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Research Article

(λ, μ)-Fuzzy Version of Ideals, Interior Ideals, Quasi-Ideals, and Bi-Ideals

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We introduced (λ, μ)-fuzzy ideals, (λ, μ)-fuzzy interior ideals, (λ, μ)-fuzzy quasi-ideals, and (λ, μ)-fuzzy bi-ideals of an ordered semigroup and studied them. When $\lambda = 0$ and $\mu = 1$, we meet the ordinary fuzzy ones. This paper can be seen as a generalization of Kehayopulu and Tsingelis (2006), Kehayopulu and Tsingelis (2007), and Yao (2009).

1. Introduction and Preliminaries

An ideal of a semigroup is a special subsemigroup satisfying certain conditions. The best way to know an algebraic structure is to begin with a special substructure of it. There are plenty of papers on ideals. After Zadeh' introduction of fuzzy set in 1965 (see [1]), the fuzzy sets have been used in the reconsideration of classical mathematics. Also, fuzzy ideals have been considered by many researchers. For example, Kim [2] studied intuitionistic fuzzy ideals of semigroups, Meng and Guo [3] researched fuzzy ideals of BCK/BCI-algebras, Koguep [4] researched fuzzy ideals of hyperlattices, and Kehayopulu and Tsingelis [5] researched fuzzy interior ideals of ordered semigroups.

Recently, Yuan et al. [6] introduced the concept of fuzzy subfield with thresholds. A fuzzy subfield with thresholds λ and μ is also called a (λ, μ)-fuzzy subfield. Yao continued to research (λ, μ)-fuzzy normal subfields, (λ, μ)-fuzzy quotient subfields, (λ, μ)-fuzzy subrings, and (λ, μ)-fuzzy ideals in [7–10]. Feng et al. researched (λ, μ)-fuzzy sublattices and (λ, μ)-fuzzy subhyperlattices in [11].

An ordered semigroup $(S, \circ, \leq)$ is a poset $(S, \leq)$ equipped with a binary operation $\circ$, such that

(1) $(S, \circ)$ is a semigroup, and

(2) if $x, a, b \in S$, then

$$a \leq b \Rightarrow \begin{cases} a \circ x \leq b \circ x \\ x \circ a \leq x \circ b. \end{cases} \tag{1.1}$$

Given an ordered semigroup S, a fuzzy subset of S (or a fuzzy set in S) is an arbitrary mapping $f : S \to [0,1]$, where $[0,1]$ is the usual closed interval of real numbers. For any $\alpha \in [0,1]$, f_α is defined by $f_\alpha = \{x \in S \mid f(x) \geq \alpha\}$. For $a \in S$, we define that $A_\alpha = \{(y,z) \in S \times S \mid a \leq yz\}$. For two fuzzy subsets f and g of S, we define the multiplication of f and g as the fuzzy subset of S defined by

$$(f * g)(a) = \begin{cases} \sup\limits_{(y,z) \in A_a} (f(y) \wedge g(z)), & \text{if } A_a \neq \emptyset, \\ 0, & \text{if } A_a = \emptyset. \end{cases} \tag{1.2}$$

In the set of fuzzy subsets of S, we define the order relation as follows: $f \subseteq g$ if and only if $f(x) \leq g(x)$ for all $x \in S$. For two fuzzy subsets f and g of S, we define

$$(f \cap g)(x) = f(x) \wedge g(x), \qquad (f \cup g)(x) = f(x) \vee g(x). \tag{1.3}$$

Note that we use $a \wedge b$ to denote $\min(a,b)$ and use $a \vee b$ to denote $\max(a,b)$.

For any $\alpha \in [0,1]$, α can be seen as a fuzzy subset of S which is defined by $\alpha(x) = \alpha$, for all $x \in S$.

In the following, we will use S or $(S, \circ, \leq)$ to denote an ordered semigroup and the multiplication of x, y will be xy instead of $x \circ y$.

In the rest of this paper, we will always assume that $0 \leq \lambda < \mu \leq 1$.

