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Research Article

A Note on *tvs*-G-Cone Metric Fixed Point Theory

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For a *tvs*-G-cone metric space (X, G) and for the family $\mathcal{A}$ of subsets of X, we introduce a new notion of the *tvs*-$\mathcal{H}$-cone metric $\mathcal{H}$ with respect to G, and we get a fixed result for the $\mathcal{CBW}$-*tvs*-G-cone-type function in a complete *tvs*-G-cone metric space $(\mathcal{A}, \mathcal{H})$. Our results generalize some recent results in the literature.

1. Introduction and Preliminaries

In 2007, Huang and Zhang [1] introduced the concept of cone metric space by replacing the set of real numbers by an ordered Banach space, and they showed some fixed point theorems of contractive-type mappings on cone metric spaces. The category of cone metric spaces is larger than metric spaces. Subsequently, many authors studied this subject and many results on fixed point theory are proved (see, e.g., [2–15]). Recently, Du [16] introduced the concept of *tvs*-cone metric and *tvs*-cone metric space to improve and extend the concept of cone metric space in the sense of Huang and Zhang [1]. Later, in the papers [16–21], the authors tried to generalize this approach by using cones in topological vector spaces *tvs* instead of Banach spaces. However, it should be noted that an old result shows that if the underlying cone of an ordered *tvs* is solid and normal, then such *tvs* must be an ordered normed space. Thus, proper generalizations when passing from norm-valued cone metric spaces to *tvs*-valued cone metric spaces can be obtained only in the case of nonnormal cones (for details, see [19]). Further, Radenović et al. [22] introduced the concept of set-valued contraction of Nadler type in the setting of *tvs*-cone spaces and proved a fixed point theorem in the setting of *tvs*-cone spaces with respect to a solid cone.

Definition 1.1 (see [22]). Let (X, d) be a *tvs*-cone metric space with a solid cone P, and let $\mathcal{A}$ be a collection of nonempty subsets of X. A map $\mathcal{H} : \mathcal{A} \times \mathcal{A} \to E$ is called a *tvs*-$\mathcal{H}$-cone metric with respect to d if for any $A_1, A_2 \in \mathcal{A}$ the following conditions hold:

(H_1) $\mathcal{H}(A_1, A_2) = \theta \Rightarrow A_1 = A_2$,

(H_2) $\mathcal{H}(A_1, A_2) = \mathcal{H}(A_2, A_1)$,

(H_3) $\forall_{\varepsilon \in E, \theta \ll \varepsilon} \forall_{x \in A_1} \exists_{y \in A_2} d(x, y) \preccurlyeq \mathcal{H}(A_1, A_2) + \varepsilon$,

(H_4) one of the following is satisfied:

(i) $\forall_{\varepsilon \in E, \theta \ll \varepsilon} \exists_{x \in A_1} \forall_{y \in A_2} \mathcal{H}(A_1, A_2) \preccurlyeq d(x, y) + \varepsilon$,

(ii) $\forall_{\varepsilon \in E, \theta \ll \varepsilon} \exists_{y \in A_2} \forall_{x \in A_1} \mathcal{H}(A_1, A_2) \preccurlyeq d(x, y) + \varepsilon$.

Theorem 1.2 (see [22]). *Let (X, d) be a tvs-cone complete metric space with a solid cone P and let $\mathcal{A}$ be a collection of nonempty closed subsets of X, $\mathcal{A} \neq \phi$, and let $\mathcal{H} : \mathcal{A} \times \mathcal{A} \to E$ be a tvs-$\mathcal{H}$-cone metric with respect to d. If the mapping $T : X \to \mathcal{A}$ the condition that exists a $\lambda \in (0, 1)$ such that for all $x, y \in X$ holds*

$$\mathcal{H}(Tx, Ty) \preccurlyeq \lambda G(x, y) \tag{1.1}$$

then T has a fixed point in X.

We recall some definitions and results of the *tvs*-cone metric spaces that introduced in [19, 23], which will be needed in the sequel.

