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Research Article

Dynamical Behaviors of Stochastic Reaction-Diffusion Cohen-Grossberg Neural Networks with Delays

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This paper investigates dynamical behaviors of stochastic Cohen-Grossberg neural network with delays and reaction diffusion. By employing Lyapunov method, Poincaré inequality and matrix technique, some sufficient criteria on ultimate boundedness, weak attractor, and asymptotic stability are obtained. Finally, a numerical example is given to illustrate the correctness and effectiveness of our theoretical results.

1. Introduction

Cohen and Grossberg proposed and investigated Cohen-Grossberg neural networks in 1983 [1]. Hopfield neural networks, recurrent neural networks, cellular neural networks, and bidirectional associative memory neural networks are special cases of this model. Since then, the Cohen-Grossberg neural networks have been widely studied in the literature, see for example, [2–12] and references therein.

Strictly speaking, diffusion effects cannot be avoided in the neural networks when electrons are moving in asymmetric electromagnetic fields. Therefore, we must consider that the activations vary in space as well as in time. In [13–19], the authors gave some stability conditions of reaction-diffusion neural networks, but these conditions were independent of diffusion effects.

On the other hand, it has been well recognized that stochastic disturbances are ubiquitous and inevitable in various systems, ranging from electronic implementations to biochemical systems, which are mainly caused by thermal noise, environmental fluctuations, as well as different orders of ongoing events in the overall systems [20, 21]. Therefore, considerable attention has been paid to investigate the dynamics of stochastic neural networks, and many results on stability of stochastic neural networks have been reported in the literature, see for example, [22–38] and references therein.

The above references mainly considered the stability of equilibrium point of neural networks. What do we study when the equilibrium point does not exist? Except for stability property, boundedness and attractor are also foundational concepts of dynamical systems, which play an important role in investigating the uniqueness of equilibrium, global asymptotic stability, global exponential stability, the existence of periodic solution, and so on [39, 40]. Recently, ultimate boundedness and attractor of several classes of neural networks with time delays have been reported. In [41], the globally robust ultimate boundedness of integrodifferential neural networks with uncertainties and varying delays was studied. Some sufficient criteria on the ultimate boundedness of deterministic neural networks with both varying and unbounded delays were derived in [42]. In [43, 44], a series of criteria on the boundedness, global exponential stability, and the existence of periodic solution for nonautonomous recurrent neural networks were established. In [45, 46], some criteria on ultimate boundedness and attractor of stochastic neural networks were derived. To the best of our knowledge, there are few results on the ultimate boundedness and attractor of stochastic reaction-diffusion neural networks.

Therefore, the arising questions about the ultimate boundedness, attractor and stability for the stochastic reaction-diffusion Cohen-Grossberg neural networks with time-varying delays are important yet meaningful.

The rest of the paper is organized as follows: some preliminaries are in Section 2, main results are presented in Section 3, a numerical example and conclusions will be drawn in Sections 4 and 5, respectively.

2. Model Description and Assumptions

Consider the following stochastic Cohen-Grossberg neural networks with delays and diffusion terms:

$$
\begin{aligned}
dy_i(t,x) &= \sum_{k=1}^{l} \frac{\partial}{\partial x_k}\left(D_{ik}\frac{\partial y_i(t,x)}{\partial x_k}\right)dt - d_i(y_i(t,x)) \\
&\quad \times \left(c_i(y_i(t,x)) - \sum_{j=1}^{n} a_{ij} f_j(y_j(t,x)) - \sum_{j=1}^{n} b_{ij} g_j(y_j(t-\tau_j(t),x)) - J_i\right)dt \\
&\quad + \sum_{j=1}^{m} \sigma_{ij}(y_j(t,x), y_j(t-\tau_j(t),x))\,dw_j(t), \quad x \in X, \\
&\qquad \frac{\partial y_i}{\partial v} := \left(\frac{\partial y_i}{\partial x_1},\ldots,\frac{\partial y_i}{\partial x_l}\right)^T = 0, \quad x \in \partial X, \\
&\qquad y_i(s,x) = \xi_i(s,x), \quad -\tau \le s \le 0, \ x \in X,
\end{aligned}
\tag{2.1}
$$

for $1 \le i \le n$ and $t \ge 0$. In the above model, $n \ge 2$ is the number of neurons in the network; x_i is space variable; $y_i(t,x)$ is the state variable of the ith neuron at time t and in space x;

