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Research Article

Positive Solutions for Neumann Boundary Value Problems of Second-Order Impulsive Differential Equations in Banach Spaces

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The existence of positive solutions for Neumann boundary value problem of second-order impulsive differential equations $-u''(t) + Mu(t) = f(t, u(t))$, $t \in J$, $t \neq t_k$, $-\Delta u'|_{t=t_k} = I_k(u(t_k))$, $k = 1, 2, \ldots, m$, $u'(0) = u'(1) = \theta$, in an ordered Banach space E was discussed by employing the fixed point index theory of condensing mapping, where $M > 0$ is a constant, $J = [0, 1]$, $f \in C(J \times K, K)$, $I_k \in C(K, K)$, $k = 1, 2, \ldots, m$, and K is the cone of positive elements in E. Moreover, an application is given to illustrate the main result.

1. Introduction

The theory of impulsive differential equations is a new and important branch of differential equation theory, which has an extensive physical, chemical, biological, engineering background and realistic mathematical model, and hence has been emerging as an important area of investigation in the last few decades; see [1]. Correspondingly, boundary value problems of second-order impulsive differential equations have been considered by many authors, and some basic results have been obtained; see [2–7]. But many of them obtained extremal solutions by monotone iterative technique coupled with the method of upper and lower solutions; see [2–4]. The research on positive solutions is seldom and most in real space $\mathbb{R}$; see [5–7].

In this paper, we consider the existence of positive solutions to the second-order impulsive differential equation Neumann boundary value problem in an ordered Banach space E:

$$\begin{gathered} -u''(t) + Mu(t) = f(t, u(t)), \quad t \in J,\ t \neq t_k, \\ -\Delta u'|_{t=t_k} = I_k(u(t_k)), \quad k = 1, 2, \ldots, m, \\ u'(0) = u'(1) = \theta, \end{gathered} \tag{1.1}$$

where $M > 0$ is a constant, $f \in C(J \times E, E)$, $J = [0,1]$; $0 < t_1 < t_2 < \cdots < t_m < 1$; $I_k \in C(E, E)$ is an impulsive function, $k = 1,2,\ldots,m$. $\Delta u'|_{t=t_k}$ denotes the jump of $u'(t)$ at $t = t_k$, that is, $\Delta u'|_{t=t_k} = u'(t_k^+) - u'(t_k^-)$, where $u'(t_k^+)$ and $u'(t_k^-)$ represent the right and left limits of $u'(t)$ at $t = t_k$, respectively.

In the special case where $E = \mathbb{R}^+ = [0, +\infty)$, $I_k = 0$, $k = 1,2,\ldots,m$, NBVP (1.1) has been proved to have positive solutions; see [8, 9]. Motivated by the aforementioned facts, our aim is to study the positive solutions for NBVP (1.1) in a Banach space by fixed point index theory of condensing mapping. Moreover, an application is given to illustrate the main result. As far as we know, no work has been done for the existence of positive solutions for NBVP (1.1) in Banach spaces.

2. Preliminaries

Let E be an ordered Banach space with the norm $\|\cdot\|$ and partial order $\leq$, whose positive cone $K = \{x \in E \mid x \geq \theta\}$ is normal with normal constant N. Let $J = [0,1]$; $0 = t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = 1$; $J_k = [t_{k-1}, t_k]$, $k = 1,2,\ldots,m+1$, $J' = J \setminus \{t_1, t_2, \ldots, t_m\}$. Let $PC^1(J,E) = \{u \in C(J,E) \mid u'(t)$ is continuous at $t \neq t_k$, and left continuous at $t = t_k$, and $u'(t_k^+)$ exists, $k = 1,2,\ldots,m\}$. Evidently, $PC^1(J,E)$ is a Banach space with the norm $\|u\|_{PC^1} = \max\{\|u(t)\|_C, \|u'\|_{PC}\}$, where $\|u\|_C = \sup_{t\in J}\|u(t)\|$, $\|u'\|_{PC} = \sup_{t\in J}\|u'(t)\|$; see [2]. An abstract function $u \in PC^1(J,E) \cap C^2(J',E)$ is called a solution of NBVP (1.1) if $u(t)$ satisfies all the equalities of (1.1).

Let $C(J,E)$ denote the Banach space of all continuous E-value functions on interval J with the norm $\|u\|_C = \sup_{t\in J}\|u(t)\|$. Let $\alpha(\cdot)$ denote the Kuratowski measure of noncompactness of the bounded set. For the details of the definition and properties of the measure of noncompactness, see [10, 11]. For any $B \subset C(J,E)$ and $t \in J$, set $B(t) = \{u(t) \mid u \in B\} \subset E$. If B is bounded in $C(J,E)$, then $B(t)$ is bounded in E, and $\alpha(B(t)) \leq \alpha(B)$.

Now, we first give the following lemmas in order to prove our main results.