Let E be be a real Hausdorff topological vector space (*tvs* for short) with the zero vector θ. A nonempty subset P of E is called a convex cone if $P + P \subseteq P$ and $\lambda P \subseteq P$ for $\lambda \geq 0$. A convex cone P is said to be pointed (or proper) if $P \cap (-P) = \{\theta\}$; P is normal (or saturated) if E has a base of neighborhoods of zero consisting of order-convex subsets. For a given cone $P \subseteq E$, we can define a partial ordering $\preccurlyeq$ with respect to P by $x \preccurlyeq y$ if and only if $y - x \in P$; $x \prec y$ will stand for $x \preccurlyeq y$ and $x \neq y$, while $x \ll y$ will stand for $y - x \in \operatorname{int} P$, where $\operatorname{int} P$ denotes the interior of P. The cone P is said to be solid if it has a nonempty interior.

In the sequel, E will be a locally convex Hausdorff *tvs* with its zero vector θ, P a proper, closed, and convex pointed cone in E with $\operatorname{int} P \neq \phi$ and $\preccurlyeq$ a partial ordering with respect to P.

Definition 1.3 (see [16, 18, 19]). Let X be a nonempty set and (E, P) an ordered *tvs*. A vector-valued function $d : X \times X \to E$ is said to be a *tvs*-cone metric, if the following conditions hold:

(C_1) $\forall_{x, y \in X, x \neq y}\ \theta \preccurlyeq d(x, y)$,

(C_2) $\forall_{x, y \in X}\ d(x, y) = \theta \Leftrightarrow x = y$,

(C_3) $\forall_{x, y \in X}\ d(x, y) = d(y, x)$,

(C_4) $\forall_{x, y, z \in X}\ d(x, z) \preccurlyeq d(x, y) + d(y, z)$.

Then the pair (X, d) is called a *tvs*-cone metric space.

Definition 1.4 (see [16, 18, 19]). Let (X, d) be a *tvs*-cone metric space, $x \in X$, and $\{x_n\}$ a sequence in X.

(1) $\{x_n\}$ *tvs*-cone converges to x whenever for every $c \in E$ with $\theta \ll c$, there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x) \ll c$ for all $n \geq n_0$. We denote this by cone-$\lim_{n\to\infty} x_n = x$;

(2) $\{x_n\}$ is a *tvs*-cone Cauchy sequence whenever for every $c \in E$ with $\theta \ll c$, there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x_m) \ll c$ for all $n, m \geq n_0$;

(3) (X, d) is *tvs*-cone complete if every *tvs*-cone Cauchy sequence in X is *tvs*-cone convergent in X.

Remark 1.5. Clearly, a cone metric space in the sense of Huang and Zhang [1] is a special case of *tvs*-cone metric spaces when (X, d) is a *tvs*-cone metric space with respect to a normal cone P.

Remark 1.6 (see [19, 22, 23]). Let (X, d) be a *tvs*-cone metric space with a solid cone P. The following properties are often used, particularly in the case when the underlying cone is nonnormal.

(p1) If $u \preccurlyeq v$ and $v \ll w$, then $u \ll w$,

(p2) If $u \ll v$ and $v \preccurlyeq w$, then $u \ll w$,

(p3) If $u \ll v$ and $v \ll w$, then $u \ll w$,

(p4) If $\theta \preccurlyeq u \ll c$ for each $c \in \operatorname{int} P$, then $u = \theta$,

(p5) If $a \preccurlyeq b + c$ for each $c \in \operatorname{int} P$, then $a \preccurlyeq b$,

(p6) If E is *tvs* with a cone P, and if $a \preccurlyeq \lambda a$ where $a \in P$ and $\lambda \in [0, 1)$, then $a = \theta$,

(p7) If $c \in \operatorname{int} P$, $a_n \in E$, and $a_n \to \theta$ in locally convex *tvsE*, then there exists $n_0 \in \mathbb{N}$ such that $a_n \ll c$ for all $n > n_0$.

Metric spaces are playing an important role in mathematics and the applied sciences. To overcome fundamental laws in Dhage's theory of generalized metric spaces [24]. In 2003, Mustafa and Sims [25] introduced a more appropriate and robust notion of a generalized metric space as follows.

Definition 1.7 (see [25]). Let X be a nonempty set, and let $G : X \times X \times X \to [0, \infty)$ be a function satisfying the following axioms:

(G1) $\forall_{x,y,z\in X}\, G(x, y, z) = 0 \Leftrightarrow x = y = z$,

(G2) $\forall_{x,y\in X, x\neq y}\, G(x, x, y) > 0$,

(G3) $\forall_{x,y,z\in X}\, G(x, y, z) \geq G(x, x, y)$,

(G4) $\forall_{x,y,z\in X}\, (x, y, z) = G(x, z, y) = G(z, y, x) = \cdots$ (symmetric in all three variables),

(G5) $\forall_{x,y,z,w\in X}\, G(x, y, z) \leq G(x, w, w) + G(w, y, z)$.