$f_j(y_j(t,x))$ and $g_j(y_j(t,x))$ denote the activation functions of the jth unit at time t and in space x; constant $D_{ik} \geq 0$; $d_i(y_i(t,x))$ presents an amplification function; $c_i(y_i(t,x))$ is an appropriately behavior function; a_{ij} and b_{ij} denote the connection strengths of the jth unit on the ith unit, respectively; $\tau_j(t)$ corresponds to the transmission delay and satisfies $0 \leq \tau_j(t) \leq \tau$; J_i denotes the external bias on the ith unit; $\sigma_{ij}(\cdot,\cdot,x)$ is the diffusion function; X is a compact set with smooth boundary ∂X and measure mes$X > 0$ in R^l; $\xi_i(s,x)$ is the initial boundary value; $w(t) = (w_1(t),\ldots,w_m(t))^T$ is m-dimensional Brownian motion defined on a complete probability space $(\Omega,\mathcal{F},\mathbb{P})$ with a natural filtration $\{\mathcal{F}_t\}_{t\geq 0}$ generated by $\{w(s) : 0 \leq s \leq t\}$, where we associate Ω with the canonical space generated by all $\{w_i(t)\}$ and denote by $\mathcal{F}$ the associated σ-algebra generated by $\{w(t)\}$ with the probability measure $\mathbb{P}$.

System (2.1) has the following matrix form:

$$\begin{aligned}
dy(t,x) = \operatorname{col}&\left\{\sum_{k=1}^{l}\frac{\partial}{\partial x_k}\left(D_{ik}\frac{\partial y_i(t,x)}{\partial x_k}\right)\right\}dt - d(y(t,x)) \\
&\times\left[c(y(t,x)) - Af(y(t,x)) - Bg(y(t-\tau(t),x)) - J\right]dt \\
&+ \sigma(y(t,x), y(t-\tau(t),x))dw(t), \quad x \in X,
\end{aligned} \tag{2.2}$$

where

$$\begin{gathered}
\operatorname{col}\left\{\sum_{k=1}^{l}\frac{\partial}{\partial x_k}\left(D_{ik}\frac{\partial y_i(t,x)}{\partial x_k}\right)\right\} = \left(\sum_{k=1}^{l}\frac{\partial}{\partial x_k}\left(D_{1k}\frac{\partial y_1(t,x)}{\partial x_k}\right),\ldots,\sum_{k=1}^{l}\frac{\partial}{\partial x_k}\left(D_{nk}\frac{\partial y_n(t,x)}{\partial x_k}\right)\right)^T, \\
A = (a_{ij})_{n\times n}, \qquad B = (b_{ij})_{n\times n}, \qquad f(y(t,x)) = (f_1(y_1(t,x)),\ldots,f_n(y_n(t,x)))^T, \\
J = (J_1,\ldots,J_n)^T, \\
g(y(t-\tau(t),x)) = \operatorname{diag}(g_1(y_1(t-\tau_1(t),x)),\ldots,g_n(y_n(t-\tau_n(t),x))), \\
d(y(t,x)) = \operatorname{diag}(d_1(y_1(t,x)),\ldots,d_n(y_n(t,x))), \\
c(y(t,x)) = \operatorname{diag}(c_1(y_1(t,x)),\ldots,c_n(y_n(t,x))), \\
\sigma(y(t,x), y(t-\tau(t),x),x) = (\sigma_{ij}(y_j(t,x), y_j(t-\tau_j(t),x),x))_{n\times m}.
\end{gathered} \tag{2.3}$$

Let $L^2(X)$ be the space of real Lebesgue measurable functions on X and a Banach space for the L_2-norm

$$\|u(t)\|_2^2 = \int_X u^2(t,x)dx. \tag{2.4}$$

Note that $\xi = \{(\xi_1(s,x),\ldots,\xi_n(s,x))^T : -\tau \leq s \leq 0\}$ is $C([-\tau,0]\times R^l; R^n)$-valued function and $\mathcal{F}_0$-measurable R^n-valued random variable, where $\mathcal{F}_0 = \mathcal{F}_s$ on $[-\tau,0]$, $C([-\tau,0]\times R^l; R^n)$ is the space of all continuous R^n-valued functions defined on $[-\tau,0]\times R^l$ with a norm $\|\xi_i(t)\|_2^2 = \int_X \xi_i^2(t,x)dx$.

The following assumptions and lemmas will be used in establishing our main results.

