Research Article

An Explicit Method for the Split Feasibility Problem with Self-Adaptive Step Sizes

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1. Introduction

Since its publication in 1994, the split feasibility problem has been studied by many authors. For some related works, please consult [1–18]. Among them, a more popular algorithm that solves the split feasibility problems is Byrne’s CQ method [2]:

\[ x_{n+1} = P_C(x_n - \tau A^*(I - P_Q)Ax_n), \]  

where \( C \) and \( Q \) are two closed convex subsets of two real Hilbert spaces \( H_1 \) and \( H_2 \), respectively, and \( A : H_1 \to H_2 \) is a bounded linear operator. The CQ algorithm only involves the computations of the projections \( P_C \) and \( P_Q \) onto the sets \( C \) and \( Q \), respectively, and is therefore implementable in the case where \( P_C \) and \( P_Q \) have closed-form expressions.

Note that CQ algorithm can be obtained from optimization. If we set

\[ f(x) := \frac{1}{2} \|Ax - P_QAx\|^2, \]  

then the convex objective \( f \) is differentiable and has a Lipschitz gradient given by

\[ \nabla f(x) = A^*(I - P_Q)A. \]
Thus, the CQ algorithm can be obtained by minimizing the following convex minimization problem

$$\min_{x \in C} f(x).$$ \hspace{1cm} (1.4)

We can use a gradient projection algorithm below to solve the split feasibility problem:

$$x_{n+1} = P_C(x_n - \tau_n \nabla f(x_n)),$$ \hspace{1cm} (1.5)

where $\tau_n$, the step size at iteration $n$, is chosen in the interval $(0, 2/L)$, where $L$ is the Lipschitz constant of $\nabla f$.

However, we observe that the determination of the step size $\tau_n$ depends on the operator (matrix) norm $\|A\|$ (or the largest eigenvalue of $A^*A$). This means that in order to implement the CQ algorithm, one has first to compute (or, at least, estimate) the matrix norm of $A$, which is in general not an easy work in practice. To overcome the above difficulty, the so-called self-adaptive method which permits step size $\tau_n$ being selected self-adaptively was developed. See, for example, [10, 14, 15, 19–23].

Inspired by the above results and the self-adaptive method, in this paper, we present an explicit iterative method with self-adaptive step sizes for solving the split feasibility problem. Convergence analysis result is given.

2. Preliminaries

Let $H_1$ and $H_2$ be two real Hilbert spaces and $C$ and $Q$ two closed convex subsets of $H_1$ and $H_2$, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator. The split feasibility problem is to find a point $x^*$ such that

$$x^* \in C, \quad Ax^* \in Q.$$ \hspace{1cm} (2.1)

Next, we use $\Gamma$ to denote the solution set of the split feasibility problem, that is, $\Gamma = \{x \in C : Ax \in Q\}$.

We know that a point $x^* \in C$ is a stationary point of problem (1.4) if it satisfies

$$\langle \nabla f(x^*), x - x^* \rangle \geq 0, \quad \forall x \in C.$$ \hspace{1cm} (2.2)

Given $x^* \in H_1$, $x^*$ solves the split feasibility problem if and only if $x^*$ solves the fixed point equation

$$x^* = P_C(x^* - \gamma A^* (I - P_Q) Ax^*).$$ \hspace{1cm} (2.3)
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Next we adopt the following notation:

(i) \( x_n \to x \) means that \( x_n \) converges strongly to \( x \);
(ii) \( x_n \rightharpoonup x \) means that \( x_n \) converges weakly to \( x \);
(iii) \( \omega_w(x_n) := \{ x : \exists x_n \to x \} \) is the weak \( \omega \)-limit set of the sequence \( \{ x_n \} \).

