$ and $p > (1-4\alpha)/3$, then (3.7) and (3.8) imply that there exists $\delta_1 = \delta_1(\alpha, p) > 0$ such that $\alpha H(1,1+x) + (1-\alpha)L(1,1+x) < M_p(1,1+x)$ for $x \in (0, \delta_1)$. If $\alpha \in (0, 3\sqrt{345}/80 - 11/16)$ and $p < (1-4\alpha)/3$, then (3.7) and (3.8) imply that there exists $\delta_2 = \delta_2(\alpha, p) > 0$ such that $\alpha H(1,1+x) + (1-\alpha)L(1,1+x) > M_p(1,1+x)$ for $x \in (0, \delta_2)$.

Finally, we prove that there exist $a_1, b_1, a_2, b_2 > 0$ with $a_1 \neq b_1$ and $a_2 \neq b_2$ such that $\alpha H(a_1, b_1) + (1-\alpha)L(a_1, b_1) < M_{(1-4\alpha)/3}(a_1, b_1)$ and $\alpha H(a_2, b_2) + (1-\alpha)L(a_2, b_2) > M_{(1-4\alpha)/3}(a_2, b_2)$ for any $3\sqrt{345}/80 - 11/16 < \alpha < 1/4$.

If $3\sqrt{345}/80 - 11/16 < \alpha < 1/4$, then from the expression of $f_2''(1)$ in (2.21) we clearly see that $f_2''(1) > 0$, which leads to the conclusion that there exists $\lambda > 1$ such that

$$f_2''(t) > 0 \tag{3.9}$$

for $t \in [1, \lambda)$.

From (2.14)–(2.18) and inequality (3.9) we know that

$$f(t) > 0 \tag{3.10}$$

for $t \in (1, \lambda)$. Equations (3.4)–(3.6) and inequality (3.10) lead to the conclusion that $\alpha H(a,b) + (1-\alpha)L(a,b) > M_{(1-4\alpha)/3}(a,b)$ for all $a/b \in (1, \lambda)$.

On the other hand, simple computations lead to

$$\begin{aligned}\lim_{x\to+\infty}&\frac{M_{(1-4\alpha)/3}(1,x)}{\alpha H(1,x)+(1-\alpha)L(1,x)}\\&=2^{3/(4\alpha-1)}\lim_{x\to+\infty}\frac{\left(1+x^{(4\alpha-1)/3}\right)^{3/(1-4\alpha)}}{2\alpha/(x+1)\ +(1-\alpha)(1-1/x)/\log x}=+\infty.\end{aligned}\tag{3.11}$$

Equation (3.11) implies that there exists $X = X(\alpha) > 1$ such that $\alpha H(a,b) + (1-\alpha)L(a,b) < M_{(1-4\alpha)/3}(a,b)$ for all $a/b \in (X,+\infty)$. □

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