Then the function G is called a generalized metric, or, more specifically a G-metric on X, and the pair (X, G) is called a G-metric space.

By using the notions of generalized metrics and *tvs*-cone metrics, we introduce the below notion of *tvs*-generalized-cone metrics.

Definition 1.8. Let X be a nonempty set and (E, P) an ordered *tvs*, and let $G : X \times X \times X \to E$ be a function satisfying the following axioms:

(G1) $\forall_{x,y,z\in X}\, G(x,y,z) = \theta$ if and only if $x = y = z$,

(G2) $\forall_{x,y\in X, x\neq y}\, \theta \ll G(x,x,y)$,

(G3) $\forall_{x,y,z\in X}\, G(x,x,y) \preccurlyeq G(x,y,z)$,

(G4) $\forall_{x,y,z\in X}\, G(x,y,z) = G(x,z,y) = G(z,y,x) = \cdots$ (symmetric in all three variables),

(G5) $\forall_{x,y,z,w\in X}\, G(x,y,z) \preccurlyeq G(x,w,w) + G(w,y,z)$.

Then the function G is called a *tvs*-generalized-cone metric, or, more specifically, a *tvs*-G-cone metric on X, and the pair (X,G) is called a *tvs*-G-cone metric space.

Definition 1.9. Let (X,G) be a *tvs*-G-cone metric space, $x \in X$, and $\{x_n\}$ a sequence in X.

(1) $\{x_n\}$*tvs*-G-cone converges to x whenever, for every $c \in E$ with $\theta \ll c$, there exists $n_0 \in \mathbb{N}$ such that $G(x_n, x_m, x) \ll c$ for all $m, n \geq n_0$. Here x is called the limit of the sequence $\{x_n\}$ and is denoted by G-cone-$\lim_{n\to\infty} x_n = x$;

(2) $\{x_n\}$ is a *tvs*-G-cone Cauchy sequence whenever, for every $c \in E$ with $\theta \ll c$, there exists $n_0 \in \mathbb{N}$ such that $G(x_n, x_m, x_l) \ll c$ for all $n, m, l \geq n_0$;

(3) (X,G) is *tvs*-G-cone complete if every *tvs*-G-cone Cauchy sequence in X is *tvs*-G-cone convergent in X.

Proposition 1.10. *Let (X,G) be a tvs-G-cone metric space, $x \in X$, and $\{x_n\}$ a sequence in X. The following are equivalent:*

(i) *$\{x_n\}$ tvs-G-cone converges to x,*

(ii) *$G(x_n, x_n, x) \to \theta$ as $n \to \infty$,*

(iii) *$G(x_n, x, x) \to$ as $n \to \infty$,*

(iv) *$G(x_n, x_m, x) \to \theta$ as $n, m \to \infty$.*

In this paper, we also introduce the below concept of the $\mathcal{CBW}$-*tvs*-G-cone-type function.

Definition 1.11. One calls $\varphi : \operatorname{int} P \cup \{\theta\} \to \operatorname{int} P \cup \{\theta\}$ a $\mathcal{CBW}$-*tvs*-G-cone-type function if the function φ satisfies the following condition

(φ_1) $\varphi(t) \ll t$ for all $t \gg \theta$ and $\varphi(\theta) = \theta$;

(φ_2) $\lim_{n\to\infty} \varphi^n(t) = \theta$ for all $t \in \operatorname{int} P \cup \{\theta\}$.

In this paeper, for a *tvs*-G-cone metric space (X,G) and for the family $\mathcal{A}$ of subsets of X, we introduce a new notion of the *tvs*-$\mathcal{H}$-cone metric $\mathcal{H}$ with respect to G, and we get a fixed result for the $\mathcal{CBW}$-*tvs*-G-cone-type function in a complete *tvs*-G-cone metric space $(\mathcal{A}, \mathcal{H})$. Our results generalize some recent results in the literature.