Recall that a function \( f : H \to \mathbb{R} \) is called convex if

\[
f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y),
\]
for all \( \lambda \in (0,1) \) and \( \forall x, y \in H \). It is known that a differentiable function \( f \) is convex if and only if there holds the relation:

\[
f(z) \geq f(x) + \langle \nabla f(x), z - x \rangle,
\]
for all \( z \in H \). Recall that an element \( g \in H \) is said to be a subgradient of \( f : H \to \mathbb{R} \) at \( x \) if

\[
f(z) \geq f(x) + \langle g, z - x \rangle,
\]
for all \( z \in H \). If the function \( f : H \to \mathbb{R} \) has at least one subgradient at \( x \) is said to be sub-differentiable at \( x \). The set of subgradients of \( f \) at the point \( x \) is called the subdifferential of \( f \) at \( x \), and is denoted by \( \partial f(x) \). A function \( f \) is called sub-differentiable if it is subdifferentiable at all \( x \in H \). If \( f \) is convex and differentiable, then its gradient and subgradient coincide. A function \( f : H \to \mathbb{R} \) is said to be weakly lower semi continuous (w-lsc) at \( x \) if \( x_n \to x \) implies

\[
f(x) \leq \liminf_{n \to \infty} f(x_n).
\]

\( f \) is said to be w-lsc on \( H \) if it is w-lsc at every point \( x \in H \).

A mapping \( T : C \to C \) is called nonexpansive if

\[
\|Tx - Ty\| \leq \|x - y\|,
\]
for all \( x, y \in C \).

Recall that the (nearest point or metric) projection from \( H \) onto \( C \), denoted \( P_C \), assigns, to each \( x \in H \), the unique point \( P_C(x) \in C \) with the property

\[
\|x - P_C(x)\| = \inf\{\|x - y\| : y \in C\}.
\]

It is well known that the metric projection \( P_C \) of \( H \) onto \( C \) has the following basic properties:

(a) \( \|P_C(x) - P_C(y)\| \leq \|x - y\| \) for all \( x, y \in H \);
(b) \( \langle x - y, P_C(x) - P_C(y) \rangle \geq \|P_C(x) - P_C(y)\|^2 \) for every \( x, y \in H \);
(c) \( \langle x - P_C(x), y - P_C(x) \rangle \leq 0 \) for all \( x \in H, y \in C \).
Lemma 2.1 (see [24]). Assume that \( \{a_n\} \) is a sequence of nonnegative real numbers such that

\[
a_{n+1} \leq (1 - \gamma_n)a_n + \delta_n,
\]

where \( \{\gamma_n\} \) is a sequence in \((0, 1)\) and \( \{\delta_n\} \) is a sequence such that

1. \( \sum_{n=1}^{\infty} \gamma_n = \infty \);
2. \( \limsup_{n \to \infty} (\delta_n / \gamma_n) \leq 0 \) or \( \sum_{n=1}^{\infty} |\delta_n| < \infty \).

Then \( \lim_{n \to \infty} a_n = 0 \).

Lemma 2.2 (see [25]). Let \( \{s_n\} \) be a sequence of real numbers that does not decrease at infinity, in the sense that there exists a subsequence \( \{s_{n_i}\} \) of \( \{s_n\} \) such that \( s_{n_i} \leq s_{n_i+1} \) for all \( i \geq 0 \). For every \( n \geq n_0 \), define an integer sequence \( \{\tau(n)\} \) as

\[
\tau(n) = \max\{k \leq n : s_{n_k} < s_{n_k+1}\}.
\]

Then \( \tau(n) \to \infty \) as \( n \to \infty \) and for all \( n \geq n_0 \)

\[
\max\{s_{\tau(n)}, s_n\} \leq s_{\tau(n)+1}.
\]

3. Main Results

In this section, we will introduce our algorithm and prove our main results.

Let \( C \) and \( Q \) be nonempty closed convex subsets of real Hilbert spaces \( H_1 \) and \( H_2 \), respectively. Let \( A : H_1 \to H_2 \) be a bounded linear operator. In the sequel, we assume that the split feasibility problem is consistent, that is \( \Gamma \neq \emptyset \).