(A1) There exist constants l_i^-, l_i^+, m_i^- and m_i^+ such that

$$l_i^- \le \frac{f_i(u) - f_i(v)}{u - v} \le l_i^+, \quad m_i^- \le \frac{g_i(u) - g_i(v)}{u - v} \le m_i^+, \quad \forall u, v \in R,\ u \ne v,\ i = 1, \ldots, n. \tag{2.5}$$

(A2) There exist constants μ and $\gamma_i > 0$ such that

$$\dot{\tau}_i(t) \le \mu, \quad y_i(t,x) c_i(y_i(t,x)) \ge \gamma_i y_i^2(t,x), \quad x \in X,\ i = 1, \ldots, n. \tag{2.6}$$

(A3) d_i is bounded, positive, and continuous, that is, there exist constants $\underline{d_i}$, $\overline{d_i}$ such that $0 < \underline{d_i} \le d_i(u) \le \overline{d_i}$, for $u \in R$, $i = 1, 2, \ldots, n$.

Lemma 2.1 (Poincaré inequality, [47]). *Assume that a real-valued function $w(x) : X \to R$ satisfies $w(x) \in D = \{w(x) \in L^2(X), (\partial w / \partial x_i) \in L^2(X)\ (1 \le i \le l), (\partial w(x)/\partial v)|_{\partial X} = 0\}$, where X is a bounded domain of R^l with a smooth boundary ∂X. Then,*

$$\lambda_1 \int_X |w(x)|^2 dx \le \int_X |\nabla w(x)|^2 dx, \tag{2.7}$$

which λ_1 is the lowest positive eigenvalue of the Neumann boundary problem:

$$-\Delta u(x) = \lambda u(x), \quad \left.\frac{\partial u(x)}{\partial v}\right|_{\partial X} = 0, \quad x \in X, \tag{2.8}$$

$\nabla = (\partial/\partial x_1, \ldots, \partial/\partial x_m)$ is the gradient operator, $\Delta = \sum_{k=1}^m (\partial^2/\partial x_k^2)$ is the Laplace operator.

Remark 2.2. Assumption (A1) is less conservative than that in [26, 28], since the constants l_i^-, l_i^+, m_i^-, and m_i^+ are allowed to be positive, negative, or zero, that is to say, the activation function in (A1) is assumed to be neither monotonic, differentiable, nor bounded. Assumption (A2) is weaker than those given in [23, 27, 30] since μ is not required to be zero or smaller than 1 and is allowed to take any value.

Remark 2.3. According to the eigenvalue theory of elliptic operators, the lowest eigenvalue λ_1 is only determined by X [47]. For example, if $X = [0, L]$, then $\lambda_1 = (\pi/L)^2$; if $X = (0,a) \times (0,b)$, then $\lambda_1 = \min\{(\pi/a)^2, (\pi/b)^2\}$.

The notation $A > 0$ (resp., $A \ge 0$) means that matrix A is symmetric-positive definite (resp., positive semidefinite). A^T denotes the transpose of the matrix A. $\lambda_{\min}(A)$ represents the minimum eigenvalue of matrix A. $\|y(t)\|^2 = \int_X y^T(t,x) y(t,x) dx = \sum_{i=1}^n \|y_i(t)\|_2^2$.

3. Main Results

Theorem 3.1. *Suppose that assumptions (A1)–(A3) hold and there exist some matrices $P = \mathrm{diag}(p_1, \ldots, p_n) > 0$, $Q_i \ge 0$, $\sigma_i > 0$, $V_i = \mathrm{diag}(v_{i1}, \ldots, v_{in}) \ge 0\ (i = 1, 2)$, $U_j = \mathrm{diag}(u_{j1}, \ldots, u_{jn}) \ge 0\ (j = 1, 2, 3)$, and σ_3 such that the following linear matrix inequality hold:*

(A4)

$$\Sigma = \begin{pmatrix} \Sigma_1 & \sigma_3 & L_2U_1 & M_2U_3 & 0 \\ * & \Sigma_2 & 0 & 0 & M_2U_2 \\ * & * & \Sigma_3 & 0 & 0 \\ * & * & * & \Sigma_4 & 0 \\ & * & * & * & \Sigma_5 \end{pmatrix} < 0,$$

$$\begin{aligned} &\operatorname{trace}\left[\sigma^T(y(t,x), y(t-\tau(t),x))P\sigma(y(t,x), y(t-\tau(t),x))\right] \\ &\quad \le y^T(t,x)\sigma_1 y(t,x) + y^T(t-\tau(t),x)\sigma_2 y(t-\tau(t),x) + 2y^T(t,x)\sigma_3 y(t-\tau(t),x), \end{aligned} \tag{3.1}$$