Lemma 2.1 (see [12]). *Let $B \subset C(J,E)$ be equicontinuous. Then $\alpha(B(t))$ is continuous on J, and*

$$\alpha(B) = \max_{t\in J} \alpha(B(t)) = \alpha(B(J)). \tag{2.1}$$

Lemma 2.2 (see [13]). *Let $B = \{u_n\} \subset C(J,E)$ be a bounded and countable set. Then $\alpha(B(t))$ is Lebesgue integral on J, and*

$$\alpha\left(\left\{\int_J u_n(t)dt \mid n \in \mathbb{N}\right\}\right) \leq 2\int_J \alpha(B(t))dt. \tag{2.2}$$

Lemma 2.3 (see [14]). *Let $D \subset E$ be bounded. Then there exists a countable set $D_0 \subset D$, such that $\alpha(D) \leq 2\alpha(D_0)$.*

To prove our main results, for any $h \in C(J, E)$, we consider the Neumann boundary value problem (NBVP) of linear impulsive differential equation in E:

$$\begin{aligned} -u''(t) + Mu(t) &= h(t), \quad t \in J', \\ -\Delta u'|_{t=t_k} &= y_k, \quad k = 1, 2, \ldots, m, \\ u'(0) &= u'(1) = \theta, \end{aligned} \tag{2.3}$$

where $M > 0,\ y_k \in E,\ k = 1, 2, \ldots, m$.

Lemma 2.4. *For any $h \in C(J, E)$, $M > 0$, and $y_k \in E$, $k = 1, 2, \ldots, m$, the linear NBVP* (2.3) *has a unique solution $u \in PC^1(J, E) \cap C^2(J', E)$ given by*

$$u(t) = \int_0^1 G(t, s)h(s)ds + \sum_{k=1}^{m} G(t, t_k)y_k, \tag{2.4}$$

where

$$G(t, s) = \begin{cases} \dfrac{\cosh\sqrt{M}(1-t)\cosh\sqrt{M}s}{\sqrt{M}\sinh\sqrt{M}}, & 0 \le s \le t \le 1, \\ \dfrac{\cosh\sqrt{M}t\cosh\sqrt{M}(1-s)}{\sqrt{M}\sinh\sqrt{M}}, & 0 \le t < s \le 1. \end{cases} \tag{2.5}$$

Proof. Suppose that $u(t)$ is a solution of (2.3); then

$$\begin{gathered} u''(t) - Mu(t) = -h(t), \\ \left[e^{-2\sqrt{M}t}\left(e^{\sqrt{M}t}u(t)\right)'\right]' = -Me^{-\sqrt{M}t}u(t) + e^{-\sqrt{M}t}u''(t) = -e^{-\sqrt{M}t}h(t). \end{gathered} \tag{2.6}$$

Let $y(t) = e^{-2\sqrt{M}t}(e^{\sqrt{M}t}u(t))'$; then

$$y'(t) = -e^{-\sqrt{M}t}h(t), \qquad \Delta y|_{t=t_k} = -e^{-\sqrt{M}t_k}y_k. \tag{2.7}$$

Integrating (2.7) from 0 to t_1, we have

$$y(t_1) - y(0) = -\int_0^{t_1} e^{-\sqrt{M}s}h(s)ds. \tag{2.8}$$

Again, integrating (2.7) from t_1 to t, where $t \in (t_1, t_2]$, then

$$y(t) = y(t_1^+) - \int_{t_1}^{t} e^{-\sqrt{M}s}h(s)ds = y(0) - \int_0^{t} e^{-\sqrt{M}s}h(s)ds - e^{-\sqrt{M}t_1}y_1. \tag{2.9}$$

Repeating the aforementioned procession, for $t \in J$, we have

$$y(t) = y(0) - \int_0^t e^{-\sqrt{M}s} h(s)ds - \sum_{0<t_k<t} e^{-\sqrt{M}t_k} y_k. \tag{2.10}$$

Hence,

$$\left(e^{\sqrt{M}t}u(t)\right)' = e^{2\sqrt{M}t}\left(y(0) - \int_0^t e^{-\sqrt{M}s} h(s)ds - \sum_{0<t_k<t} e^{-\sqrt{M}t_k} y_k\right). \tag{2.11}$$

For $t \in J$, integrating (2.11) from 0 to t, we have

$$\begin{aligned} u(t) &= e^{-\sqrt{M}t}\left(u(0) + \int_0^t e^{2\sqrt{M}s} y(0)ds - \int_0^t e^{2\sqrt{M}s}\int_0^s e^{-\sqrt{M}\tau}h(\tau)d\tau\, ds \right.\\ &\qquad\left. - \int_0^t e^{2\sqrt{M}s}\sum_{0<t_k<s} e^{-\sqrt{M}t_k} y_k ds\right)\\ &= e^{-\sqrt{M}t}\left\{u(0) + \frac{1}{2\sqrt{M}}\left[y(0)\left(e^{2\sqrt{M}t}-1\right) - e^{2\sqrt{M}t}\int_0^t e^{-\sqrt{M}s}h(s)ds\right.\right.\\ &\qquad\left.\left. + \int_0^t e^{\sqrt{M}s}h(s)ds - \sum_{0<t_k<t}\left(e^{2\sqrt{M}t} - e^{2\sqrt{M}t_k}\right)e^{-\sqrt{M}t_k}y_k\right]\right\}. \end{aligned} \tag{2.12}$$