Algorithm 3.1. For \( u \in C \) and given \( x_0 \in C \), let the sequence \( \{x_{n+1}\} \) defined by

\[
y_n = \alpha_n u + (1 - \alpha_n)x_n,
\]

\[
x_{n+1} = P_C\left( y_n - \tau_n \frac{f(y_n)\nabla f(y_n)}{\|\nabla f(y_n)\|^2} \right), \quad n \geq 0,
\]

where \( \{\alpha_n\} \subset (0, 1) \) and \( \{\tau_n\} \subset (0, 2) \).

Remark 3.2. In the sequel, we may assume that \( \nabla f(y_n) \neq 0 \) for all \( n \). Note that this fact can be guaranteed if the sequence \( \{y_n\} \) is infinite; that is, Algorithm 3.1 does not terminate in a finite number of iterations.

Theorem 3.3. Assume that the following conditions are satisfied:

(i) \( \lim_{n \to \infty} \alpha_n = 0 \) and \( \sum_{n=1}^{\infty} \alpha_n = \infty \);
(ii) \( \inf_n \tau_n (2 - \tau_n) > 0 \).

Then \( \{x_n\} \) defined by (3.1) converges strongly to \( P_{\Gamma}(u) \).
Proof. Let $\nu \in \Gamma$. It follows that $\nabla f(\nu) = 0$ for all $\nu \in \Gamma$. From (2.5), we deduce that

$$f(y_n) = f(y_n) - f(\nu) \leq \langle \nabla f(y_n), y_n - \nu \rangle. \quad (3.2)$$

Thus, by (3.1) and (3.2), we have

$$\|x_{n+1} - \nu\|^2 = \left\| P_{C} \left( y_n - \tau_n \frac{f(y_n)\nabla f(y_n)}{\|\nabla f(y_n)\|^2} \right) - \nu \right\|^2$$

$$\leq \left\| y_n - \tau_n \frac{f(y_n)\nabla f(y_n)}{\|\nabla f(y_n)\|^2} \right\|^2 - \nu \right\|^2$$

$$= \|y_n - \nu\|^2 - 2\tau_n \frac{f(y_n)}{\|\nabla f(y_n)\|^2} \langle \nabla f(y_n), y_n - \nu \rangle + \tau_n^2 \frac{f^2(y_n)}{\|\nabla f(y_n)\|^2}$$

$$\leq \|y_n - \nu\|^2 - 2\tau_n \frac{f^2(y_n)}{\|\nabla f(y_n)\|^2} + \tau_n^2 \|\nabla f(y_n)\|^2$$

$$\leq \|y_n - \nu\|^2 - \tau_n (2 - \tau_n) \frac{f^2(y_n)}{\|\nabla f(y_n)\|^2},$$

$$\|y_n - \nu\|^2 = \|\alpha_n(u - \nu) + (1 - \alpha_n)(x_n - \nu)\|^2 \leq \alpha_n\|u - \nu\|^2 + (1 - \alpha_n)\|x_n - \nu\|^2.$$ 

It follows that

$$\|x_{n+1} - \nu\|^2 \leq \alpha_n\|u - \nu\|^2 + (1 - \alpha_n)\|x_n - \nu\|^2 - \tau_n (2 - \tau_n) \frac{f^2(y_n)}{\|\nabla f(y_n)\|^2}$$

$$\leq \alpha_n\|u - \nu\|^2 + (1 - \alpha_n)\|x_n - \nu\|^2$$

$$\leq \max \left\{ \|u - \nu\|^2, \|x_n - \nu\|^2 \right\}. \quad (3.4)$$

By induction, we deduce

$$\|x_{n+1} - \nu\| \leq \max \{\|u - \nu\|, \|x_0 - \nu\|\}. \quad (3.5)$$

Hence, $\{x_n\}$ is bounded.

