Hindawi Publishing Corporation
International Journal of Differential Equations
Volume 2012, Article ID 417261, 16 pages
doi:10.1155/2012/417261

Review Article

Life Span of Positive Solutions for the Cauchy Problem for the Parabolic Equations

Yusuke Yamauchi

Department of Applied Physics, Waseda University, 3-4-1 Okubo, Shinjuku-ku, Tokyo 169-8555, Japan

Correspondence should be addressed to Yusuke Yamauchi, yamauchi@aoni.waseda.jp

Received 15 December 2011; Accepted 30 January 2012

Academic Editor: Ebrahim Momoniat

Since 1960's, the blow-up phenomena for the Fujita type parabolic equation have been investigated by many researchers. In this survey paper, we discuss various results on the life span of positive solutions for several superlinear parabolic problems. In the last section, we introduce a recent result by the author.

1. Introduction

1.1. Fujita Type Results

We first recall the result on the Cauchy problem for a semilinear heat equation:

$$\begin{gathered}\frac{\partial u}{\partial t} = \Delta u + u^p, \quad (x,t) \in \mathbf{R}^n \times (0,\infty), \\ u(x,0) = \phi(x) \geq 0, \quad x \in \mathbf{R}^n,\end{gathered} \tag{1.1}$$

where $n \in \mathbf{N}$, Δ is the n-dimensional Laplacian, and $p > 1$. Let ϕ be a bounded continuous function on $\mathbf{R}^n$.

In pioneer work [1], Fujita showed that the exponent $p_F = 1 + 2/n$ plays the crucial role for the existence and nonexistence of the solutions of (1.1). Let G denote the Gaussian heat kernel: $G_t(x) = (4\pi t)^{-n/2} \exp(-|x|^2/4t)$.

Theorem 1.1 (see [1]). *Suppose that $\phi \in C^2(\mathbf{R}^n)$ and that its all derivatives are bounded.*

(i) *Let $p < p_F$. Then there is no global solution of* (1.1) *satisfying that*

$$u(x,t) \leq M \exp\left(|x|^{\beta}\right), \tag{1.2}$$

for $M > 0$ and $0 < \beta < 2$.

(ii) *Let $p > p_F$. Then for any $\tau > 0$ there exists $\delta > 0$ with the following property: if*

$$\phi(x) \leq \delta G_{\tau}(x), \tag{1.3}$$

then there exists a global solution of (1.1) *satisfying*

$$u(x,t) \leq M \exp\left(|x|^{\beta}\right) \tag{1.4}$$

for $M > 0$ and $0 < \beta < 2$.

In [2], Hayakawa showed first that there is no global solution of (1.1) in the critical case $p = p_F$ when $n = 1$ or 2.

Theorem 1.2 (see [2]). *In case of $n = 2$, $p = p_F = 2$ or $n = 1$, $p = p_F = 3$,* (1.1) *has no global solutions for any nontrivial initial data.*

In genaral space dimensions, Kobayashi et al. [3] consider the following problem:

$$\begin{gathered}\frac{\partial u}{\partial t} = \Delta u + f(u), \quad (x,t) \in \mathbf{R}^n \times (0,\infty), \\ u(x,0) = \phi(x) \geq 0, \quad x \in \mathbf{R}^n,\end{gathered} \tag{1.5}$$

where $n \in \mathbf{N}$ and $p > 1$. Let ϕ be a bounded continuous function on $\mathbf{R}^n$.

Theorem 1.3 (see [3]). *Suppose that f satisfies the following three conditions:*

(a) *f is a locally Lipschitz continuous and nondecreasing function in $[0,\infty)$ with $f(0) = 0$ and $f(\lambda) > 0$ for $\lambda > 0$,*

(b) *$\int_{0+}^{\varepsilon} f(\lambda)/\lambda^{2+(2+n)} d\lambda$ for some $\varepsilon > 0$,*

(c) *there exists a positive constant $c \leq 1$ such that*

$$\begin{gathered}f(\lambda\mu) \geq \mu^{1+(2+n)} f(\lambda) \quad (0 < \lambda \leq \mu, \lambda < c), \\ f(\lambda\mu) \geq \mu^{2+(2+n)} f(\lambda) \quad (0 < \mu \leq \lambda < c).\end{gathered} \tag{1.6}$$

Then each positive solution of (1.5) *blows up in finite time.*

Remark 1.4. (i) We remark that the proofs of the theorems in [2, 3] are mainly based on the iterated estimate from below obtained by the following integral equation:

$$u(x,t) = (G_t * \phi)(x) + \int_0^t (G_{t-s} * u^p(s))(x)ds. \tag{1.7}$$

(ii) The critical nonlinearity of power type $f(u) = u^{1+2/n}$ satisfies the assumptions (a), (b), and (c) in [3].

Weissler proved the nonexistence of global solution in L^p-framework in [4]. The proof is quite short and elegant.