In this paper, we introduced (λ, μ)-fuzzy ideals, (λ, μ)-fuzzy interior ideals, (λ, μ)-fuzzy quasi-ideals and (λ, μ)-fuzzy bi-ideals of an ordered semigroup. We obtained the followings:

(1) in an ordered semigroup, every (λ, μ)-fuzzy ideal is a (λ, μ)-fuzzy interior ideal;

(2) in an ordered semigroup, every (λ, μ)-fuzzy right (resp. left) ideal is a (λ, μ)-fuzzy quasi-ideal;

(3) in an ordered semigroup, every (λ, μ)-fuzzy quasi-ideal is a (λ, μ)-fuzzy bi-ideal;

(4) in a regular ordered semigroup, the (λ, μ)-fuzzy quasi-ideals and the (λ, μ)-fuzzy bi-ideals coincide.

2. (λ, μ)-Fuzzy Ideals and (λ, μ)-Fuzzy Interior Ideals

Definition 2.1. Let $(S, \cdot, \leq)$ be an ordered semigroup. A fuzzy subset f of S is called a (λ, μ)-*fuzzy right ideal* (resp. (λ, μ)-*fuzzy left ideal*) of S if

(1) $f(xy) \vee \lambda \geq f(x) \wedge \mu$ (resp. $f(xy) \vee \lambda \geq f(y) \wedge \mu$) for all $x, y \in S$, and

(2) if $x \leq y$, then $f(x) \vee \lambda \geq f(y) \wedge \mu$ for all $x, y \in S$.

A fuzzy subset f of S is called a (λ, μ)-fuzzy ideal of S if it is both a (λ, μ)-fuzzy right and a (λ, μ)-fuzzy left ideal of S.

Example 2.2. Let $(S, *, \leq)$ be an ordered semigroup where $S = \{e, a, b\}$ and $e \leq a \leq b$. The multiplication table is defined by the following:

$*$	e	a	b
e	e	a	b
a	a	b	e
b	b	e	a

(2.1)

A fuzzy set f is defined as follows:

S	e	a	b
f	0.1	0.2	0.3

(2.2)

Then, f is a $(0.3, 0.7)$-fuzzy ideal of S. But it is not a fuzzy ideal of S.

Definition 2.3 (see [12]). If $(S, \circ, \leq)$ is an ordered semigroup, a nonempty subset A of S is called an interior ideal of S if

(1) $SAS \subseteq A$, and

(2) if $a \in A, b \in S$, and $b \leq a$, then $b \in A$.

Definition 2.4. If $(S, \circ, \leq)$ is an ordered semigroup, a fuzzy subset f of S is called a (λ, μ)-fuzzy interior ideal of S if

(1) $f(xay) \vee \lambda \geq f(a) \wedge \mu$ for all $x, a, y \in S$, and

(2) if $x \leq y$, then $f(x) \vee \lambda \geq f(y) \wedge \mu$.

In the previous example, f is also a $(0.3, 0.7)$-fuzzy interior ideal of S. In fact, every fuzzy ideal of an ordered semigroup is a fuzzy interior.

Theorem 2.5. *Let $(S, \circ, \leq)$ be an ordered semigroup and f a (λ, μ)-fuzzy ideal of S, then f is a (λ, μ)-fuzzy interior ideal of S.*

Proof. Let $x, a, y \in S$. Since f is a (λ, μ)-fuzzy left ideal of S and $x, ay \in S$, we have

$$f(x(ay)) \vee \lambda \geq f(ay) \wedge \mu. \tag{2.3}$$

Since f is a (λ, μ)-fuzzy right ideal of S, we have

$$f(ay) \vee \lambda \geq f(a) \wedge \mu. \tag{2.4}$$

From (2.3) and (2.4) we know that $f(xay) \vee \lambda = (f(x(ay)) \vee \lambda) \vee \lambda \geq (f(ay) \wedge \mu) \vee \lambda = (f(ay) \vee \lambda) \wedge (\mu \vee \lambda) \geq f(a) \wedge \mu$. □

Theorem 2.6. *Let $(S, \circ, \leq)$ be an ordered semigroup, then f is a (λ, μ)-fuzzy interior ideal of S if and only if f_α is an interior ideal of S for all $\alpha \in (\lambda, \mu]$.*

Proof. Let f be a (λ, μ)-fuzzy interior ideal of S and $\alpha \in (\lambda, \mu]$.