where $x \in X, *$ *means the symmetric term,*

$$\begin{gathered} \Sigma_1 = -2\lambda_1 PD - 2\gamma\underline{d}P + 3\overline{d}^2 P + M_3V_1M_3 + \sigma_1 + Q_1 - 2L_1U_1 - 2M_1U_3, \\ \Sigma_2 = M_3V_2M_3 + \sigma_2 - (1-\mu)Q_1 - 2M_1U_2, \\ \Sigma_3 = A^TPA - 2U_1, \qquad \Sigma_4 = Q_2 - V_1 - 2U_3, \\ \Sigma_5 = B^TPB - (1-\mu)Q_2 - V_2 - 2U_2, \\ D = \operatorname{diag}(D_1,\dots,D_n), \qquad D_i = \min_{1\le k\le l}\{D_{ik}\}, \qquad \gamma = \operatorname{diag}(\gamma_1,\dots,\gamma_n), \\ \overline{d} = \operatorname{diag}\left(\overline{d}_1,\dots,\overline{d}_n\right), \qquad \underline{d} = \operatorname{diag}\left(\underline{d}_1,\dots,\underline{d}_n\right), \\ L_1 = \operatorname{diag}(l_1^-l_1^+,\dots,l_n^-l_n^+), \qquad L_2 = \operatorname{diag}(l_1^- + l_1^+,\dots,l_n^- + l_n^+), \\ M_1 = \operatorname{diag}(m_1^-m_1^+,\dots,m_n^-m_n^+), \qquad M_2 = \operatorname{diag}(m_1^- + m_1^+,\dots,m_n^- + m_n^+), \\ M_3 = \operatorname{diag}(\max\{|m_1^-|,|m_1^+|\},\dots,\max\{|m_n^-|,|m_n^+|\}). \end{gathered} \tag{3.2}$$

Then system (2.1) *is stochastically ultimately bounded, that is, if for any* $\varepsilon \in (0,1)$*, there is a positive constant* $C = C(\varepsilon)$ *such that the solution* $y(t,x)$ *of system* (2.1) *satisfies*

$$\limsup_{t\to\infty} P\{\|y(t)\| \le C\} \ge 1 - \varepsilon. \tag{3.3}$$

Proof. If $\mu \le 1$, then it follows from (A4) that there exists a sufficiently small $\lambda > 0$ such that

$$\Delta = \begin{pmatrix} \Delta_1 & \sigma_3 & L_2U_1 & M_2U_3 & 0 \\ * & \Delta_2 & 0 & 0 & M_2U_2 \\ * & * & \Delta_3 & 0 & 0 \\ * & * & * & \Delta_4 & 0 \\ & * & * & * & \Delta_5 \end{pmatrix} < 0, \tag{3.4}$$

where

$$\begin{gathered}
\Delta_1 = -2\lambda_1 PD - 2\gamma \underline{d} P + \lambda P + 3\overline{d}^2 P + 2\lambda I + M_3 V_1 M_3 + \sigma_1 + Q_1 - 2L_1 U_1 - 2M_1 U_3, \\
\Delta_2 = \lambda I + M_3 V_2 M_3 + \sigma_2 - (1-\mu) e^{-\lambda\tau} Q_1 - 2M_1 U_2, \\
\Delta_3 = \lambda I + A^T P A - 2U_1, \qquad \Delta_4 = \lambda I + Q_2 - V_1, \\
\Delta_5 = \lambda I + B^T P B - (1-\mu) e^{-\lambda\tau} Q_2 - V_2 - 2U_2.
\end{gathered} \tag{3.5}$$

If $\mu > 1$, then it follows from (A4) that there exists a sufficiently small $\lambda > 0$ such that

$$\overline{\Delta} = \begin{pmatrix} \Delta_1 & \sigma_3 & L_2 U_1 & M_2 U_3 & 0 \\ * & \overline{\Delta}_2 & 0 & 0 & M_2 U_2 \\ * & * & \Delta_3 & 0 & 0 \\ * & * & * & \Delta_4 & 0 \\ & * & * & * & \overline{\Delta}_5 \end{pmatrix} < 0, \tag{3.6}$$

where Δ_1, Δ_3, and Δ_4 are the same as in (3.4),

$$\begin{gathered}
\overline{\Delta}_2 = \lambda I + M_3 V_2 M_3 + \sigma_2 - (1-\mu) Q_1 - 2M_1 U_2, \\
\overline{\Delta}_5 = \lambda I + B^T P B - (1-\mu) Q_2 - V_2 - 2U_2.
\end{gathered} \tag{3.7}$$

Consider the following Lyapunov functional:

$$\begin{aligned}
V(y(t)) = {} & \int_X e^{\lambda t} y^T(t,x) P y(t,x) dx \\
& + \int_X \int_{t-\tau(t)}^{t} e^{\lambda s} \left[y^T(s,x) Q_1 y(s,x) + g^T(y(s,x)) Q_2 g(y(s,x)) \right] ds\, dx.
\end{aligned} \tag{3.8}$$