Notice that $y(0) = \sqrt{M}u(0) + u'(0)$; thus, for $t \in J$, we have

$$\begin{aligned} u(t) &= \frac{1}{2\sqrt{M}}\left[e^{-\sqrt{M}t}2\sqrt{M}u(0) + \left(\sqrt{M}u(0) + u'(0)\right)e^{\sqrt{M}t}\right.\\ &\qquad - \left(\sqrt{M}u(0) + u'(0)\right)e^{-\sqrt{M}t} + e^{-\sqrt{M}t}\int_0^t e^{\sqrt{M}s}h(s)ds\\ &\qquad\left. - e^{\sqrt{M}t}\int_0^t e^{-\sqrt{M}s}h(s)ds - \sum_{0<t_k<t}\left(e^{2\sqrt{M}t} - e^{2\sqrt{M}t_k}\right)e^{-\sqrt{M}(t+t_k)}y_k\right]\\ &= \frac{1}{2\sqrt{M}}\left[\left(\sqrt{M}u(0) - u'(0)\right)e^{-\sqrt{M}t} + \left(\sqrt{M}u(0) + u'(0)\right)e^{\sqrt{M}t}\right. \end{aligned}$$

$$
\begin{aligned}
&+ e^{-\sqrt{M}t}\int_0^t e^{\sqrt{M}s}h(s)ds - e^{\sqrt{M}t}\int_0^t e^{-\sqrt{M}s}h(s)ds \\
&- \sum_{0<t_k<t}\left(e^{\sqrt{M}(t-t_k)}\, y_k - e^{\sqrt{M}(t_k-t)}y_k\right)\Bigg],
\end{aligned}
\tag{2.13}
$$

$$
\begin{aligned}
u'(t) = \frac{1}{2}\Bigg[&-\left(\sqrt{M}u(0) - u'(0)\right)e^{-\sqrt{M}t} + \left(\sqrt{M}u(0) + u'(0)\right)e^{\sqrt{M}t} \\
&- e^{-\sqrt{M}t}\int_0^t e^{\sqrt{M}s}h(s)ds - e^{\sqrt{M}t}\int_0^t e^{-\sqrt{M}s}h(s)ds \\
&- \sum_{0<t_k<t}\left(e^{\sqrt{M}(t-t_k)}y_k + e^{-\sqrt{M}(t-t_k)}y_k\right)\Bigg].
\end{aligned}
\tag{2.14}
$$

In view of that $u'(0) = u'(1) = \theta$, we have

$$
\begin{aligned}
u(0) &= \int_0^1 \frac{e^{\sqrt{M}(1-s)} + e^{-\sqrt{M}(1-s)}}{\sqrt{M}\left(e^{\sqrt{M}} - e^{-\sqrt{M}}\right)}h(s)ds + \sum_{k=1}^{m}\frac{e^{\sqrt{M}(1-t_k)} + e^{-\sqrt{M}(1-t_k)}}{\sqrt{M}\left(e^{\sqrt{M}} - e^{-\sqrt{M}}\right)} \\
&= \int_0^1 \frac{\cosh\sqrt{M}(1-s)}{\sqrt{M}\sinh\sqrt{M}}h(s)ds + \sum_{k=1}^{m}\frac{\cosh\sqrt{M}(1-t_k)}{\sqrt{M}\sinh\sqrt{M}}y_k.
\end{aligned}
\tag{2.15}
$$

Substituting (2.15) into (2.13), for $t \in J$, we obtain

$$
\begin{aligned}
u(t) = &\int_0^t \frac{\left(e^{\sqrt{M}(1-t)} + e^{-\sqrt{M}(1-t)}\right)\left(e^{\sqrt{M}s} + e^{-\sqrt{M}s}\right)}{2\sqrt{M}\left(e^{\sqrt{M}} - e^{-\sqrt{M}}\right)}h(s)ds \\
&+ \sum_{0<t_k<t}\frac{\left(e^{\sqrt{M}(1-t)} + e^{-\sqrt{M}(1-t)}\right)\left(e^{\sqrt{M}t_k} + e^{-\sqrt{M}t_k}\right)}{2\sqrt{M}\left(e^{\sqrt{M}} - e^{-\sqrt{M}}\right)}y_k \\
&+ \int_t^1 \frac{\cosh\sqrt{M}t\cosh\sqrt{M}(1-s)}{\sqrt{M}\sinh\sqrt{M}}h(s)ds \\
&+ \sum_{t\le t_k<1}\frac{\cosh\sqrt{M}t\cosh\sqrt{M}(1-t_k)}{\sqrt{M}\sinh\sqrt{M}}y_k \\
= &\int_0^1 G(t,s)h(s)ds + \sum_{k=1}^{m}G(t,t_k)y_k.
\end{aligned}
\tag{2.16}
$$

Inversely, we can verify directly that the function $u \in PC^1(J,E) \cap C^2(J',E)$ defined by (2.4) is a solution of the linear NBVP (2.3). Therefore, the conclusion of Lemma 2.4 holds. □

By (2.5), it is easy to verify that $G(t,s)$ has the following property:

$$\frac{1}{\sqrt{M}\sinh\sqrt{M}} \le G(t,s) \le \frac{\cosh^2\sqrt{M}}{\sqrt{M}\sinh\sqrt{M}}. \tag{2.17}$$

Evidently, $C(J,E)$ is also an ordered Banach space with the partial order $\le$ reduced by the positive cone $C(J,K) = \{u \in C(J,E) \mid u(t) \ge \theta,\ t \in J\}$. $C(J,K)$ is also normal with the same normal constant N.