Theorem 1.5 (see [4]). *Suppose $p = p_F$ and that $\phi \geq 0$ in $L^q(\mathbf{R}^n)$ $(q \geq 1)$ is not identically zero. Then there is no nonnegative global solution $u : [0,\infty) \to L^q$ to the integral* (1.7) *with initial value ϕ.*

The outline of the proof is as follows. First we assume that there is a global solution. From the fact that the solution $u(\varepsilon, x) \geq kG_\alpha(x)(k, \alpha > 0)$ for some $\varepsilon > 0$, we can obtain that $\|u(t)\|_{L^1(\mathbf{R}^n)} \geq C(k,p,n)\int_0^t (s+\alpha)^{-1}ds$. This contradicts the boundedness of $\|u(t)\|_{L^1(\mathbf{R}^n)}$ for large $t > 0$. Hence the solution is not global.

Existence and nonexistence results for time-global solutions of (1.1) are summarized as follows.

(i) Let $p \in (1, p_F]$. Then every nontrivial solution of (1.1) blows up in finite time.

(ii) Let $p \in (p_F, \infty)$. Then (1.1) has a time-global classical solution for small initial data ϕ and has a blowing-up solution for large initial data ϕ.

In several papers [5–7], the results for the critical exponent are extended to the more general equations.

In [6], Qi consider the following Cauchy problem of porous medium equation:

$$\begin{gathered} \frac{\partial u}{\partial t} = \Delta u^m + t^s|x|^\sigma u^p, \quad (x,t) \in \mathbf{R}^n \times (0,\infty), \\ u(x,0) = \phi(x) \geq 0, \quad x \in \mathbf{R}^n, \end{gathered} \tag{1.8}$$

where $m > \max\{(n-2)/n, 0\}$, $\sigma > \max\{-n,-2\}$ and $p > \max\{m,1\}$. They showed that the critical exponent is $p_{c_1} := m + (m-1)s + (2+2s+\sigma)/n$.

Theorem 1.6 (see [6]). (i) *If $1 < p \leq p_{c_1}$, then every nontrivial solution of* (1.8) *blows up in finite time.*

(ii) *If $p > p_{c_1}$, then* (1.8) *has global classical solutions for small initial data.*

Another extension is the following quasilinear parabolic equation:

$$\begin{gathered} \frac{\partial u}{\partial t} = \operatorname{div}\left(|\nabla u|^{m-1}\nabla u\right) + t^s|x|^\sigma u^p, \quad (x,t) \in \mathbf{R}^n \times (0,\infty), \\ u(x,0) = \phi(x) \geq 0, \quad x \in \mathbf{R}^n, \end{gathered} \tag{1.9}$$

where $(n-1)(n+1) < m < 1$, $s \geq 0$, $p > 1$ and $\sigma > n(1-m) - (1+m+2s)$. In [7], the authors showed that the critical exponent is $p_{c_2} := m + (1+m+2s+\sigma)/n$.

Theorem 1.7 (see [7]). (i) *If* $1 < p \leq p_{c_2}$, *then every nontrivial solution of the* (1.9) *blows up in finite time.*

(ii) *If* $p > p_{c_2}$, *then* (1.9) *has global classical solutions for small initial data.*

Remark 1.8. (i) (Sublinear case). In [8], Aguirre and Escobedo proved that if $0 < p < 1$, then every solution of (1.1) is global.

(ii) (System of equations). There are various extensions of Fujita type results to the system of equations. See, for example, papers [9–13], surveys [14, 15], and references therein. In [9, 13], the authors investigated the following system:

$$\begin{gathered}\frac{\partial u}{\partial t} = \Delta u + |x|^{\sigma_1} u^{p_{11}} v^{p_{12}}, \quad (x,t) \in \mathbf{R}^n \times (0,\infty), \\ \frac{\partial v}{\partial t} = \Delta v + |x|^{\sigma_1} u^{p_{21}} v^{p_{22}}, \quad (x,t) \in \mathbf{R}^n \times (0,\infty), \\ u(x,0) = \phi(x) \geq 0, \quad x \in \mathbf{R}^n, \\ v(x,0) = \psi(x) \geq 0, \quad x \in \mathbf{R}^n,\end{gathered} \tag{1.10}$$

where $\sigma_j > \max\{-2, -n\}$ and $p_{jk} \geq 0$ $(j, k = 1, 2)$.

1.2. Blow-up Results for Slowly Decaying Initial Data

Especially for slowly decaying initial data, it was shown that the solution of (1.1) blows up in finite time for any $p > 1$. In [16] Lee and Ni showed that a sufficient condition for finite time blows up on the decay order of initial data. We note that the slow decay of initial data in all directions was assumed. Here, let μ_R be the first Dirichlet eigenvalue of $-\Delta$ in the ball B_R.