Applying Itô formula in [48] to $V(y(t))$ along (2.2), one obtains

$$\begin{aligned}
dV(y(t)) = {} & \int_X \lambda e^{\lambda t} y^T(t,x) P y(t,x) dx\, dt \\
& + 2\sum_{i=1}^{n} p_i e^{\lambda t} \int_X y_i(t,x) \sum_{k=1}^{l} \frac{\partial}{\partial x_k} \left(D_{ik} \frac{\partial y_i}{\partial x_k} \right) dx\, dt \\
& - 2e^{\lambda t} \int_X y^T(t,x) P d(y(t,x)) \left[c(y(t,x)) - A f(y(t,x)) - B g(y(t-\tau(t),x)) - J \right] dx\, dt \\
& + e^{\lambda t} \int_X \operatorname{trace}\left[\sigma^T(y(t,x), y(t-\tau(t),x), x) P \sigma(y(t,x), y(t-\tau(t),x), x) \right] dx\, dt \\
& + 2e^{\lambda t} \int_X y^T(t,x) P \sigma(y(t,x), y(t-\tau(t),x), x) dx\, dw(t)
\end{aligned}$$

$$
\begin{aligned}
&+\int_X e^{\lambda t}\Big[y^T(t,x)Q_1 y(t,x) + g^T(y(t,x))Q_2 g(y(t,x))\Big]dx\,dt \\
&-(1-\dot{\tau}(t))e^{\lambda(t-\tau(t))}\Big[y^T(t-\tau(t),x)Q_1 y(t-\tau(t),x) \\
&\qquad\qquad +g^T(y(t-\tau(t),x))Q_2 g(y(t-\tau(t),x))\Big]dx\,dt.
\end{aligned}
\tag{3.9}
$$

From assumptions (A1)–(A4), one obtains

$$
\begin{aligned}
&2\int_X y^T(t,x)Pd(y(t,x))c(y(t,x))dx \geq 2\int_X y^T(t,x)P\underline{d}\gamma y(t,x)dx, \\
&2\int_X y^T(t,x)Pd(y(t,x))Af(y(t,x))dx \\
&\quad = 2\int_X y^T(t,x)d(y(t,x))PAf(y(t,x))dx \\
&\quad \leq \int_X y^T(t,x)d^2(y(t,x))Py(t,x) + f^T(y(t,x))A^T PAf(y(t,x))dx \\
&\quad \leq \int_X y^T(t,x)\overline{d}^2 Py(t,x) + f^T(y(t,x))A^T PAf(y(t,x))dx, \\
&2\int_X y^T(t,x)Pd(y(t,x))Bg(y(t-\tau(t),x))dx \\
&\quad \leq \int_X y^T(t,x)\overline{d}^2 Py(t,x) + g^T(y(t-\tau(t),x))B^T PBg(y(t-\tau(t),x))dx, \\
&2\int_X y^T(t,x)Pd(y(t,x))Jdx \leq \int_X y^T(t,x)\overline{d}^2 Py(t,x) + J^T PJdx.
\end{aligned}
\tag{3.10}
$$

From the boundary condition and Lemma 2.1, one obtains

$$
\begin{aligned}
&\sum_{k=1}^{l}\int_X y_i \frac{\partial}{\partial x_k}\left(D_{ik}\frac{\partial y_i}{\partial x_k}\right)dx \\
&\quad = \int_X y_i \nabla\cdot\left(D_{ik}\frac{\partial y_i}{\partial x_k}\right)_{k=1}^{l} dx \\
&\quad = \int_X \nabla\cdot\left(y_i D_{ik}\frac{\partial y_i}{\partial x_k}\right)_{k=1}^{l} dx - \int_X \left(D_{ik}\frac{\partial y_i}{\partial x_k}\right)_{k=1}^{l}\cdot\nabla y_i dx \\
&\quad = \sum_{k=1}^{l}\int_{\partial X}\left(y_i D_{ik}\frac{\partial y_i}{\partial x_k}\right)_{k=1}^{l}\cdot ds - \sum_{k=1}^{l}\int_X D_{ik}\left(\frac{\partial y_i}{\partial x_k}\right)^2 dx
\end{aligned}
$$

$$= -\sum_{k=1}^{l}\int_X D_{ik}\left(\frac{\partial y_i}{\partial x_k}\right)^2 dx \leq \sum_{k=1}^{l}\int_X D_i\left(\frac{\partial y_i}{\partial x_k}\right)^2 dx$$

$$= -D_i\int_X |\nabla y_i|^2 dx \leq -\lambda_1 D_i\int_X |y_i|^2 dx = -\lambda_1 D_i\|y_i\|_2^2, \tag{3.11}$$

where "$\cdot$" is inner product, $D_i = \min_{1\leq k\leq l}\{D_{ik}\}$,

$$\left(D_{ik}\frac{\partial y_i}{\partial x_k}\right)_{k=1}^{l} = \left(\left(D_{i1}\frac{\partial y_i}{\partial x_1}\right),\ldots,\left(D_{il}\frac{\partial y_i}{\partial x_l}\right)\right)^T. \tag{3.12}$$