2. Main Results

Let E be a locally convex Hausdorff *tvs* with its zero vector θ, P a proper, closed and convex pointed cone in E with $\operatorname{int} P \neq \phi$, and $\preccurlyeq$ a partial ordering with respect to P. We introduce the below notion of the *tvs*-$\mathcal{H}$-cone metric $\mathcal{H}$ with respect to *tvs*-G-cone metric G.

Definition 2.1. Let (X, G) be a *tvs-G*-cone metric space with a solid cone P, and let $\mathcal{A}$ be a collection of nonempty subsets of X. A map $\mathcal{H} : \mathcal{A} \times \mathcal{A} \times \mathcal{A} \to E$ is called a *tvs-*$\mathcal{H}$*-*cone metric with respect to G if for any $A_1, A_2, A_3 \in \mathcal{A}$ the following conditions hold:

(H_1) $\mathcal{H}(A_1, A_2, A_3) = \theta \Rightarrow A_1 = A_2 = A_3$,

(H_2) $\mathcal{H}(A_1, A_2, A_3) = \mathcal{H}(A_2, A_1, A_3) = \mathcal{H}(A_1, A_3, A_2) = \cdots$ (symmetry in all variables),

(H_3) $\mathcal{H}(A_1, A_1, A_2) \preccurlyeq \mathcal{H}(A_1, A_2, A_3)$,

(H_4) $\forall_{\varepsilon \in E, \theta \ll \varepsilon} \forall_{x \in A_1, y \in A_2} \exists_{z \in A_3} G(x, y, z) \preccurlyeq \mathcal{H}(A_1, A_2, A_3) + \varepsilon$,

(H_5) one of the following is satisfied:

(i) $\forall_{\varepsilon \in E, \theta \ll \varepsilon} \exists_{x \in A_1} \forall_{y \in A_2, z \in A_3} \mathcal{H}(A_1, A_2, A_3) \preccurlyeq G(x, y, z) + \varepsilon$,

(ii) $\forall_{\varepsilon \in E, \theta \ll \varepsilon} \exists_{y \in A_2} \forall_{x \in A_1, z \in A_3} \mathcal{H}(A_1, A_2, A_3) \preccurlyeq G(x, y, z) + \varepsilon$,

(iii) $\forall_{\varepsilon \in E, \theta \ll \varepsilon} \exists_{z \in A_3} \forall_{y \in A_2, x \in A_1} \mathcal{H}(A_1, A_2, A_3) \preccurlyeq G(x, y, z) + \varepsilon$.

We will prove that a *tvs-*$\mathcal{H}$-cone metric satisfies the conditions of (G_1)–(G_5).

Lemma 2.2. *Let* (X, G) *be a tvs-G-cone metric space with a solid cone* P*, and let* $\mathcal{A}$ *be a collection of nonempty subsets of* X*,* $\mathcal{A} \neq \phi$*. If* $\mathcal{H} : \mathcal{A} \times \mathcal{A} \times \mathcal{A} \to E$ *is a tvs-*$\mathcal{H}$*-cone metric with respect to* G*, then pair* $(\mathcal{A}, \mathcal{H})$ *is a tvs-G-cone metric space.*

Proof. Let $\{\varepsilon_n\} \subset E$ be a sequence such that $\theta \ll \varepsilon_n$ for all $n \in \mathbb{N}$ and G-*cone*-$\lim_{n\to\infty} \varepsilon_n = \theta$. Take any $A_1, A_2, A_3 \in \mathcal{A}$ and $x \in A_1$, $y \in A_2$. From (H_4), for each $n \in \mathbb{N}$, there exists $z_n \in A_3$ such that

$$G(x, y, z_n) \preccurlyeq \mathcal{H}(A_1, A_2, A_3) + \varepsilon_n. \tag{2.1}$$

Therefore, $\mathcal{H}(A_1, A_2, A_3) + \varepsilon_n \in P$ for each $n \in \mathbb{N}$. By the closedness of P, we conclude that $\theta \preccurlyeq \mathcal{H}(A_1, A_2, A_3)$.

Assume that $A_1 = A_2 = A_3$. From H_5, we obtain $\mathcal{H}(A_1, A_2, A_3) \preccurlyeq \varepsilon_n$ for any $n \in \mathbb{N}$. So $\mathcal{H}(A_1, A_2, A_3) = \theta$.