Theorem 1.9 (see [16]). *The solution of* (1.1) *blows up in finite time if*

$$\liminf_{x \to \infty} |x|^{2/(p-1)} \phi(x) > \mu_1^{1/(p-1)}. \tag{1.11}$$

We put $\Omega = \{(r,\omega) \in (0,\infty) \times \mathbf{S}^{n-1};\ r > R,\ d(\omega,\omega_0) < cr^{-\mu}\}$ for some $R > 0$, $c > 0$, $\omega_0 \in \mathbf{S}^{n-1}$, and $0 \leq \mu < 1$, where $d(\cdot,\cdot)$ denotes the usual distance on the unit sphere $\mathbf{S}^{n-1}$. Mizoguchi and Yanagida [17] showed that a sufficient condition for finite time blows up on the decay order of initial data in Ω. The authors consider the following problem:

$$\begin{gathered}\frac{\partial u}{\partial t} = \Delta u + |u|^{p-1} u, \quad (x,t) \in \mathbf{R}^n \times (0,\infty), \\ u(x,0) = \phi(x), \quad x \in \mathbf{R}^n,\end{gathered} \tag{1.12}$$

where $n \in \mathbf{N}$ and $p > 1$. The following results indicate that the slow decay of initial data in all directions is not necessary for finite time blow-up.

Theorem 1.10 (see [17]). (i) *Assume that initial data ϕ may change sign. Suppose that $\phi \in W^{1,\infty}(\mathbf{R}^n)$ satisfies the following:*

$$\phi \geq K_1 r^{-\alpha} \quad \text{in } \Omega \text{ for some } \alpha > 0, \;\; K_1 > 0,$$
$$\left|\nabla \phi\right| \geq K_2 r^{-\alpha-1+\mu} \quad \text{in } \Omega \text{ for some } \alpha > 0, \;\; K_2 > 0, \tag{1.13}$$

with $0 < \alpha < 2(1-\mu)/(p-1)$. Then the solution of (1.12) *blows up in finite time.*

(ii) *(Positive solutions). Assume that initial data ϕ is nonnegative. Suppose that $\phi \in L^\infty(\mathbf{R}^n)$ satisfies that*

$$\phi \geq K_1 r^{-\alpha} \quad \text{in } \Omega \text{ for some } \alpha > 0 \;\; K_1 > 0, \tag{1.14}$$

with $0 < \alpha < 2(1-\mu)/(p-1)$. Then the solution of (1.1) *blows up in finite time.*

Remark 1.11. (i) From the theorem, in particular, for nondecaying initial data the solution of (1.1) blows up in finite time.

(ii) For sign changing solutions, Fujita type results are discussed in [18–22], for instance.

(iii) Many blow-up results, for instance [1, 5–7, 12, 13], are based on Kaplan's method (the eigenfunction method). See [23].

In the remainder of this paper, we discuss various studies for the lifespan of the positive solutions of the parabolic problems. In Sections 2 and 3, we introduce the results of asymptotics of life span with respect to the size of initial data and to the size of diffusion constant, and the results of minimal time blow up, respectively. In Section 4, we shall show an upper bound of the life span of the solution for (1.1).

2. Asymptotics of Life Span

2.1. Life Span for the Equation with Large or Small Initial Data

Recently, several studies have been made on the life span of solutions for (1.1). See [16, 19, 24–38], and references therein. In this section, we mainly consider the following Cauchy problems:

$$\frac{\partial u}{\partial t} = \Delta u + u^p, \quad (x,t) \in \mathbf{R}^n \times (0,\infty),$$
$$u(x,0) = \lambda\psi(x) \geq 0, \quad x \in \mathbf{R}^n, \tag{2.1}$$

$$\frac{\partial u}{\partial t} = \Delta u + |u|^{p-1}u, \quad (x,t) \in \mathbf{R}^n \times (0,\infty),$$
$$u(x,0) = \lambda\psi(x), \quad x \in \mathbf{R}^n, \tag{2.2}$$

where $n \in \mathbf{N}$, $p > 1$. Let ψ be a bounded continuous function on $\mathbf{R}^n$ and λ be a positive parameter.

In this paper, we define the lifespan (or blow-up time) $T_{\max}$ as

$$T_{\max} := \sup\{T > 0 \mid . \text{ The problem possesses a unique classical solution in } \mathbf{R}^n \times [0, T)\}. \tag{2.3}$$

We first introduce the result about asymptotic behavior of the life span $T_{\max}(\lambda)$ for (2.1) as large or small λ by Lee and Ni [16]. The authors showed the order of the asymptotics of the life span.

Theorem 2.1 (see [16]). *Assume that φ is nonnegative.*

(i) *There exist constants $C_1 > 0$ and $C_2 > 0$ such that $C_1\lambda^{1-p} \le T_{\max}(\lambda) \le C_2\lambda^{1-p}$ for large λ.*

(ii) *If $\liminf_{|x|\to\infty}\varphi(x) > 0$, then there exists constants $C_1 > 0$ and $C_2 > 0$ such that $C_1\lambda^{1-p} \le T_{\max}(\lambda) \le C_2\lambda^{1-p}$ for small λ.*

In [29], Gui and Wang obtained more detailed information of the asymptotics for (2.1). They showed that for large λ the supremum of initial data ϕ is dominant in the asymptotics, and that for small λ the limiting value of ϕ at space infinity is dominant.