Combining (3.10) and (3.11) into (3.9), we have

$$\begin{aligned}
dV(y(t)) \leq{} & \int_X e^{\lambda t}y^T(t,x)\left[\lambda P - 2\lambda_1 PD - 2P\underline{d}\gamma + 3\overline{d}^2 P\right]y(t,x)dx\,dt \\
&+ \int_X e^{\lambda t}\left[f^T(y(t,x))A^T PAf(y(t,x)) + g^T(y(t-\tau(t),x))B^T PBg(y(t-\tau(t),x))\right]dx\,dt \\
&+ \int_X e^{\lambda t}J^T PJ dx + \int_X e^{\lambda t}\Big[y^T(t,x)\sigma_1 y(t,x) + y^T(t-\tau(t),x)\sigma_2 y(t-\tau(t),x) \\
&\qquad +2y^T(t,x)\sigma_3 y(t-\tau(t),x)\Big]dx\,dt \\
&+ \int_X 2e^{\lambda t}y^T(t,x)P\sigma(y(t,x),y(t-\tau(t),x),x)dx\,dw(t) \\
&+ \int_X\Big\{e^{\lambda t}\left[y^T(t,x)Q_1 y(t,x) + g^T(y(t,x))Q_2 g(y(t,x))\right] - (1-\mu)h(\mu)e^{\lambda t} \\
&\qquad \times\left[y^T(t-\tau(t),x)Q_1 y(t-\tau(t),x) + g^T(y(t-\tau(t),x))Q_2 g(y(t-\tau(t),x))\right]\Big\}dx\,dt,
\end{aligned} \tag{3.13}$$

where $h(\mu) = e^{-\lambda\tau}$ $(\mu \leq 1)$ or 1 $(\mu > 1)$.

In addition, it follows from (A1) that

$$y^T(t,x)M_3V_1M_3y(t,x) - g^T(y(t,x))V_1g(y(t,x)) \geq 0,$$

$$y^T(t-\tau(t),x)M_3V_2M_3y(t-\tau(t),x) - g^T(y(t-\tau(t),x))V_2g(y(t-\tau(t),x)) \geq 0,$$

$$\begin{aligned}
0 \le\ & -2\sum_{i=1}^{n} u_{1i}\left[f_i(y_i(t,x)) - f_i(0) - l_i^{+} y_i(t,x)\right]\left[f_i(y_i(t,x)) - f_i(0) - l_i^{-} y_i(t,x)\right] \\
=\ & -2\sum_{i=1}^{n} u_{1i}\left[f_i(y_i(t,x)) - l_i^{+} y_i(t,x)\right]\left[f_i(y_i(t,x)) - l_i^{-} y_i(t,x)\right] \\
& -2\sum_{i=1}^{n} u_{1i} f_i^2(0) + 2\sum_{i=1}^{n} u_{1i} f_i(0)\left[2 f_i(y_i(t,x)) - (l_i^{+} + l_i^{-}) y_i(t,x)\right] \\
\le\ & -2\sum_{i=1}^{n} u_{1i}\left[f_i(y_i(t,x)) - l_i^{+} y_i(t,x)\right]\left[f_i(y_i(t,x)) - l_i^{-} y_i(t,x)\right] \\
& + \sum_{i=1}^{n}\left[\lambda f_i^2(y_i(t,x)) + 4\lambda^{-1} f_i^2(0) u_{1i}^2 + \lambda y_i^2(t,x) + \lambda^{-1} f_i^2(0) u_{1i}^2 (l_i^{+} + l_i^{-})^2\right].
\end{aligned} \tag{3.14}$$