Define an operator $A : C(J,K) \to C(J,K)$ as follows:

$$Au(t) = \int_0^1 G(t,s)f(s,u(s))ds + \sum_{k=1}^{m} G(t,t_k)I_k(u(t_k)). \tag{2.18}$$

Clearly, $A : C(J,K) \to C(J,K)$ is continuous, and the positive solution of NBVP (1.1) is the nontrivial fixed point of operator A. However, the integral operator A is noncompactness in general Banach space. In order to employ the topological degree theory and the fixed point theory of condensing mapping, there demands that the nonlinear f and impulsive function I_k satisfy some noncompactness measure condition. Thus, we suppose the following.

$(P0)$ For any $R > 0$, $f(J \times K_R)$ and $I_k(K_R)$ are bounded and

$$\alpha\big(f(t,D)\big) \le L\alpha(D), \quad \alpha(I_k(D)) \le M_k\alpha(D), \quad k = 1,2,\ldots,m, \tag{2.19}$$

where $K_R = K \cap B(\theta,R)$, $D \subset K$ is arbitrarily countable set, $L > 0$ and $M_k \ge 0$ are counsants and satisfy $(4L/M) + (2\cosh^2\sqrt{M}\ \sum_{k=1}^{m} M_k/\sqrt{M}\ \sinh\sqrt{M}) < 1$.

Lemma 2.5. *Suppose that condition $(P0)$ is satisfied; then $A : C(J,K) \to C(J,K)$ is condensing.*

Proof. Since $A(B)$ is bounded and equicontinuous for any bounded and nonrelative compact set $B \subset C(J,K)$, by Lemma 2.3, there exists a countable set $B_1 = \{u_n\} \subset B$, such that

$$\alpha(A(B)) \le 2\alpha(A(B_1)). \tag{2.20}$$

By assumption $(P0)$ and Lemma 2.1,

$$\begin{aligned}
\alpha(A(B_1)(t)) &= \alpha\left(\left\{\int_0^1 G(t,s)f(s,u_n(s))ds + \sum_{k=1}^{m} G(t,t_k)I_k(u_n(t_k))\right\}\right) \\
&\le \alpha\left(\left\{\int_0^1 G(t,s)f(s,u_n(s))ds\right\}\right) + \alpha\left(\left\{\sum_{k=1}^{m} G(t,t_k)I_k(u_n(t_k))\right\}\right) \\
&\le 2\int_0^1 G(t,s)\alpha\big(f(s,B_1(s))\big)ds + \sum_{k=1}^{m} G(t,t_k)\alpha(I_k(B_1(t_k)))
\end{aligned}$$

$$\begin{aligned}
&\leq 2\int_0^1 G(t,s)L\alpha(B_1(s))ds + \sum_{k=1}^{m} G(t,t_k)M_k\alpha(B_1(t_k)) \\
&\leq 2L\int_0^1 G(t,s)ds\alpha(B_1) + \sum_{k=1}^{m} M_k G(t,t_k)\alpha(B_1) \\
&\leq \frac{2L}{M}\alpha(B_1) + \frac{\cosh^2\sqrt{M}\sum_{k=1}^{m} M_k}{\sqrt{M}\sinh\sqrt{M}}\alpha(B_1) \\
&= \left(\frac{2L}{M} + \frac{\cosh^2\sqrt{M}\sum_{k=1}^{m} M_k}{\sqrt{M}\sinh\sqrt{M}}\right)\alpha(B_1).
\end{aligned} \tag{2.21}$$

Since $A(B_1)$ is equicontinuous, by Lemma 2.1, we have

$$\alpha(A(B_1)) = \max_{t\in J}\alpha(A(B_1)(t)) \leq \left(\frac{2L}{M} + \frac{\cosh^2\sqrt{M}\sum_{k=1}^{m} M_k}{\sqrt{M}\sinh\sqrt{M}}\right)\alpha(B_1). \tag{2.22}$$

Combining (2.20) and $(P0)$, we have

$$\alpha(A(B)) \leq 2\alpha(A(B_1)) \leq \left(\frac{4L}{M} + \frac{2\cosh^2\sqrt{M}\sum_{k=1}^{m} M_k}{\sqrt{M}\sinh\sqrt{M}}\right)\alpha(B). \tag{2.23}$$

Hence, $A : C(J,K) \to C(J,K)$ is condensing. □

Let P be a cone in $C(J,K)$ defined by

$$P = \{u \in C(J,K) \mid u(t) \geq \sigma u(\tau),\ \forall t,\tau \in J\}, \tag{2.24}$$

where $\sigma = 1/\cosh^2\sqrt{M}$.