Theorem 2.2 (see [29]). *Assume that φ is nonnegative.*

(i) *We have*

$$\lim_{\lambda\to\infty} T_{\max}(\lambda)\cdot\lambda^{p-1} = \frac{1}{p-1}\|\varphi\|_{L^\infty(\mathbf{R}^n)}^{1-p}. \tag{2.4}$$

(ii) *If $\lim_{|x|\to\infty}\varphi(x) = \varphi_\infty > 0$, then*

$$\lim_{\lambda\to 0} T_{\max}(\lambda)\cdot\lambda^{p-1} = \frac{1}{p-1}\varphi_\infty^{1-p}. \tag{2.5}$$

It is noteworthy that the limiting values as $\lambda \to 0$ and $\lambda \to \infty$ in the theorem are different. The proof of the theorem is also based on Kaplan's method, and the assumption $\lim_{|x|\to\infty}\varphi(x) = \varphi_\infty$ plays an important role in the proof.

Thereafter, the results in [16, 29] are extended by several authors. Pinsky considered the following problem:

$$\begin{gathered}\frac{\partial u}{\partial t} = \Delta u + a(x)u^p, \quad (x,t) \in \mathbf{R}^n \times (0,\infty),\\ u(x,0) = \lambda\varphi(x) \ge 0, \quad x \in \mathbf{R}^n,\end{gathered} \tag{2.6}$$

where $a(x) \geq 0$ is Hölder continuous, $0 \leq \varphi \in BC(\mathbf{R}^n)$, $\varphi \not\equiv 0$, $p > 1$ and $\lambda > 0$ is a parameter. The author treated the initial data in the following two classes:

$$\begin{aligned} &\text{Class S } 0 \leq \varphi(x) \leq \delta \exp\left(-\gamma |x|^2\right), \qquad \text{where } \delta, \gamma > 0. \\ &\text{Class L } c_1 \leq \varphi(x) \leq c_2, \qquad \text{where } c_1, c_2 > 0. \end{aligned} \tag{2.7}$$

Here, we introduce the results in [34] with $a(x) \sim |x|^m$ $(|x| \gg 1)$ for $m > \max\{-2, -n\}$.

Theorem 2.3 (see [34] Class S for small λ). *Assume that φ belongs to Class S.*

(i) *Let $p \in (1, 1 + (2+m)/n)$. Then*

$$T_{\max}(\lambda) \sim \lambda^{2(1-p)/(2+m-n(1-p))}, \qquad \text{as } \lambda \longrightarrow 0. \tag{2.8}$$

(ii) *Let $p = 1 + (2+m)/n$. Then there exists a constant $c_1 > 0$ and for every $\varepsilon > 0$, a constant $c_2 > 0$ such that*

$$c_1 \lambda^{-(2+m)/n} \leq \log T_{\max}(\lambda) \leq c_2 \lambda^{-\varepsilon-(2+m)/n}, \qquad \text{for small } \lambda. \tag{2.9}$$

Theorem 2.4 (see [34] Class L for small λ). *Assume that φ belongs to Class L. Then*

$$T_{\max}(\lambda) \sim \lambda^{2(1-p)/(2+m)}, \qquad \text{as } \lambda \longrightarrow 0. \tag{2.10}$$

Theorem 2.5 (see [34], for large λ). *Assume that a is bounded and let φ be arbitrary initial data.*

(i) *If there exists an $x_0 \in \mathbf{R}^n$ such that $a(x_0), \varphi(x_0) > 0$, then*

$$T_{\max}(\lambda) \sim \lambda^{1-p}, \qquad \text{as } \lambda \longrightarrow \infty. \tag{2.11}$$

(ii) *If $\operatorname{dist}(\operatorname{supp}(a), \operatorname{supp}(\varphi)) > 0$, then*

$$T_{\max}(\lambda) \sim (\log \lambda)^{-1}, \qquad \text{as } \lambda \longrightarrow \infty. \tag{2.12}$$

Kobayashi extended the results to the following system of equations:

$$\begin{aligned} \frac{\partial u}{\partial t} &= \Delta u + a(x) v^p, \quad (x,t) \in \mathbf{R}^n \times (0,\infty), \\ \frac{\partial v}{\partial t} &= \Delta v + b(x) u^q, \quad (x,t) \in \mathbf{R}^n \times (0,\infty), \\ u(x,0) &= \lambda^{\mu} \phi(x), \quad x \in \mathbf{R}^n, \\ v(x,0) &= \lambda^{\nu} \psi(x), \quad x \in \mathbf{R}^n, \end{aligned} \tag{2.13}$$

where $p, q > 1$, $\mu, \nu > 0$, and $\lambda > 0$ are parameters. See [30].

On the other hand, Mukai et al. showed the results for the following equation of porous medium type in [31]:

$$\frac{\partial u}{\partial t} = \Delta u^m + u^p, \quad (x,t) \in \mathbf{R}^n \times (0,\infty),$$
$$u(x,0) = \lambda\varphi(x), \quad x \in \mathbf{R}^n, \tag{2.14}$$

where $1 < m < p$, and φ is a nonnegative and bounded function.