Similarly, one obtains

$$\begin{aligned}
0 \le\ & -2\sum_{i=1}^{n} u_{2i}\left[g_i(y_i(t-\tau_i(t),x)) - g_i(0) - m_i^{+} y_i(t-\tau_i(t),x)\right] \\
& \times\left[g_i(y_i(t-\tau_i(t),x)) - g_i(0) - m_i^{-} y_i(t-\tau_i(t),x)\right] \\
\le\ & -2\sum_{i=1}^{n} u_{2i}\left[g_i(y_i(t-\tau_i(t),x)) - m_i^{+} y_i(t-\tau_i(t),x)\right] \\
& \times\left[g_i(y_i(t-\tau_i(t),x)) - m_i^{-} y_i(t-\tau_i(t),x)\right] \\
& + \sum_{i=1}^{n}\left[\lambda g_i^2(y_i(t-\tau_i(t),x)) + 4\lambda^{-1} g_i^2(0) u_{2i}^2 + \lambda y_i^2(t-\tau_i(t),x) + \lambda^{-1} g_i^2(0) u_{2i}^2 (m_i^{+} + m_i^{-})^2\right], \\
0 \le\ & -2\sum_{i=1}^{n} u_{3i}\left[g_i(y_i(t,x)) - g_i(0) - m_i^{+} y_i(t,x)\right]\left[g_i(y_i(t,x)) - g_i(0) - m_i^{-} y_i(t,x)\right] \\
\le\ & -2\sum_{i=1}^{n} u_{3i}\left[g_i(y_i(t,x)) - m_i^{+} y_i(t,x)\right]\left[g_i(y_i(t,x)) - m_i^{-} y_i(t,x)\right] \\
& + \sum_{i=1}^{n}\left[\lambda g_i^2(y_i(t,x)) + 4\lambda^{-1} g_i^2(0) u_{3i}^2 + \lambda y_i^2(t,x) + \lambda^{-1} g_i^2(0) u_{3i}^2 (m_i^{+} + m_i^{-})^2\right].
\end{aligned} \tag{3.15}$$

From (3.13)–(3.15), one derives

$$\begin{aligned}
dV(y(t)) \le\ & \int_X 2e^{\lambda t} y^T(t,x) P\sigma(y(t,x),\, y(t-\tau(t),x), x)\,dx\,dw(t) \\
& + \int_X e^{\lambda t}\eta^T(t,x)\Delta\eta(t,x)\,dx + e^{\lambda t} C_1,
\end{aligned} \tag{3.16}$$

or

$$dV(y(t)) \leq \int_X 2e^{\lambda t} y^T(t,x) P\sigma(y(t,x),\, y(t-\tau(t),x),x)\,dx\,dw(t) \\ + \int_X e^{\lambda t}\eta^T(t,x)\overline{\Delta}\eta(t,x)dx + e^{\lambda t}C_1, \tag{3.17}$$

where $\eta(t,x) = (y^T(t,x), y^T(t-\tau(t),x), f^T(y(t,x)),\ g^T(y(t,x)), g^T(y(t-\tau(t),x)))^T$,

$$C_1 = \int_X \Bigg\{ J^T P J + \sum_{i=1}^{n} \Big[4\lambda^{-1} f_i^2(0) u_{1i}^2 + \lambda^{-1} f_i^2(0) u_{1i}^2 (l_i^+ + l_i^-)^2 \\ + 4\lambda^{-1} g_i^2(0)\left(u_{2i}^2 + u_{3i}^2\right) + \lambda^{-1} g_i^2(0)\left(u_{2i}^2 + u_{3i}^2\right)(m_i^+ + m_i^-)^2 \Big] \Bigg\} dx. \tag{3.18}$$

Thus, one obtains

$$\lambda_{\min}(P)e^{\lambda t} E\|y(t)\|^2 \leq EV(y(t)) \leq EV(y(0)) + \lambda^{-1}e^{\lambda t}C_1, \tag{3.19}$$

$$E\|y(t)\|^2 \leq \frac{e^{-\lambda t}EV(y(0)) + \lambda^{-1}C_1}{\lambda_{\min}(P)}. \tag{3.20}$$

For any $\varepsilon > 0$, set $C = \sqrt{\lambda^{-1}C_1/\lambda_{\min}(P)\varepsilon}$. By Chebyshev's inequality and (3.20), we obtain

$$\limsup_{t\to\infty} P\{\|y(t)\| > C\} \leq \frac{\limsup_{t\to\infty} E\|y(t)\|^2}{C^2} = \varepsilon, \tag{3.21}$$

which implies

$$\limsup_{t\to\infty} P\{\|y(t)\| \leq C\} \geq 1 - \varepsilon. \tag{3.22}$$

The proof is completed. □

Theorem 3.1 shows that there exists $t_0 > 0$ such that for any $t \geq t_0$, $P\{\|y(t)\| \leq C\} \geq 1-\varepsilon$. Let B_C be denoted by

$$B_C = \{y \mid \|y(t)\| \leq C, t \geq t_0\}. \tag{3.23}$$

Clearly, B_C is closed, bounded, and invariant. Moreover,

$$\limsup_{t\to\infty} \inf_{z\in B_C} \|y(t) - z\| = 0 \tag{3.24}$$

with no less than probability $1-\varepsilon$, which means that B_C attracts the solutions infinitely many times with no less than probability $1-\varepsilon$, so we may say that B_C is a weak attractor for the solutions.