Let $A_1, A_2, A_3, A_4 \in \mathcal{A}$. Assume that A_1, A_2, A_3 satisfy condition (H_5)(i). Then, for each $n \in \mathbb{N}$, there exists $x_n \in A_1$ such that $\mathcal{H}(A_1, A_2, A_3) \preccurlyeq G(x_n, y, z) + \varepsilon_n$ for all $y \in A_2$ and $z \in A_3$. From (H_4), there exists a sequence $\{w_n\} \subset A_4$ satisfying $G(x_n, w_n, w_n) \preccurlyeq \mathcal{H}(A_1, A_4, A_4) + \varepsilon_n$ for every $n \in \mathbb{N}$. Obviously, for any $y \in A_2$ and any $z \in A_3$ and $n \in \mathbb{N}$, we have

$$\begin{aligned} \mathcal{H}(A_1, A_2, A_3) &\preccurlyeq G(x_n, y, z) + \varepsilon_n \\ &\preccurlyeq G(x_n, w_{\mathrm{n}}, w_n) + G(w_n, y, z) + \varepsilon_n. \end{aligned} \tag{2.2}$$

Now for each $n \in \mathbb{N}$, there exists $y_n \in A_2$, $z_n \in A_3$ such that $G(w_n, y_n, z_n) \preccurlyeq \mathcal{H}(A_4, A_2, A_3) + \varepsilon_n$.

Consequently, we obtain that for each $n \in \mathbb{N}$

$$\mathcal{H}(A_1, A_2, A_3) \preccurlyeq \mathcal{H}(A_1, A_4, A_4) + \mathcal{H}(A_4, A_2, A_3) + 3\varepsilon_n. \tag{2.3}$$

Therefore,

$$\mathcal{H}(A_1, A_2, A_3) \preccurlyeq \mathcal{H}(A_1, A_4, A_4) + \mathcal{H}(A_4, A_2, A_3). \tag{2.4}$$

In the case when (H_5)(ii) or (H_5)(iii) holds, we use the analogous method. □

In the sequel, we denote by Θ the class of functions $\varphi : \operatorname{int} P \cup \{\theta\} \to \operatorname{int} P \cup \{\theta\}$ satisfying the following conditions:

(C_1) φ is a $\mathcal{CBW}$-*tvs*-*G*-cone-type-function;

(C_2) φ is subadditive, that is, $\varphi(u_1 + u_2) \preccurlyeq \varphi(u_1) + \varphi(u_2)$ for all $u_1, u_2 \in \operatorname{int} P$.

Our main result is the following.

Theorem 2.3. *Let (X, G) be a tvs-G-cone complete metric space with a solid cone P, let $\mathcal{A}$ be a collection of nonempty closed subsets of X, $\mathcal{A} \neq \phi$, and let $\mathcal{H} : \mathcal{A} \times \mathcal{A} \times \mathcal{A} \to E$ be a tvs-$\mathcal{H}$-cone metric with respect to G. If the mapping $T : X \to \mathcal{A}$ satisfies the condition that exists a $\varphi \in \Theta$ such that for all $x, y, z \in X$ holds*

$$\mathcal{H}(Tx, Ty, Tz) \preccurlyeq \varphi(G(x, y, z)), \tag{2.5}$$

then T has a fixed point in X.

Proof. Let us choose $\varepsilon \in \operatorname{int} P$ arbitrarily, and let $\varepsilon_n \in E$ be a sequence such that $\theta \ll \varepsilon_n$ and $\varepsilon_n \preccurlyeq \varepsilon/3^n$. Let us choose $x_0 \in X$ arbitrarily and $x_1 \in Tx_0$. If $G(x_0, x_0, x_1) = \theta$, then $x_0 = x_1 \in T(x_0)$, and we are done. Assume that $G(x_0, x_0, x_1) \gg \theta$. Taking into account (2.5) and (H_4), there exists $x_2 \in Tx_1$ such that

$$\begin{aligned} G(x_1, x_1, x_2) &\preccurlyeq \mathcal{H}(Tx_0, Tx_0, Tx_1) + \varepsilon_1 \\ &\preccurlyeq \varphi(G(x_0, x_0, x_1)) + \varepsilon_1. \end{aligned} \tag{2.6}$$