Theorem 2.6 (see [31]). (i) *Let $p > m$. Suppose that $\|\varphi\|_{L^\infty(\mathbf{R}^n)} = \varphi(0) > 0$. Then*

$$\lim_{\lambda\to\infty} \lambda^{p-1} T_{\max}(\lambda) = \frac{1}{p-1}\varphi(0)^{1-p}. \tag{2.15}$$

(ii) *Let $p > m$. Suppose that $\|\varphi\|_{L^\infty(\mathbf{R}^n)} = \lim_{|x|\to\infty}\varphi(x) > 0$. Then*

$$\lim_{\lambda\to 0} \lambda^{p-1} T_{\max}(\lambda) = \frac{1}{p-1}\|\varphi\|^{1-p}_{L^\infty(\mathbf{R}^n)}. \tag{2.16}$$

(iii) *Assume that $a \geq 0$. Let $m < p < m + 2/n$ or $a < 2/(p-m)$. Suppose that $\varphi(x) \sim |x|^{-a}$ for large $|x|$. Then*

$$T_{\max}(\lambda) \sim \lambda^{(1-p)/(1-\min\{a,n\}(p-m)/2)} \quad \text{as } \lambda \longrightarrow 0. \tag{2.17}$$

Now we turn to the problems (2.1) and (2.2). In [39], Mizoguchi and Yanagida determined the higher-order term of the life span $T_{\max}(\lambda)$ for (2.2) as $\lambda \to \infty$. The authors introduced the following function space:

$$D = \left\{\varphi \in BC(\mathbf{R}^n) : \exp\left(-c|x|^2\right)\nabla\varphi \in L^2(\mathbf{R}^n) \text{ for some } c > 0\right\}. \tag{2.18}$$

Theorem 2.7 (see [39]). *Suppose that $\varphi \in D$ satisfies the following assumptions.*

(A1) *$|\varphi|$ attains its maximum at some point $x = a$, and φ satisfies that*

$$\left|\varphi(x) - \varphi(a)\right| = |x-a|^k \tilde{\varphi}(x) + o\left(|x-a|^{k+2}\right), \tag{2.19}$$

at $x = a$ with some $k > -2$, where $\tilde{\varphi}$ is twice continuously differentiable at $x = a$ and satisfies $\tilde{\varphi}(a) = 0$, $\nabla\tilde{\varphi}(a) = 0$ and $\Delta\tilde{\varphi}(a) \geq 0$.

(A2) *There exist $R > 0$ and $\delta > 0$ such that*

$$\left|\varphi(x)\right| < \|\varphi\|_{L^\infty(\mathbf{R}^n)} - \delta \quad \forall |x| > R. \tag{2.20}$$

If $|\psi|$ attains its maximum at only one point $x = a$, then

$$\begin{aligned} T_{\max}(\lambda) &= \frac{1}{p-1}\|\psi\|_{L^\infty(\mathbf{R}^n)}^{1-p}\lambda^{1-p} \\ &\quad + K\|\psi\|_{L^\infty(\mathbf{R}^n)}^{(2+k/2)(1-p)-1}\Delta\widetilde{\psi}(a)\lambda^{(2+k/2)(1-p)} + o\left(\lambda^{(2+k/2)(1-p)}\right), \end{aligned} \tag{2.21}$$

as $\lambda \to \infty$ with

$$K = \frac{2^{k+1}\Gamma((k+n+2)/2)}{(p-1)^{1+k/2}\Gamma((n+2)/2)}. \tag{2.22}$$

In particular, if ψ is smooth at $x = a$, then

$$\begin{aligned} T_{\max}(\lambda) &= \frac{1}{p-1}\|\psi\|_{L^\infty(\mathbf{R}^n)}^{1-p}\lambda^{1-p} \\ &\quad + \frac{2}{p-1}\|\psi\|_{L^\infty(\mathbf{R}^n)}^{2(1-p)-1}\Delta\widetilde{\psi}(a)\lambda^{2(1-p)} + o\left(\lambda^{2(1-p)}\right), \end{aligned} \tag{2.23}$$

as $\lambda \to \infty$.

2.2. Life Span for the Equation with Large Diffusion

We shall consider the following Cauchy problem:

$$\begin{aligned} \frac{\partial u}{\partial t} &= D\Delta u + |u|^{p-1}u, \quad (x,t) \in \mathbf{R}^n \times (0,\infty), \\ u(x,0) &= \lambda + \phi(x), \qquad x \in \mathbf{R}^n, \end{aligned} \tag{2.24}$$

where $D > 0$, $p > 1$, $n \geq 3$, $\lambda > 0$, and $\phi \in L^\infty(\mathbf{R}^n) \cap L^1(\mathbf{R}^n, (1+|x|)^2 dx)$.