Theorem 3.2. *Suppose that all conditions of Theorem 3.1 hold. Then there exists a weak attractor B_C for the solutions of system* (2.1).

Theorem 3.3. *Suppose that all conditions of Theorem 3.1 hold and $c(0) = f(0) = g(0) = J = 0$. Then zero solution of system* (2.1) *is mean square exponential stability.*

Remark 3.4. Assumption (A4) depends on λ_1 and μ, so the criteria on the stability, ultimate boundedness, and weak attractor depend on diffusion effects and the derivative of the delays and are independent of the magnitude of the delays.

4. An Example

In this section, a numerical example is presented to demonstrate the validity and effectiveness of our theoretical results.

Example 4.1. Consider the following system

$$\begin{aligned} dy(t,x) = \operatorname{col}&\left\{\sum_{k=1}^{l}\frac{\partial}{\partial x_k}\left(D_{ik}\frac{\partial y_i(t,x)}{\partial x_k}\right)\right\}dt - d\big(y(t,x)\big) \\ &\times\big[c\big(y(t,x)\big) - Af\big(y(t,x)\big) - Bg\big(y(t-\tau(t),x)\big) - J\big]dt \\ &+ \big[Gy(t,x) + Hy(t-\tau(t),x)\big]dw(t), \quad x \in X, \end{aligned} \tag{4.1}$$

where $n = 2$, $l = m = 1$, $X = [0,\pi]$, $D_{11} = D_{21} = 0.5$, $d_1(y_1(t)) = 0.3 + 0.1\cos y_1(t)$, $d_2(y_2(t)) = 0.3 + 0.1\sin y_2(t)$, $c(y(t)) = \gamma y(t)$, $f(y) = g(y) = 0.1\tanh(y)$,

$$\begin{gathered} A = \begin{pmatrix} -0.5 & 0.4 \\ 0.2 & -0.5 \end{pmatrix}, \qquad B = \begin{pmatrix} 0.4 & -0.7 \\ -0.8 & 0.4 \end{pmatrix}, \qquad \gamma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\ J = \begin{pmatrix} 0.01 \\ 0.01 \end{pmatrix}, \qquad G = H = \begin{pmatrix} 0.2 & 0.1 \\ 0.1 & 0.2 \end{pmatrix}, \end{gathered} \tag{4.2}$$

$w(t)$ is one-dimensional Brownian motion. Then we compute that $\lambda_1 = 1$, $D = \operatorname{diag}(0.5, 0.5)$, $L_1 = M_1 = 0$, $L_2 = M_2 = M_3 = \operatorname{diag}(0.1, 0.1)$, $\underline{d} = \operatorname{diag}(0.2, 0.2)$, $\overline{d} = \operatorname{diag}(0.4, 0.4)$,

$\sigma_1 = G^T PG$, $\sigma_2 = H^T PH$, and $\sigma_3 = G^T PH$. By using the Matlab LMI Toolbox, for $\mu = 0.1$, based on Theorem 3.1, such system is stochastically ultimately bounded when

$$P = \begin{pmatrix} 23.9409 & 0 \\ 0 & 24.5531 \end{pmatrix}, \qquad U_1 = \begin{pmatrix} 13.8701 & 0 \\ 0 & 15.0659 \end{pmatrix},$$
$$U_2 = \begin{pmatrix} 7.5901 & 0 \\ 0 & 6.4378 \end{pmatrix}, \qquad U_3 = \begin{pmatrix} 11.8008 & 0 \\ 0 & 11.6500 \end{pmatrix}$$
$$Q_1 = \begin{pmatrix} 13.7292 & -0.0345 \\ -0.0345 & 13.9274 \end{pmatrix}, \qquad Q_2 = \begin{pmatrix} 16.9580 & -4.6635 \\ -4.6635 & 16.5060 \end{pmatrix}, \tag{4.3}$$
$$V_1 = \begin{pmatrix} 15.1844 & 0 \\ 0 & 15.1109 \end{pmatrix}, \qquad V_2 = \begin{pmatrix} 13.0777 & 0 \\ 0 & 12.4917 \end{pmatrix}.$$

5. Conclusion

In this paper, new results and sufficient criteria on the ultimate boundedness, weak attractor, and stability are established for stochastic reaction-diffusion Cohen-Grossberg neural networks with delays by using Lyapunov method, Poincaré inequality and matrix technique. The criteria depend on diffusion effect and derivative of the delays and are independent of the magnitude of the delays.

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