Lemma 2.6. *For any $f(J,K) \subset K$, $A(C(J,K)) \subset P$.*

Proof. For any $u \in C(J,K)$, $t,\tau \in J$, by (2.18) and the second inequality of (2.17), we have

$$\begin{aligned}
A(u(\tau)) &= \int_0^1 G(\tau,s)f(s,u(s))ds + \sum_{k=1}^{m} G(\tau,t_k)I_k(u(t_k)) \\
&\leq \frac{\cosh^2\sqrt{M}}{\sqrt{M}\sinh\sqrt{M}}\left(\int_0^1 f(s,u(s))ds + \sum_{k=1}^{m} I_k(u(t_k))\right).
\end{aligned} \tag{2.25}$$

By this, (2.18), and the first inequality of (2.17), we have

$$\begin{aligned} A(u(t)) &= \int_0^1 G(t,s)f(s,u(s))ds + \sum_{k=1}^{m} G(t,t_k)I_k(u(t_k)) \\ &\geq \frac{1}{\sqrt{M}\sinh\sqrt{M}}\left(\int_0^1 f(s,u(s))ds + \sum_{k=1}^{m} I_k(u(t_k))\right) \\ &\geq \sigma A(u(\tau)). \end{aligned} \tag{2.26}$$

Hence, $A(C(J,K)) \subset P$. □

Thus, for any $f(J,K) \subset K$, $A : P \rightarrow P$ is condensing mapping; the positive solution of NBVP (1.1) is equivalent to the nontrivial fixed point of A in P. For $0 < r < R < \infty$, let $P_r = \{u \in P \mid \|u\|_C < r\}$, and $\partial P_r = \{u \in P \mid \|u\|_C = r\}$, which is the relative boundary bound of P_r in P. Denote that $P_{r,R} = P_R \setminus \overline{P_r}$; then the fixed point of A in $P_{r,R}$ is the positive solution of NBVP (1.1). We will use the fixed point theory of condensing mapping to find the fixed point of A in $P_{r,R}$.

Let X be a Banach space and let $P \subset X$ be a cone in X. Assume that Ω is a bounded open subset of X and let $\partial\Omega$ be its bound. Let $Q : P \cap \overline{\Omega} \rightarrow P$ be a condensing mapping. If $Qu \neq u$ for every $u \in P \cap \partial\Omega$, then the fixed point index $i(Q, P \cap \Omega, P)$ is defined. If $i(Q, P \cap \Omega, P) \neq 0$, then Q has a fixed point in $P \cap \Omega$. As the fixed point index theory of completely continuous mapping, see [10, 11], we have the following lemmas that are needed in our argument for condensing mapping.

Lemma 2.7. *Let $Q : P \rightarrow P$ be condensing mapping; if*

$$u \neq \lambda Au, \quad \forall u \in \partial P_r,\ 0 < \lambda \leq 1, \tag{2.27}$$

then $i(Q, P_r, P) = 1$.

Lemma 2.8. *Let $Q : P \rightarrow P$ be condensing mapping; if there exists $v_0 \in P$, $v_0 \neq \theta$, such that*

$$u - Au \neq \tau v_0, \quad \forall u \in \partial P_r,\ \tau \geq 0, \tag{2.28}$$

then $i(Q, P_r, P) = 0$.

3. Main Results

$(P1)$ (i) There exist $\delta > 0$, $a, a_k > 0$, such that for all $x \in P_\delta$ and $t \in J$, $f(t,x) \leq ax$, $I_k(x) \leq a_k x$, and $a + (\sum_{k=1}^{m} a_k/\sigma^2) < M$.
(ii) There exist $b, b_k > 0$, $h_0 \in C(J,K)$, and $y_k \in K$, such that for all $x \in P$ and $t \in J$, $f(t,x) \geq bx - h_0(t)$, $I_k(x) \geq b_k x - y_k$, and $b + \sigma^2 \sum_{k=1}^{m} b_k > M$.

$(P2)$ (i) There exist $\delta > 0$, $b, b_k > 0$, such that for all $x \in P_\delta$ and $t \in J$, $f(t,x) \geq bx$, $I_k(x) \geq b_k x$, and $b + \sigma^2 \sum_{k=1}^{m} b_k > M$.
(ii) There exist $a, a_k > 0$, $h_0 \in C(J,K)$, and $y_k \in K$, such that for all $x \in P$ and $t \in J$, $f(t,x) \leq ax + h_0(t)$, $I_k(x) \leq a_k x + y_k$, and $a + (\sum_{k=1}^{m} a_k/\sigma^2) < M$.

Theorem 3.1. *Let E be an ordered Banach space, whose positive* cone *K is normal, $f \in C(J \times K, K)$, and $I_k \in C(K, K)$, $k = 1, 2, \ldots, m$. Suppose that conditions $(P0)$ and $(P1)$ or $(P2)$ are satisfied; then the NBVP* (1.1) *has at least one positive solution.*

Proof. We show, respectively, that the operator A defined by (2.18) has a nontrivial fixed point in two cases that $(P1)$ is satisfied and $(P2)$ is satisfied.