In [24, 25] Fujishima and Ishige obtained the asymptotics of the life span $T_{\max}(D)$ of the solution of (2.24) as $D \to \infty$. The situation is divided into three cases:

$$\int_{\mathbf{R}^n} \phi(x)dx > 0, \qquad \int_{\mathbf{R}^n} \phi(x)dx = 0, \qquad \int_{\mathbf{R}^n} \phi(x)dx < 0. \tag{2.25}$$

We prepare some notation. Put the following:

$$M(\phi) := \int_{\mathbf{R}^n} \phi(x)dx, \qquad \Xi(\phi) := \int_{\mathbf{R}^n} x\phi(x)dx, \qquad S_\lambda := \frac{\lambda^{1-p}}{p-1}. \tag{2.26}$$

Theorem 2.8 (see [24, 25]). (i) *Assume that* $M(\phi) > 0$. *Then* $T_{\max}(D) \le S_\lambda$ *for any* $D > 0$, *and*

$$S_\lambda - T_{\max}(D) = (4\pi S_\lambda)^{-n/2}\lambda^{-p}D^{-n/2}\left[M(\phi) + O\left(D^{-1}\right)\right] \tag{2.27}$$

as $D \to \infty$.

(ii) *Assume that* $M(\phi) = 0$. *Then* $T_{\max}(D) \le S_\lambda$ *for any* $D > 0$, *and*

$$S_\lambda - T_{\max}(D) = \frac{(4\pi S_\lambda)^{-n/2}}{\lambda^p\sqrt{2eS_\lambda}}D^{(-n-1)/2}\left[\Xi(\phi) + O\left(D^{-1/2}\right)\right], \tag{2.28}$$

as $D \to \infty$.

(iii) *Assume that* $M(\phi) < 0$. *Then* $T_{\max}(D) \le S_\lambda$ *for any* $D > 0$, *and*

$$S_\lambda - T_{\max}(D) = O\left(D^{-n/2-1}\right), \tag{2.29}$$

as $D \to \infty$.

Remark 2.9. (i) Changing variable, we can easily see that the problem with large diffusion is equivalent to the equation with small initial data (cf. [29]. Theorem 1.1(ii)).

(ii) In [24, 25], the behavior of the blow-up set of the solution as $D \to \infty$ was also studied.

3. Minimal Time Blow-up Results

3.1. Minimal Time Blow-up Results

In this section, we discuss the life span for the following parabolic equations (cf. [26–28, 35]):

$$\begin{gathered} \frac{\partial u}{\partial t} = \Delta u + f(u), \quad (x,t) \in \mathbf{R}^n \times (0,\infty), \\ u(x,0) = \phi(x) \ge 0, \quad x \in \mathbf{R}^n, \end{gathered} \tag{3.1}$$

where ϕ is a bounded continuous function on $\mathbf{R}^n$. Suppose that

$$\begin{gathered} f \text{ is locally Lipschitz function in } [0,\infty), \\ f(\xi) > 0 \ (\xi > 0), \\ \int_1^\infty \frac{d\xi}{f(\xi)} < \infty. \end{gathered} \tag{3.2}$$

Applying the comparison principle to (3.1), we always have

$$T_{\max} \ge \int_{\|\phi\|_{L^\infty(\mathbf{R}^n)}}^\infty \frac{d\xi}{f(\xi)}. \tag{3.3}$$

When $f(u) = u^p$, we always have

$$T_{\max} \geq \frac{1}{p-1} \|\phi\|_{L^\infty(\mathbf{R}^n)}^{1-p}. \tag{3.4}$$

A solution u to (3.1) with initial data ϕ is said to *blow up at minimal blow-up time* provided that

$$T_{\max} = \int_{\|\phi\|_{L^\infty(\mathbf{R}^n)}}^{\infty} \frac{d\xi}{f(\xi)}. \tag{3.5}$$

We put $\rho(x) := e^{-|x|} / (\int_{\mathbf{R}^n} e^{-|y|} dy)$ and $A_\rho(x;\phi) := \int_{\mathbf{R}^n} \rho(y-x)\phi(y)dy$. The necessary and sufficient conditions of initial data ϕ for blowing up at minimal blow-up time are following.

Theorem 3.1 (see [26]). *Let u be a solution of* (3.1). *Assume that there exist constants $\xi_0 > 0$ and $p > 1$ such that $f(\xi)/\xi^p$ is nondecreasing for $\xi \geq \xi_0$. Then u blows up at minimal blow-up time if and only if one of the following two conditions for initial data ϕ holds*

$$\begin{gathered} \text{There exists a sequence } \{x_n\} \subset \mathbf{R}^n \text{ such that} \\ |x_n| \longrightarrow \infty \text{ and } \phi(x + x_n) \longrightarrow \|\phi\|_{L^\infty(\mathbf{R}^n)} \text{ a.e. in } \mathbf{R}^n \text{ as } n \longrightarrow \infty; \end{gathered} \tag{3.6}$$

$$\sup_{x \in \mathbf{R}^n} A_\rho(x;\phi) = \|\phi\|_{L^\infty(\mathbf{R}^n)}. \tag{3.7}$$