First of all, we need to show that $xay \in f_\alpha$, for all $a \in f_\alpha$, $x, y \in S$.

From $f(xay) \vee \lambda \geq f(a) \wedge \mu \geq \alpha \wedge \mu = \alpha$ and $\lambda < \alpha$, we conclude that $f(xay) \geq \alpha$, that is, $xay \in f_\alpha$.

Then, we need to show that $b \in f_\alpha$ for all $a \in f_\alpha, b \in S$ such that $b \leq a$.

From $b \leq a$ we know that $f(b) \vee \lambda \geq f(a) \wedge \mu$ and from $a \in f_\alpha$ we have $f(a) \geq \alpha$. Thus, $f(b) \vee \lambda \geq \alpha \wedge \mu = \alpha$. Notice that $\lambda < \alpha$, then we conclude that $f(b) \geq \alpha$, that is, $b \in f_\alpha$.

Conversely, let f_α be an interior ideal of S for all $\alpha \in (\lambda, \mu]$.

If there are $x_0, a_0, y_0 \in S$, such that $f(x_0 a_0 y_0) \vee \lambda < \alpha = f(a_0) \wedge \mu$, then $\alpha \in (\lambda, \mu], f(a_0) \geq \alpha$ and $f(x_0 a_0 y_0) < \alpha$. That is $a_0 \in f_\alpha$ and $x_0 a_0 y_0 \notin f_\alpha$. This is a contradiction with that f_α is an interior ideal of S. Hence $f(xay) \vee \lambda \geq f(a) \wedge \mu$ holds for all $x, a, y \in S$.

If there are $x_0, y_0 \in S$ such that $x_0 \leq y_0$ and $f(x_0) \vee \lambda < \alpha = f(y_0) \wedge \mu$, then $\alpha \in (\lambda, \mu], f(y_0) \geq \alpha$, and $f(x_0) < \alpha$, that is, $y_0 \in f_\alpha$ and $x_0 \notin f_\alpha$. This is a contradiction with that f_α is an interior ideal of S. Hence if $x \leq y$, then $f(x) \vee \lambda \geq f(y) \wedge \mu$. □

3. (λ, μ)-Fuzzy Quasi-Ideals and (λ, μ)-Fuzzy Bi-Ideals

Definition 3.1. Let $(S, \circ, \leq)$ be an ordered semigroup. A subset A of S is called a quasi-ideal of S if

(1) $AS \cap SA \subseteq S$, and

(2) if $x \in S$ and $x \leq y \in A$, then $x \in A$.

Definition 3.2. A nonempty subset A of an ordered semigroup S is called a bi-ideal of S if it satisfies

(1) $ASA \subseteq A$, and

(2) $x \in S$ and $x \leq y \in A$, then $x \in A$.

Definition 3.3. Let $(S, \circ, \leq)$ be an ordered semigroup. A fuzzy subset f of S is called a (λ, μ)-fuzzy quasi-ideal of S if

(1) $f \cup \lambda \supseteq (f * 1) \cap (1 * f) \cap \mu$, and

(2) if $x \leq y$, then $f(x) \vee \lambda \geq f(y) \wedge \mu$ for all $x, y \in S$.

Definition 3.4. Let $(S, \circ, \leq)$ be an ordered semigroup. A fuzzy subset f of S is called a (λ, μ)-fuzzy bi-ideal of S if for all $x, y, z \in S$,

(1) $f(xyz) \vee \lambda \geq (f(x) \wedge f(z)) \wedge \mu$, and

(2) if $x \leq y$, then $f(x) \vee \lambda \geq f(y) \wedge \mu$.

Remark 3.5. It is easy to see that a fuzzy quasi-ideal [13] of S is a $(0,1)$-fuzzy quasi-ideal of S, and a fuzzy bi-ideal [13] of S is a $(0,1)$-fuzzy bi-idealof S.