Case 1. Assume that $(P1)$ is satisfied; let $0 < r < \delta$, where δ is the constant in condition $(P1)$, to prove that A satisfies

$$u \neq \lambda A u, \quad \forall u \in \partial P_r, \ 0 < \lambda \leq 1. \tag{3.1}$$

If (3.1) is not true, then there exist $u_0 \in \partial P_r$ and $0 < \lambda_0 \leq 1$, such that $u_0 = \lambda_0 A u_0$; by the definition of A, $u_0(t)$ satisfies

$$\begin{gathered} -u_0''(t) + M u_0(t) = \lambda_0 f(t, u_0(t)), \quad t \in J, \ t \neq t_k, \\ -\Delta u_0'|_{t=t_k} = \lambda_0 I_k(u_0(t_k)), \quad k = 1, 2, \ldots, m, \\ u_0'(0) = u_0'(1) = \theta. \end{gathered} \tag{3.2}$$

Integrating (3.2) from 0 to 1, using (i) of assumption $(P1)$, we have

$$(M - a) \int_0^1 u_0(t) dt \leq \sum_{k=1}^{m} a_k u_0(t_k). \tag{3.3}$$

Since $u_0 \in P$, for any $t, s \in J$, by the definition of P, we have $u_0(t) \geq \sigma u_0(s)$, $u_0(t_k) \leq (1/\sigma) u_0(s)$, and thus

$$\sigma(M - a) u_0(s) \leq \frac{\sum_{k=1}^{m} a_k}{\sigma} u_0(s), \tag{3.4}$$

that is; $(M - (a + (\sum_{k=1}^{m} a_k / \sigma^2)))\ u_0(s) \leq \theta$. So we obtain that $u_0(s) \leq \theta$ in J, which contracts with $u_0 \in \partial P_r$. Hence (3.1) is satisfied; by Lemma 2.7, we have

$$i(A, P_r, P) = 1. \tag{3.5}$$

Let $e \in C(J, K)$, $\|e\| = 1$, $v_0(t) \equiv e$, and obviously $v_0 \in P$. We show that if R is large enough, then

$$u - A u \neq \tau v_0, \quad \forall u \in \partial P_R, \ \tau \geq 0. \tag{3.6}$$

In fact, if there exist $u_0 \in \partial P_R$, $\tau_0 \geq 0$ such that $u_0 - Au_0 = \tau_0 v_0$, then $Au_0 = u_0 - \tau_0 v_0$; by the definition of A, $u_0(t)$ satisfies

$$\begin{gathered} -u_0''(t) + Mu_0(t) - M\tau_0 v_0 = f(t, u_0(t)), \quad t \in J,\ t \neq t_k, \\ -\Delta u_0'|_{t=t_k} = I_k(u_0(t_k)), \quad k = 1, 2, \ldots, m, \\ u_0'(0) = u_0'(1) = \theta. \end{gathered} \tag{3.7}$$

By (ii) of assumption $(P1)$, we have

$$-u_0''(t) + Mu_0(t) = f(t, u_0(t)) + M\tau_0 v_0 \geq bu_0(t) - h_0(t), \quad t \in J'. \tag{3.8}$$

Integrating on J and using (ii) of assumption $(P1)$, we have

$$(b - M)\int_0^1 u_0(t)dt + \sum_{k=1}^m b_k u_0(t_k) \leq \int_0^1 h_0(t)dt + \sum_{k=1}^m y_k. \tag{3.9}$$

If $b > M$, for any $t, s \in J$, by the definition of P, we have $u_0(t) \geq \sigma u_0(s)$; $u_0(t_k) \geq \sigma u_0(s)$; thus

$$\left(\sigma(b - M) + \sigma\sum_{k=1}^m b_k\right)u_0(s) \leq \int_0^1 h_0(t)dt + \sum_{k=1}^m y_k. \tag{3.10}$$

By $\sigma(b - M) + \sigma\sum_{k=1}^m b_k > 0$ and the normality of cone K, we have

$$\|u_0\|_C \leq \frac{N\left(\|h_0\|_C + \sum_{k=1}^m \|y_k\|\right)}{\sigma(b - M) + \sigma\sum_{k=1}^m b_k} \triangleq R_1. \tag{3.11}$$

If $b \leq M$, then for any $t, s \in J$, by the definition of P, we have $u_0(t) \leq (1/\sigma)u_0(s)$, $u_0(t_k) \geq \sigma u_0(s)$; thus

$$\left(\frac{(b - M)}{\sigma} + \sigma\sum_{k=1}^m b_k\right)u_0(s) \leq \int_0^1 h_0(t)dt + \sum_{k=1}^m y_k. \tag{3.12}$$

By $b + \sigma^2\sum_{k=1}^m b_k > M$, and the normality of K, we have

$$\|u_0\|_C \leq \frac{N\left(\|h_0\|_C + \sum_{k=1}^m \|y_k\|\right)}{((b - M)/\sigma) + \sigma\sum_{k=1}^m b_k} \triangleq R_2. \tag{3.13}$$

Let $R > \max\{R_1, R_2, r\}$; then (3.6) is satisfied; by Lemma 2.8, we have

$$i(A, P_R, P) = 0. \tag{3.14}$$

Combining (3.5), (3.14), and the additivity of fixed point index, we have

$$i(A, P_{r,R}, P) = i(A, P_R, P) - i(A, P_r, P) = -1 \neq 0. \tag{3.15}$$

Therefore A has a fixed point in $P_{r,R}$, which is the positive solution of NBVP (1.1).