In [36], Seki et al. consider the following quasilinear equations:

$$\begin{gathered} \frac{\partial u}{\partial t} = \Delta\Phi(u) + f(u), \quad (x,t) \in \mathbf{R}^n \times (0,\infty), \\ u(x,0) = \phi(x) \geq 0, \quad x \in \mathbf{R}^n, \end{gathered} \tag{3.8}$$

where ϕ is a bounded continuous function on $\mathbf{R}^n$. Suppose that

$$\begin{gathered} \Phi(\xi), \qquad f(\xi) \in C^1[0,\infty) \cap C^\infty(0,\infty), \\ \Phi(\xi) > 0, \quad \Phi'(\xi) > 0, \quad \Phi''(\xi) \geq 0 \ (\xi > 0), \\ f(\xi) > 0 \quad (\xi > 0), \\ \Phi(0) = 0. \end{gathered} \tag{3.9}$$

The authors proved that if there exist a function $\Psi(\eta)$ and constants $c > 0$ and $\eta_1 > 0$ such that

$$\Psi(\eta) > 0, \quad \Psi'(\eta) \geq 0, \quad \Psi''(\eta) \geq 0 \quad (\eta > \eta 1),$$
$$\int_{\eta 1+1}^{\infty} \frac{d\eta}{\Psi(\eta)} < \infty, \tag{3.10}$$
$$\left\{ f\left(\Phi^{-1}(\eta)\right)\right\}' \Psi(\eta) - f\left(\Phi^{-1}(\eta)\right)\Psi'(\eta) \geq c\Psi(\eta)\Psi'(\eta) \quad (\eta > \eta 1),$$

where $\xi = \Phi^{-1}(\eta)$ is the inverse function of $\eta = \Phi(\xi)$, then the same result as the previous theorem holds. For example, the result can be applied to the equation $u_t = \Delta u^m + u^p (p > m)$.

4. Estimates of Lifespan

4.1. Upper Bound of the Iife Span

In this section, we shall show an upper bound of the life span of positive solutions of the Cauchy problem for a semilinear heat equation:

$$\frac{\partial u}{\partial t} = \Delta u + f(u), \quad (x,t) \in \mathbf{R}^n \times (0,\infty),$$
$$u(x,0) = \phi(x) \geq 0, \quad x \in \mathbf{R}^n, \tag{4.1}$$

where $n \in \mathbf{N}$, and ϕ is a bounded continuous function on $\mathbf{R}^n$. We assume that $F(u)$ satisfies

$$f(u) \geq u^p \quad \text{for } u \geq 0, \tag{4.2}$$

with $p > 1$.

In order to state the results, we prepare several notations. For $\xi' \in \mathbf{S}^{n-1}$, and $\delta \in (0, \sqrt{2})$, we set neighborhood $S_{\xi'}(\delta)$:

$$S_{\xi'}(\delta) := \left\{ \eta' \in \mathbf{S}^{n-1};\ \left|\eta' - \xi'\right| < \delta \right\}. \tag{4.3}$$

Define

$$M_\infty := \sup_{\xi' \in \mathbf{S}^{n-1}, \delta > 0} \left\{ \operatorname*{ess.inf}_{x' \in S_{\xi'}(\delta)} \left(\liminf_{r \to +\infty} \phi(rx') \right) \right\}. \tag{4.4}$$

Theorem 4.1 (cf. [33, 37, 38]). *Let $n \geq 2$. Assume that $M_\infty > 0$. Then the classical solution for* (1.1) *blows up in finite time, and the blow-up time is estimated as follows:*

$$T_{\max} \leq \frac{1}{p-1} M_\infty^{1-p}. \tag{4.5}$$

This result allows us to remove the assumption of Gui and Wang [29] on uniformity of initial data at space infinity; $\lim_{|x|\to\infty}\psi(x) = \psi_\infty$, and to give information of the life span for initial data of intermediate size.

Here, we show some examples of the values M_∞ and the initial data ϕ in space dimensions $n = 2$. For simplicity, we employ polar coordinates.

(1) $\phi(r,\theta) = 1 - \exp(-r^2)$.
Since $\liminf_{r\to+\infty}\phi(rx') = 1$, we have $M_\infty = 1$.

(2) $\phi(r,\theta) = \{1 - \exp(-r^2)\}(1 + \cos\theta)$.
Since $\liminf_{r\to+\infty}\phi(rx') = 1 + \cos\theta$, we have $M_\infty = 2$.

(3) $\phi(r,\theta) = \{1 - \exp(-r^2)\}(1 + \cos\theta)(2 - \cos r)$.
Since $\liminf_{r\to+\infty}\phi(rx') = 1 + \cos\theta$, we have $M_\infty = 2$.

Once we admit the theorem, we can prove the following corollary immediately.

Corollary 4.2. *Let $n \geq 2$. Suppose that $M_\infty = \|\phi\|_{L^\infty(\mathbf{R}^n)}$. Then the solution u blows up at minimal blow-up time.*

Remark 4.3. For the examples of the initial data 1 and 2, $\|\phi\|_{L^\infty(\mathbf{R}^n)} = M_\infty$ holds. Hence, the solutions blow up at minimal blow-up time. However, for the example 3, we cannot specify the life span $T_{\max}$.