Theorem 3.6. *Let $(S, \circ, \leq)$ be an ordered semigroup, then f is a (λ, μ)-fuzzy quasi-ideal of S if and only if f_α is a quasi-ideal of S for all $\alpha \in (\lambda, \mu]$.*

Proof. Let f be a (λ, μ)-fuzzy quasi-ideal of S and $\alpha \in (\lambda, \mu]$.

First of all, we need to show that $Sf_\alpha \cap f_\alpha S \subseteq f_\alpha$.

If $x \in Sf_\alpha \cap f_\alpha S$, then $x = st_1 = t_2 s$ for some $t_1, t_2 \in f_\alpha$ and $s \in S$.

From $f \cup \lambda \supseteq (f*1) \cap (1*f) \cap \mu$, we conclude that $f(x) \vee \lambda \geq (f*1)(x) \wedge (f*1)(x) \wedge \mu \geq f(t_1) \wedge f(t_2) \wedge \mu \geq \alpha \wedge \mu = \alpha$. Thus, $f(x) \geq \alpha$, and so $x \in f_\alpha$. Hence, $S * f_\alpha \cap f_\alpha * S \subseteq f_\alpha$.

Next, we need to show that $b \in f_\alpha$ for all $a \in f_\alpha, b \in S$ such that $b \leq a$.

From $b \leq a$ we know that $f(b) \vee \lambda \geq f(a) \wedge \mu$ and from $a \in f_\alpha$ we have $f(a) \geq \alpha$. Thus, $f(b) \vee \lambda \geq \alpha \wedge \mu = \alpha$. Notice that $\lambda < \alpha$, we conclude that $f(b) \geq \alpha$, that is, $b \in f_\alpha$.

Conversely, let f_α be a quasi-ideal of S for all $\alpha \in (\lambda, \mu]$. Then, $f_\alpha S \cap Sf_\alpha \subseteq f_\alpha$.

If there is $x_0 \in S$, such that $f(x_0) \vee \lambda < \alpha = (f*1)(x) \wedge (1*f)(x) \wedge \mu$, then $\alpha \in (\lambda, \mu]$, $f(x_0) < \alpha$, $(f*1)(x_0) \geq \alpha$ and $(1*f)(x_0) \geq \alpha$. That is $x_0 \notin f_\alpha$, $\sup_{x_0 \leq x_1 x_2} f(x_1) \geq \alpha$ and $\sup_{x_0 \leq x_1 x_2} f(x_2) \geq \alpha$.

From $f_\alpha S \cap Sf_\alpha \subseteq f_\alpha$ and $x_0 \notin f_\alpha$, we obtain that $x_0 \notin f_\alpha S \cap Sf_\alpha$.

From $\sup_{x_0 \leq x_1 x_2} f(x_1) \geq \alpha$ and $\alpha \neq 0$, we know that there exists at least one pair $(x_1, x_2) \in S \times S$ such that $x_0 \leq x_1 x_2$ and $f(x_1) \geq \alpha$. Thus, $x_0 \leq x_1 x_2 \in f_\alpha S$. Hence, $x_0 \in f_\alpha S$.

Similarly, we can prove that $x_0 \in Sf_\alpha$.

So $x_0 \in f_\alpha S \cap Sf_\alpha$. This is a contradiction.

Hence, $f \cup \lambda \supseteq (f*1) \cap (1*f) \cap \mu$ holds.

If there are $x_0, y_0 \in S$ such that $x_0 \leq y_0$ and $f(x_0) \vee \lambda < \alpha = f(y_0) \wedge \mu$, then $\alpha \in (\lambda, \mu]$, $f(y_0) \geq \alpha$ and $f(x_0) < \alpha$, that is, $y_0 \in f_\alpha$ and $x_0 \notin f_\alpha$. This is a contradiction with that f_α is a quasi-ideal of S. Hence if $x \leq y$, then $f(x) \vee \lambda \geq f(y) \wedge \mu$. □

Theorem 3.7. *Let $(S, \circ, \leq)$ be an ordered semigroup, then f is a (λ, μ)-fuzzy bi-ideal of S if and only if f_α is a bi-ideal of S for all $\alpha \in (\lambda, \mu]$.*

Proof. The proof of this theorem is similar to the proof of the previous theorem. □