Case 2. Assume that $(P2)$ is satisfied; let $0 < r < \delta$, where δ is the constant in condition $(P2)$, to proof that A satisfies

$$u - Au \neq \tau v_0, \quad \forall u \in \partial P_r,\ \tau \geq 0, \tag{3.16}$$

where $v_0(t) = e \in P,\ e \neq \theta$. In fact, if there exists $u_0 \in \partial P_r$ and $\tau_0 \geq 0$, such that $u_0 - Au_0 = \tau_0 v_0$, then u_0 satisfies (3.7) and (i) of condition $(P2)$, and we have

$$-u_0''(t) + Mu_0(t) = f(t, u_0(t)) + M\tau_0 v_0 \geq bu_0(t), \quad t \in J'. \tag{3.17}$$

Integrating on J and using (i) of assumption $(P2)$, we have

$$(b - M)\int_0^1 u_0(t)dt + \sum_{k=1}^{m} b_k u_0(t_k) \leq \theta. \tag{3.18}$$

If $b > M$, for any $t, s \in J$, by the definition of P, we have $u_0(t) \geq \sigma u_0(s),\ u_0(t_k) \geq \sigma u_0(s)$, for all $t, s \in J$, and thus

$$\left(\sigma(b - M) + \sigma\sum_{k=1}^{m} b_k\right)u_0(s) \leq \theta. \tag{3.19}$$

By $\sigma(b - M) + \sigma\sum_{k=1}^{m} b_k > 0$, we obtain that $u_0(s) \leq \theta$, which contracts with $u_0 \in \partial P_r$. Hence (3.16) is satisfied.

If $b \leq M$, for any $t, s \in J$, by the definition of P, we have $u_0(t) \leq (1/\sigma)u_0(s)$, $u_0(t_k) \geq \sigma u_0(s)$, for all $t, s \in J$, and thus

$$\left(\frac{1}{\sigma}(b - M) + \sigma\sum_{k=1}^{m} b_k\right)u_0(s) \leq \theta. \tag{3.20}$$

By $b + \sigma^2 \sum_{k=1}^{m} b_k > M$, we obtain that $u_0(s) \leq \theta$, which contracts with $u_0 \in \partial P_r$. Hence (3.16) is satisfied.

Hence, by Lemma 2.8, we have

$$i(A, P_r, P) = 0. \tag{3.21}$$

Next, we show that if R is large enough, then

$$u \neq \lambda Au, \quad \forall u \in \partial P_R,\ 0 < \lambda \leq 1. \tag{3.22}$$

In fact, if there exists $u_0 \in \partial P_R$ and $0 < \lambda_0 \leq 1$ such that $u_0 = \lambda_0 A u_0$, then u_0 satisfies (3.2). Integrating (3.2) on J, and using (ii) of $(P2)$, we have

$$(M-a)\int_0^1 u_0(t)dt - \sum_{k=1}^{m} a_k u_0(t_k) \leq \int_0^1 h_0(t)dt + \sum_{k=1}^{m} y_k. \tag{3.23}$$

For any $t, s \in J$, by the definition of P, we have $u_0(t) \geq \sigma u_0(s)$, $u_0(t_k) \leq (1/\sigma)u_0(s)$, for all $t, s \in J$, and thus

$$\left(\sigma(M-a) - \frac{1}{\sigma}\sum_{k=1}^{m} a_k\right) u_0(s) \leq \int_0^1 h_0(t)dt + \sum_{k=1}^{m} y_k. \tag{3.24}$$

By $a + (\sum_{k=1}^{m} a_k/\sigma^2) < M$, we have

$$u_0(s) \leq \frac{\int_0^1 h_0(t)dt + \sum_{k=1}^{m} y_k}{\sigma(M-a) - (1/\sigma)\sum_{k=1}^{m} a_k}. \tag{3.25}$$

By the normality of K, we have

$$\|u_0(s)\| \leq \frac{N\left\|\int_0^1 h_0(t)dt + \sum_{k=1}^{m} y_k\right\|}{\sigma(M-a) - (1/\sigma)\sum_{k=1}^{m} a_k}. \tag{3.26}$$

Thus

$$\|u_0\|_C \leq \frac{N\left(\|h_0\|_C + \sum_{k=1}^{m}\|y_k\|\right)}{\sigma(M-a) - (1/\sigma)\sum_{k=1}^{m} a_k} \triangleq R_3. \tag{3.27}$$

Let $R > \max\{R_3, r\}$; then (3.22) is satisfied; by Lemma 2.7, we have

$$i(A, P_R, P) = 1. \tag{3.28}$$

Combining (3.21), (3.28), and the additivity of fixed index, we have

$$i(A, P_{r,R}, P) = i(A, P_R, P) - i(A, P_r, P) = 1. \tag{3.29}$$

Therefore A has a fixed point in $P_{r,R}$, which is the positive solution of NBVP (1.1). □

Remark 3.2. The conditions $(P1)$ and $(P2)$ are a natural extension of suplinear condition and sublinear condition in Banach space E. Hence if $I_k = \theta$, then Theorem 3.1 improves and extends the main results in [8, 9].