Taking into account (2.5), (2.6), and *(H_4)* and since $\varphi \in \Theta$, there exists $x_3 \in Tx_2$ such that

$$\begin{aligned} G(x_2, x_2, x_3) &\preccurlyeq \mathcal{H}(Tx_1, Tx_1, Tx_2) + \varepsilon_2 \\ &\preccurlyeq \varphi(G(x_1, x_1, x_2)) + \varepsilon_2 \\ &\preccurlyeq \varphi\big(\varphi(G(x_0, x_0, x_1)) + \varepsilon_1\big) + \varepsilon_2 \\ &\preccurlyeq \varphi\big(\varphi(G(x_0, x_0, x_1))\big) + \varphi(\varepsilon_1) + \varepsilon_2 \\ &\ll \varphi^2(G(x_0, x_0, x_1)) + \varepsilon_1 + \varepsilon_2 \\ &\preccurlyeq \varphi^2(G(x_0, x_0, x_1)) + \frac{\varepsilon}{3} + \frac{\varepsilon}{3^2}. \end{aligned} \tag{2.7}$$

We continue in this manner. In general, for x_n, $n \in \mathbb{N}$, x_{n+1} is chosen such that $x_{n+1} \in Tx_n$ and

$$\begin{aligned}
G(x_n, x_n, x_{n+1}) &\preccurlyeq \mathscr{H}(Tx_{n-1}, Tx_{n-1}, Tx_n) + \varepsilon_n \\
&\ll \varphi(G(x_{n-1}, x_{n-1}, x_n)) + \varepsilon_n \\
&\preccurlyeq \varphi^2(G(x_{n-2}, x_{n-2}, x_{n-1})) + \varepsilon_{n-1} + \varepsilon_n \\
&\preccurlyeq \cdots\cdots \\
&\preccurlyeq \varphi^n(G(x_0, x_0, x_1)) + \sum_{i=1}^{n} \varepsilon_i \\
&\preccurlyeq \varphi^n(G(x_0, x_0, x_1)) + \sum_{i=1}^{n} \frac{\varepsilon}{3^i} \\
&\preccurlyeq \varphi^n(G(x_0, x_0, x_1)) + \frac{1}{2}\varepsilon.
\end{aligned} \tag{2.8}$$

Since ε is arbitrary, letting $\varepsilon \to \theta$ and by the definition of the $\mathcal{CBW}$-*tvs*-*G*-cone-type function, we obtain that

$$\lim_{n \to \infty} G(x_n, x_n, x_{n+1}) = \theta. \tag{2.9}$$

Next, we let $c_m = G(x_m, x_{m+1}, x_{m+1})$, and we claim that the following result holds: for each $\gamma \gg \theta$, there is $n_0(\varepsilon) \in N$ such that for all m, $n \geq n_0(\gamma)$,

$$G(x_m, x_{m+1}, x_{m+1}) \ll \gamma, \tag{2.10}$$

We will prove (2.10) by contradiction. Suppose that (2.10) is false. Then there exists some $\gamma \gg \theta$ such that for all $p \in \mathbb{N}$, there are $m_p, n_p \in \mathbb{N}$ with $m_p > n_p \geq p$ satisfying

(i) m_p is even and n_p is odd,

(ii) $G(x_{m_p}, x_{n_p}, x_{n_p}) \succcurlyeq \gamma$, and

(iii) m_p is the smallest even number such that conditions (i), (ii) hold.

Since $c_m \downarrow \theta$, by (ii), we have that $\lim_{p \to \infty} G(x_{m_p}, x_{n_p}, x_{n_p}) = \gamma$ and

$$\begin{aligned}
\gamma &\preccurlyeq G\left(x_{m_p}, x_{n_p}, x_{n_p}\right) \\
&\preccurlyeq G\left(x_{m_p}, x_{m_p+1}, x_{m_p+1}\right) + G\left(x_{m_p+1}, x_{n_p+1}, x_{n_p+1}\right) + G\left(x_{n_p+1}, x_{n_p}, x_{n_p}\right).
\end{aligned} \tag{2.11}$$

It follows from (H_4); let us choose $\varepsilon \in E$ arbitrarily such that

$$G\left(x_{n_p+1}, x_{n_p+1}, x_{m_p+1}\right) \preccurlyeq \mathscr{H}\left(Tx_{n_p+1}, Tx_{n_p+1}, Tx_{m_p+1}\right) + \varepsilon. \tag{2.12}$$