Theorem 3.8. *Let $(S, \circ, \leq)$ be an ordered semigroup, then the (λ, μ)-fuzzy right (resp. left) ideals of S are (λ, μ)-fuzzy quasi-ideals of S.*

Proof. Let f be a (λ, μ)-fuzzy right ideal of S and $x \in S$. First we have

$$((f*1) \cap (1*f))(x) = (f*1)(x) \wedge (1*f)(x). \tag{3.1}$$

If $A_x = \emptyset$, then we have $(f*1)(x) = 0 = (1*f)(x)$. So $f(x) \vee \lambda \geq 0 = (f*1)(x) \wedge (1*f)(x) \wedge \mu$. Thus, $f \cup \lambda \supseteq (f*1) \cap (1*f) \cap \mu$.

If $A_x \neq \emptyset$, then

$$(f*1)(x) = \sup_{(u,v) \in A_x} (f(u) \wedge 1(v)). \tag{3.2}$$

On the other hand, $f(x) \vee \lambda \geq f(u) \wedge 1(v) \wedge \mu$, for all $(u, v) \in A_x$.

Indeed, if $(u, v) \in A_x$, then $x \leq uv$, thus $f(x) \vee \lambda = f(x) \vee \lambda \vee \lambda \geq (f(uv) \wedge \mu) \vee \lambda = (f(uv) \vee \lambda) \wedge (\lambda \vee \mu) \geq (f(u) \wedge \mu) \wedge \mu = f(u) \wedge \mu = f(u) \wedge 1(v) \wedge \mu$.

Hence, we have that $f(x) \vee \lambda \geq (\sup_{(u,v)\in A_x}(f(u) \wedge 1(v))) \wedge \mu = (f * 1)(x) \wedge \mu \geq (f * 1)(x) \wedge (1 * f)(x) \wedge \mu$. Thus, $f \cup \lambda \supseteq (f * 1) \cap (1 * f) \cap \mu$.

Therefore, f is a (λ, μ)-fuzzy quasi-ideal of S. □

Theorem 3.9. *Let $(S, \circ, \leq)$ be an ordered semigroup, then the (λ, μ)-fuzzy quasi-ideals of S are (λ, μ)-fuzzy bi-ideals of S.*

Proof. Let f be a (λ, μ)-fuzzy quasi-ideal of S and $x, y, z \in S$. Then we have that

$$f(xyz) \vee \lambda \geq (f * 1)(xyz) \wedge (1 * f)(xyz) \wedge \mu. \tag{3.3}$$

From $(x, yz) \in A_{xyz}$, we have that $(f * 1)(xyz) \geq f(x) \wedge 1(yz) = f(x)$.

From $(xy, z) \in A_{xyz}$, we have that $(1 * f)(xyz) \geq 1(xy) \wedge f(z) = f(z)$.

Thus, $f(xyz) \vee \lambda \geq f(x) \wedge f(z) \wedge \mu$.

Therefore, f is a (λ, μ)-fuzzy bi-ideal of S. □

Definition 3.10 (see [5]). An ordered semigroup $(S, \circ, \leq)$ is called regular if for all $a \in S$ there exists $x \in S$ such that $a \leq axa$.

Theorem 3.11. *In a regular ordered semigroup S, the (λ, μ)-fuzzy quasi-ideals and the (λ, μ)-fuzzy bi-ideals coincide.*

Proof. Let f be a (λ, μ)-fuzzy bi-ideal of S and $x \in S$. We need to prove that

$$f(x) \vee \lambda \geq (f * 1)(x) \wedge (1 * f) \wedge \mu. \tag{3.4}$$

If $A_x = \emptyset$, it is easy to verify that condition (3.4) is satisfied.

Let $A_x \neq \emptyset$.

(1) If $(f * 1)(x) \wedge \mu \leq f(x) \vee \lambda$, then we have that $f(x) \vee \lambda \geq (f * 1)(x) \wedge \mu \geq (f * 1)(x) \wedge (1 * f)(x) \wedge \mu$. Thus, condition (3.4) is satisfied.