At the same time, we note that

$$\|y_n - \nu\|^2 = \|\alpha_n(u - \nu) + (1 - \alpha_n)(x_n - \nu)\|^2 \leq (1 - \alpha_n)\|x_n - \nu\|^2 + 2\alpha_n\langle u - \nu, y_n - \nu \rangle. \quad (3.6)$$
Therefore,

\[
\|x_{n+1} - v\|^2 \leq (1 - \alpha_n)\|x_n - v\|^2 + 2\alpha_n \langle u - v, y_n - v \rangle - \tau_n(2 - \tau_n) \frac{f^2(y_n)}{\|\nabla f(y_n)\|^2} \\
= (1 - \alpha_n)\|x_n - v\|^2 + 2\alpha_n \langle u - v, \alpha_n(u - v) + (1 - \alpha_n)(x_n - v) \rangle \\
- \tau_n(2 - \tau_n) \frac{f^2(y_n)}{\|\nabla f(y_n)\|^2} \\
= (1 - \alpha_n)\|x_n - v\|^2 + 2\alpha_n^2\|u - v\|^2 + 2\alpha_n(1 - \alpha_n)\langle u - v, x_n - v \rangle \\
- \tau_n(2 - \tau_n) \frac{f^2(y_n)}{\|\nabla f(y_n)\|^2}.
\]

(3.7)

It follows that

\[
\|x_{n+1} - v\|^2 - \|x_n - v\|^2 + \alpha_n \left(\|x_n - v\|^2 - 2\alpha_n\|u - v\|^2 + 2\alpha_n(u - v, x_n - v)\right) \\
+ \tau_n(2 - \tau_n) \frac{f^2(y_n)}{\|\nabla f(y_n)\|^2} \leq 2\alpha_n(u - v, x_n - v).
\]

(3.8)

Next, we will prove that \(x_n \to v\). Set \(\omega_n = \|x_n - v\|^2\) for all \(n \geq 0\). Since \(\alpha_n \to 0\) and \(\inf_n \tau_n(2 - \tau_n) > 0\), we may assume without loss of generality that \(\tau_n(2 - \tau_n) \geq \sigma\) for some \(\sigma > 0\). Thus, we can rewrite (3.8) as

\[
\omega_{n+1} - \omega_n + \alpha_n U_n + \frac{\sigma f^2(y_n)}{\|\nabla f(y_n)\|^2} \leq 2\alpha_n(u - v, x_n - v),
\]

(3.9)

where \(U_n = \|x_n - v\|^2 - 2\alpha_n\|u - v\|^2 + 2\alpha_n(u - v, x_n - v)\).

Now, we consider two possible cases.

**Case 1.** Assume that \(\{\omega_n\}\) is eventually decreasing; that is, there exists \(N > 0\) such that \(\{\omega_n\}\) is decreasing for \(n \geq N\). In this case, \(\{\omega_n\}\) must be convergent and from (3.9) it follows that

\[
0 \leq \frac{\sigma f^2(y_n)}{\|\nabla f(y_n)\|^2} \leq \omega_n - \omega_{n+1} + \alpha_n U_n + 2\alpha_n\|u - v\|\|x_n - v\| \leq \omega_n - \omega_{n+1} + M\alpha_n,
\]

(3.10)

where \(M > 0\) is a constant such that \(\sup_n \{2\|u - v\|\|x_n - v\| + \|U_n\|\} \leq M\). Letting \(n \to \infty\) in (3.10), we get

\[
\lim_{n \to \infty} f(y_n) = 0.
\]

(3.11)
Since $\{y_n\}$ is bounded, there exists a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ converging weakly to $\tilde{x} \in C$. Since, $x_n - y_n \to 0$, we also have $\{x_{n_k}\}$ of $\{x_n\}$ converging weakly to $\tilde{x} \in C$. From the weak lower semicontinuity of $f$, we have

$$0 \leq f(\tilde{x}) \leq \liminf_{k \to \infty} f(y_{n_k}) = \lim_{n \to \infty} f(y_n) = 0. \tag{3.12}$$