4. Example

We provide an example to illustrate our main result.

Example 4.1. Consider the following problem:

$$
\begin{gathered}
-\frac{\partial^2}{\partial t^2}w(t,x)+w(t,x)=\int_0^1 e^{(t-s)}w^2(s,x)ds+\frac{1}{100}w(t,x), \quad t\in J',\ x\in I,\\
-\Delta\frac{\partial}{\partial t}w(t,x)|_{t=1/2}=\frac{1}{100}w\left(\frac{1}{2},x\right), \quad x\in I,\\
\frac{\partial}{\partial t}w(0,x)=\frac{\partial}{\partial t}w(1,x)=0, \quad x\in I,
\end{gathered}
\tag{4.1}
$$

where $J=[0,1]$, $J'=J\setminus\{1/2\}$, $I=[0,T]$, and $T>0$ is a constant.

Conclusion

Problem (4.1) has at least one positive solution.

Proof. Let $E=C(I)$, and $K=\{w\in C(I)\mid w(x)\geq 0,\ x\in I\}$; then E is a Banach space with norm $\|w\|=\max_{t\in I}|w(x)|$, and K is a positive cone of E. Let $u(t)=w(t,\cdot)$; then the problem (4.1) can be transformed into the form of NBVP (1.1), where $M=1$, $f(t,u)=\int_0^1 e^{(t-s)}u^2(s)ds+(1/100)u(t)$, and $I_1(u(1/2))=(1/100)u(1/2)$. Evidently $C(J,E)$ is a Banach space with norm $\|u\|_C=\max_{t\in J}\|u(t)\|$, and $C(J,K)$ is positive cone of $C(J,E)$. Let $P=\{u\in C(J,K)\mid u(t)\geq \sigma u(s),\ t,s\in J\}$, where $\sigma=1/\cosh^2 1$; then P is a cone in $C(J,K)$, and for any $u\in P$, $t\in J$, we have $\sigma\|u\|_C\leq u(t)\leq\|u\|_C$.

Next, we will verify that the conditions $(P0)$ and $(P1)$ in Theorem 3.1 are satisfied.

It is easy to verify that for any $R>0$, $f(J\times K_R)$ and $I_1(K_R)$ are bounded. Let $g(t,u)=\int_0^1 e^{(t-s)}u^2(s)ds$; then g is completely continuous. So for any countable bounded set $D\subset K$, we have

$$
\alpha\big(f(t,D)\big)\leq\alpha\big(g(t,D)\big)+\frac{1}{100}\alpha(D)=\frac{1}{100}\alpha(D), \quad \alpha(I_1(D))=\frac{1}{100}\alpha(D),
\tag{4.2}
$$

and for $L=1/100$, and $M_1=1/100$, by simple calculations, $(4L/M)+(2\cosh^2\sqrt{M}\,M_1/\sqrt{M}\,\sinh\sqrt{M})<1$. So condition $(P0)$ is satisfied.

Let $\delta=1/100$; then for $u\in P_\delta$, we have

$$
\begin{gathered}
f(t,u)\leq\frac{\delta\int_0^1 e^{(t-s)}ds}{\sigma}u(t)+\frac{1}{100}u(t)\leq\left(\frac{\delta(e-1)}{\sigma}+\frac{1}{100}\right)u(t),\\
I_1\left(u\left(\frac{1}{2}\right)\right)=\frac{1}{100}u\left(\frac{1}{2}\right)\leq\frac{1}{100\sigma}u(t).
\end{gathered}
\tag{4.3}
$$

Let $a=(\delta(e-1)/\sigma)+(1/100)$, $a_1=(1/100)\sigma$; by simple calculations, we have $a+(a_1/\sigma^2)<1$. So (i) of condition $(P1)$ is satisfied.

Let $R = 100$, for $u \in P$, $\|u\|_C \geq R$; we have $f(t,u) \geq (1/2\,\sigma^2 R + (1/100))u(t)$. For $u \in P$, $0 \leq \|u\|_C \leq R$, we have $f(t,u) \leq R^2 e^t + 1$. Hence, let $h_0(t) = 10000e^t + 1$, $y_1 = 0$; then for any $u \in P$, we have

$$f(t,u) \geq \left(\frac{1}{2}\sigma^2 R + \frac{1}{100}\right)u(t) - h_0(t)$$
$$I_1\left(u\left(\frac{1}{2}\right)\right) \geq \frac{1}{100}\sigma u(t). \tag{4.4}$$

Let $b = (1/2)\sigma^2 R + (1/100)$, and $b_1 = (1/100)\sigma$; by simple calculations, we have $b + \sigma^2 b_1 > 1$, so (ii) of condition $(P1)$ is satisfied.

By Theorem 3.1, Problem (4.1) has at least one positive solution. □

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