(2) If $(f * 1)(x) \wedge \mu > f(x) \vee \lambda$, then there exists at least one pair $(z, w) \in A_x$ such that $f(z) \wedge 1(w) \wedge \mu > f(x) \vee \lambda$. That is $z, w \in S$, $x \leq zw$ and $f(z) \wedge \mu > f(x) \vee \lambda$.

We will prove that $(1 * f)(x) \wedge \mu \leq f(x) \vee \lambda$. Then, $f(x) \vee \lambda \geq (1 * f)(x) \wedge \mu \geq (f * 1)(x) \wedge (1 * f)(x) \wedge \mu$, and condition (3.4) is satisfied.

For any $(u, v) \in A_x$, we need to show that $1(u) \wedge f(v) \wedge \mu \leq f(x) \vee \lambda$.

Let $(u, v) \in A_x$, then $x \leq uv$ for some $u, v \in S$. Since S is regular, there exists $s \in S$ such that $x \leq xsx$.

From $x \leq xsx$, $x \leq zw$ and $x \leq uv$, we obtain that $x \leq zwsuv$. Since f is a (λ, μ)-fuzzy bi-ideal of S, we have that

$$f(x) \vee \lambda \geq (f(zwsuv) \wedge \mu) \vee \lambda = (f(zwsuv) \vee \lambda) \wedge (\mu \vee \lambda) \geq f(z) \wedge f(v) \wedge \mu. \tag{3.5}$$

Note that $f(z) \wedge \mu > f(x) \vee \lambda$. Thus, $f(x) \vee \lambda \geq f(v) \wedge \mu = 1(u) \wedge f(v) \wedge \mu$. □

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References

[1] L. A. Zadeh, "Fuzzy sets," *Information and Computation*, vol. 8, pp. 338–353, 1965.

[2] K. H. Kim, "Intuitionistic fuzzy ideals of semigroups," *Indian Journal of Pure and Applied Mathematics*, vol. 33, no. 4, pp. 443–449, 2002.

[3] J. Meng and X. Guo, "On fuzzy ideals of BCK/BCI-algebras," *Fuzzy Sets and Systems*, vol. 149, no. 3, pp. 509–525, 2005.

[4] B. B. N. Koguep, "On fuzzy ideals of hyperlattice," *International Journal of Algebra*, vol. 2, no. 15, pp. 739–750, 2008.

[5] N. Kehayopulu and M. Tsingelis, "Fuzzy interior ideals in ordered semigroups," *Lobachevskii Journal of Mathematics*, vol. 21, pp. 65–71, 2006.

[6] X. Yuan, C. Zhang, and Y. Ren, "Generalized fuzzy groups and many-valued implications," *Fuzzy Sets and Systems*, vol. 138, no. 1, pp. 205–211, 2003.

[7] B. Yao, "(λ,μ)-fuzzy normal subgroups and (λ,μ)-fuzzy quotient subgroups," *Journal of Fuzzy Mathematics*, vol. 13, no. 3, pp. 695–705, 2005.

[8] B. Yao, "(λ,μ)-fuzzy subrings and (λ,μ)-fuzzy ideals," *Journal of Fuzzy Mathematics*, vol. 15, no. 4, pp. 981–987, 2007.

[9] B. Yao, *Fuzzy Theory on Group and Ring*, Science and Technology Press, Beijing, China, 2008.

[10] B. Yao, "(λ,μ)-fuzzy ideals in semigroups," *Fuzzy Systems and Mathematics*, vol. 23, no. 1, pp. 123–127, 2009.

[11] Y. Feng, H. Duan, and Q. Zeng, "(λ,μ)-fuzzy sublattices and (λ,μ)-fuzzy sub- hyperlattices," *Fuzzy Information and Engineering*, vol. 78, pp. 17–26, 2010.

[12] N. Kehayopulu, "Note on interior ideals, ideal elements in ordered semigroups," *Scientiae Mathematicae*, vol. 2, no. 3, pp. 407–409, 1999.

[13] N. Kehayopulu and M. Tsingelis, "Fuzzy ideals in ordered semigroups," *Quasigroups and Related Systems*, vol. 15, no. 2, pp. 185–195, 2007.