Outline of the Proof of the Theorem

The proof is based on a slight modification of Kaplan's method. We first prepare the sequence $\{w_j(t)\}$. For $\xi' \in \mathbf{S}^{n-1}$ and $\delta > 0$, we first determine the sequences $\{a_j\} \subset \mathbf{R}^n$ and $\{R_j\} \subset (0,\infty)$. Let $\{a_j\} \subset \mathbf{R}^n$ be a sequence satisfying that $|a_j| \to \infty$ as $j \to \infty$, and that $a_j/|a_j| = \xi'$ for any $j \in \mathbf{N}$. Put $R_j = (\delta\sqrt{4-\delta^2}/2)|a_j|$. For $R_j > 0$, let ρ_{R_j} be the first eigenfunction of $-\Delta$ on $B_{R_j}(0) = \{x \in \mathbf{R}^n;\ |x| < R_j\}$ with zero Dirichlet boundary condition under the normalization $\int_{B_{R_j}(0)} \rho_{R_j}(x)dx = 1$. Moreover, let μ_{R_j} be the corresponding first eigenvalue. For the solutions for (1.1), define

$$w_j(t) := \int_{B_{R_j}(0)} u(x + a_j, t)\rho_{R_j}(x)dx. \tag{4.6}$$

Now we introduce the following two lemmas.

Lemma 4.4 (see [13]). *The blow-up time of w_j is estimated from above as follows:*

$$T^*_{w_j} \leq \frac{\log\left(1 - \mu_{R_j} w_j^{1-p}(0)\right)}{-(p-1)\mu_{R_j}} \tag{4.7}$$

for large j.

Lemma 4.5 (cf. [13]). (i) *We have*

$$\liminf_{j\to+\infty} w_j(0) \geq \operatorname*{ess.inf}_{x'\in S_{\xi'}(\delta)} \phi_\infty(x'). \tag{4.8}$$

(ii) *We have*

$$\lim_{j\to+\infty} \frac{\log\left(1-\mu_{R_j} w_j^{1-p}(0)\right)}{-\mu_{R_j} w_j^{1-p}(0)} = 1. \tag{4.9}$$

From the definition of $w_j(t)$, $T_{\max} \leq T^*_{w_j}$ holds for large j. Using the lemmas, we obtain that

$$\begin{aligned}
T_{\max} &\leq \limsup_{j+\to\infty} T^*_{w_j} \\
&\leq \limsup_{j+\to\infty} \frac{\log\left(1-\mu_{R_j} w_j^{1-p}(0)\right)}{-(p-1)\mu_{R_j}} \\
&= \frac{1}{p-1} \lim_{j\to+\infty} \frac{\log\left(1-\mu_{R_j} w_j^{1-p}(0)\right)}{-\mu_{R_j} w_j^{1-p}(0)} \cdot \left(\liminf_{j\to+\infty} w_j(0)\right)^{1-p} \\
&\leq \frac{1}{p-1}\left(\operatorname*{ess.inf}_{x'\in S_{\xi'}(\delta)} \phi_\infty(x')\right)^{1-p} \\
&\leq \frac{1}{p-1} M_\infty^{1-p}.
\end{aligned} \tag{4.10}$$

This completes the proof.

Remark 4.6. For the problem (1.1), it is well known that

$$\begin{aligned}
\overline{u}(x,t) &= (e^{t\Delta}\phi)(x)\left[1-(p-1)\int_0^t \left\|e^{s\Delta}\phi\right\|_{L^\infty(\mathbf{R}^n)}^{p-1} ds\right]^{-1/(p-1)}, \\
\underline{u}(x,t) &= \left(\left(e^{t\Delta}\phi\right)(x)^{1-p}-(p-1)t\right)^{-1/(p-1)}
\end{aligned} \tag{4.11}$$

are a supersolution and a subsolution, respectively, where $e^{t\Delta}\phi = \int_{\mathbf{R}^n} G_t(x-y)\phi(y)dx$. Hence, we note that if there exist constants $T_1, T_2 > 0$ satisfying

$$\begin{aligned}
1 &= (p-1)\int_0^{T_1} \left\|e^{s\Delta}\phi\right\|_{L^\infty(\mathbf{R}^n)}^{p-1} ds, \\
\left\|e^{T_2\Delta}\phi\right\|_{L^\infty(\mathbf{R}^n)}^{1-p} &= (p-1)T_2,
\end{aligned} \tag{4.12}$$

then we have $T_1 \leq T_{\max} \leq T_2$.

At last, we introduce the result in $n = 1$. The proof does work in the same way as in $n \geq 2$.

Theorem 4.7 (see [13]). *Let $n = 1$. Assume that*

$$\max\left\{\liminf_{x\to-\infty} \phi(x), \quad \liminf_{x\to+\infty} \phi(x)\right\} > 0. \tag{4.13}$$

Then the classical solution for (1.1) *blows up in finite time, and the blow-up time is estimated as follows:*

$$T_{\max} \leq \frac{1}{p-1}\left(\max\left\{\liminf_{x\to-\infty} \phi(x), \quad \liminf_{x\to+\infty} \phi(x)\right\}\right)^{1-p}. \tag{4.14}$$

Acknowledgment

The author would like to thank Professor Tohru Ozawa for his helpful advice and suggestions.

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