Taking into account (2.5), (2.11), and (2.12), we have that

$$\begin{aligned}\gamma &\preccurlyeq G\left(x_{m_p}, x_{n_p}, x_{n_p}\right) \\ &\preccurlyeq G\left(x_{m_p}, x_{m_p+1}, x_{m_p+1}\right) + \mathscr{H}\left(Tx_{n_p+1}, Tx_{n_p+1}, Tx_{m_p+1}\right) + \varepsilon + G\left(x_{n_p+1}, x_{n_p}, x_{n_p}\right) \\ &\preccurlyeq G\left(x_{m_p}, x_{m_p+1}, x_{m_p+1}\right) + \varphi\left(G\left(x_{n_p}, Gx_{n_p}, Gx_{m_p}\right)\right) + \varepsilon + G\left(x_{n_p+1}, x_{n_p}, x_{n_p}\right) \\ &\ll G\left(x_{m_p}, x_{m_p+1}, x_{m_p+1}\right) + G\left(x_{n_p}, Gx_{n_p}, Gx_{m_p}\right) + \varepsilon + G\left(x_{n_p+1}, x_{n_p}, x_{n_p}\right).\end{aligned} \tag{2.13}$$

Since ε is arbitrarily, letting $\varepsilon \to \theta$ and by letting $p \to \infty$, we have

$$\gamma \ll \theta + \lim_{p\to\infty} G\left(x_{m_p}, x_{n_p}, x_{n_p}\right) + \theta + \theta = \gamma, \tag{2.14}$$

a contradiction. So $\{x_n\}$ is a *tvs-G*-cone Cauchy sequence. Since (X, G) is a *tvs-G*-cone complete metric space, $\{x_n\}$ is *tvs-G*-cone convergent in X and G-cone-$\lim_{n\to\infty} x_n = x$. Thus, for every $\tau \in \operatorname{int} P$ and sufficiently large n, we have that

$$\mathscr{H}(Tx_n, Tx_n, Tx) \preccurlyeq \varphi(G(x_n, x_n, x)) \ll G(x_n, x_n, x) \ll \frac{\tau}{3}. \tag{2.15}$$

Since for $n \in \mathbb{N} \cup \{0\}$, $x_{n+1} \in Tx_n$, by (H_4) we obtain that for all $n \in \mathbb{N}$ there exists $y_n \in Tx$ such that

$$\begin{aligned}G(x_{n+1}, x_{n+1}, y_{n+1}) &\preccurlyeq \mathscr{H}(Tx_n, Tx_n, Tx) + \varepsilon_{n+1} \\ &\preccurlyeq \varphi(G(x_n, x_n, x)) + \frac{\varepsilon}{3^{n+1}} \\ &\ll G(x_n, x_n, x) + \frac{\varepsilon}{3^{n+1}}.\end{aligned} \tag{2.16}$$

Since $\varepsilon/3^{n+1} \to \theta$, then for sufficiently large n, we obtain that

$$G(y_{n+1}, x, x) \preccurlyeq G(y_{n+1}, x_{n+1}, x_{n+1}) + G(x_{n+1}, x, x) \ll \frac{2\tau}{3} + \frac{\tau}{3} = \tau, \tag{2.17}$$

which implies G-cone-$\lim_{n\to\infty} y_n = x$. Since Tx is closed, we obtain that $x \in Tx$. □

For the case $\varphi(t) = kt$, $k \in (0, 1)$, then $\varphi \in \Theta$ and it is easy to get the following corollary.

Corollary 2.4. *Let (X, G) be a tvs-G-cone complete metric space with a solid cone P, let $\mathscr{A}$ be a collection of nonempty closed subsets of X, $\mathscr{A} \neq \phi$, and let $\mathscr{H} : \mathscr{A} \times \mathscr{A} \times \mathscr{A} \to E$ be a tvs-$\mathscr{H}$-cone* *metric with respect to* G*. If the mapping* $T : X \to \mathcal{A}$ *satisfies the condition that exists* $k \in (0,1)$ *such that for all* $x, y, z \in X$ *holds*

$$\mathcal{H}(Tx, Ty, Tz) \preccurlyeq k \cdot G(x, y, z), \tag{2.18}$$

then T *has a fixed point in* X.

Remark 2.5. Following Corollary 2.4, it is easy to get Theorem 1.2. So our results generalize some recent results in the literature (e.g., [22]).

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