Hence, $f(\tilde{x}) = 0$; that is, $A\tilde{x} \in Q$. This indicates that

$$\omega_w(y_n) = \omega_w(x_n) \subset \Gamma. \tag{3.13}$$

Furthermore, by using the property of the projection $(c)$, we deduce

$$\limsup_{n \to \infty} \langle u - \nu, x_n - \nu \rangle = \max_{\tilde{x} \in \omega_w(x_n)} \langle u - P_\Gamma(u), \tilde{x} - P_\Gamma(u) \rangle \leq 0. \tag{3.14}$$

From (3.8), we obtain

$$\omega_{n+1} \leq (1 - \alpha_n)\omega_n + \alpha_n \left(2\alpha_n\|u - \nu\|^2 + 2(1 - \alpha_n)\langle u - \nu, x_n - \nu \rangle \right). \tag{3.15}$$

This together with Lemma 2.1 imply that $\omega_n \to 0$.

**Case 2.** Assume that $\{\omega_n\}$ is not eventually decreasing. That is, there exists an integer $n_0$ such that $\omega_{n_0} \leq \omega_{n_0 + 1}$. Thus, we can define an integer sequence $\{\tau_n\}$ for all $n \geq n_0$ as follows:

$$\tau(n) = \max\{k \in \mathbb{N} \mid n_0 \leq k \leq n, \omega_k \leq \omega_{k+1} \}. \tag{3.16}$$

Clearly, $\tau(n)$ is a nondecreasing sequence such that $\tau(n) \to +\infty$ as $n \to \infty$ and

$$\omega_{\tau(n)} \leq \omega_{\tau(n)+1}, \tag{3.17}$$

for all $n \geq n_0$. In this case, we derive from (3.10) that

$$\frac{\sigma f^2(y_{\tau(n)})}{\|\nabla f(y_{\tau(n)})\|^2} \leq M\alpha_{\tau(n)} \to 0. \tag{3.18}$$

It follows that

$$\lim_{n \to \infty} f(y_{\tau(n)}) = 0. \tag{3.19}$$
This implies that every weak cluster point of \( \{ y_{\tau(n)} \} \) is in the solution set \( \Gamma \); that is, \( \omega_w(y_{\tau(n)}) \subset \Gamma \). So, \( \omega_w(x_{\tau(n)}) \subset \Gamma \). On the other hand, we note that

\[
\| y_{\tau(n)} - x_{\tau(n)} \| = \alpha_{\tau(n)} \| u - x_{\tau(n)} \| \to 0,
\]

\[
\| x_{\tau(n)+1} - y_{\tau(n)} \| \leq \frac{\tau_{\tau(n)} f(y_{\tau(n)})}{\| \nabla f(y_{\tau(n)}) \|} \to 0.
\]

From which we can deduce that

\[
\limsup_{n \to \infty} \langle u - \nu, x_{\tau(n)} - \nu \rangle = \max_{\tilde{x} \in \omega_w(x_{\tau(n)})} \langle u - P_{\Gamma}(u), \tilde{x} - P_{\Gamma}(u) \rangle \leq 0.
\]

Since \( \omega_{\tau(n)} \leq \omega_{\tau(n)+1} \), we have from (3.9) that

\[
\omega_{\tau(n)} \leq (1 - 2\alpha_{\tau(n)}) \langle u - \nu, x_{\tau(n)} - \nu \rangle + 2\alpha_{\tau(n)} \| u - \nu \|^2.
\]

Combining (3.21) and (3.22) yields

\[
\limsup_{n \to \infty} \omega_{\tau(n)} \leq 0,
\]

and hence

\[
\lim_{n \to \infty} \omega_{\tau(n)} = 0.
\]

From (3.15), we have

\[
\limsup_{n \to \infty} \omega_{\tau(n)+1} \leq \limsup_{n \to \infty} \omega_{\tau(n)}.
\]

Thus,

\[
\lim_{n \to \infty} \omega_{\tau(n)+1} = 0.
\]

From Lemma 2.2, we have

\[
0 \leq \omega_n \leq \max\{ \omega_{\tau(n)}, \omega_{\tau(n)+1} \}.
\]

Therefore, \( \omega_n \to 0 \). That is, \( x_n \to \nu \). This completes the proof.

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References