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Research Article

A Note on Impulsive Fractional Evolution Equations with Nondense Domain

Zufeng Zhang[1, 2] and Bin Liu[1]

[1] *School of Mathematics and Statistics, Huazhong University of Science and Technology, Wuhan 430074, China*
[2] *School of Mathematics and Statistics, Suzhou University, Suzhou 234000, China*

Correspondence should be addressed to Bin Liu, binliu@mail.hust.edu.cn

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This paper is concerned with the existence of integral solutions for nondensely defined fractional functional differential equations with impulse effects. Some errors in the existing paper concerned with nondensely defined fractional differential equations are pointed out, and correct formula of integral solutions is established by using integrated semigroup and some probability densities. Sufficient conditions for the existence are obtained by applying the Banach contraction mapping principle. An example is also given to illustrate our results.

1. Introduction

The aim in this paper is to study the existence of the integral solutions for the fractional semilinear differential equations of the form

$$\begin{gathered} D^q y(t) = Ay(t) + f(t, y_t), \quad t \in J := [0, b], \quad t \neq t_k, \quad k = 1, \dots, m, \\ \Delta y\big|_{t=t_k} = I_k\left(y\left(t_k^-\right)\right), \quad k = 1, \dots, m, \\ y(t) = \phi(t), \quad t \in [-\tau, 0], \end{gathered} \tag{1.1}$$

where $0 < q < 1$, D^q is the Caputo fractional derivative. $f : J \times \mathfrak{D} \to E$ is a given function, $\mathfrak{D} = \{\psi : [-\tau, 0] \to E, \psi$ is continuous everywhere except for a finite number of points s at which $\psi(s^-), \psi(s^+)$ exist and $\psi(s^-) = \psi(s)\}$, and E is a real Banach space with the norm $|\cdot|$. Denoting the domain of A by $D(A)$, $A : D(A) \subset E \to E$ is nondensely closed linear operator on E, $\phi \in \mathfrak{D}$. $I_k : E \to E$, $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = b$, $y(t_k^+)$ and $y(t_k^-)$ represent the right and left limits at t_k of $y(t)$ as usual; we assume $y(t_k^-) = y(t_k)$. $\Delta y|_{t=t_k} = y(t_k^+) - y(t_k^-)$ represents the jump in the state y at time t_k. Moreover, for any $t \in J$, the histories y_t belong to $\mathfrak{D}$ defined by $y_t(\varsigma) = y(t+\varsigma), \varsigma \in [-\tau, 0]$.

In the past decades, the theory of fractional differential equations has become an important area of investigation because of its wide applicability in many branches of physics, economics, and technical sciences [1–10]. In recent years, many authors were devoted to mild solutions to fractional evolution equations, and there have been a lot of interesting works. For instance, in [11], El-Borai discussed the following equation in Banach space X:

$$\begin{gathered} D^q u(t) = Au(t) + B(t)u(t), \\ u(0) = u_0, \end{gathered} \tag{1.2}$$

where A generates an analytic semigroup, and the solution was given in terms of some probability densities. In [12], Zhou and Jiao concerned the existence and uniqueness of mild solutions for fractional evolution equations by some fixed point theorems. Cao et al. [13] studied the α-mild solutions for a class of fractional evolution equations and optimal controls in fractional powder space. For more information on this subject, the readers may refer to [14–16] and the references therein.

Research on integer order differential evolution equations including a nondensely defined operator was initialed by Da Prato and Sinestrari [17] and has been extensively investigated by many authors [18–25]. The main methods used in their work are based on integrated semigroup theory. Recently, existence results for integral solutions of nondensely defined fractional evolution equations were established in some papers [9, 26]. But there are some errors in transforming integral solution into an available form. For example, definition of integral solution [9] is given by

$$x(t) = S(t)\big(x_0 - g(x)\big) + \lim_{\lambda\to\infty} \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} S(t-s) B(\lambda, A) f(s, x(s)) ds, \quad t \ge 0. \tag{1.3}$$

Here $D(A) \subset E$ and $\overline{D(A)} \neq E$. $B(\lambda, A) := \lambda(\lambda I - A)^{-1}$ will be introduced in next section. If we let f take values in $\overline{D(A)}$, then (1.3) becomes

$$x(t) = S(t)\big(x_0 - g(x)\big) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} S(t-s) f(s, x(s)) ds. \tag{1.4}$$

According to [19], integral solution should be mild solution in this case. But as pointed in [14], (1.4) is not the mild solution.

Motivated by these papers and the fact that impulse effects exist widely in the realistic situations, we give the definition of integral solution and prove the existence results for impulsive semilinear fractional differential equations with nondensely defined operators. The rest of the paper will be organized as follows. In Section 2, we will recall some basic definitions and preliminary facts from integrated semigroups and fractional derivation and integration which would be used later. Section 3 is devoted to the existence of integral solutions of problem (1.1). We present an example to illustrate our results in Section 4. At last, we end the paper with a conclusion.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary results which would be used in the rest of the paper.

We denote by $C([0,b];E)$ the Banach space of all continuous functions from $[0,b]$ into E with the norm

$$\|x\|_\infty = \sup\{|y(t)| : t \in [0,b]\}. \tag{2.1}$$

For $\phi \in \mathfrak{D}$ the norm of ϕ is defined by

$$\|\phi\|_{\mathfrak{D}} = \sup\{|\phi(\varsigma)| : \varsigma \in [-\tau,0]\}. \tag{2.2}$$

$B(E)$ denotes the Banach space of bounded linear operators from E into E, with the norm

$$\|N\| = \sup\{|N(y)| : |y| = 1\}, \tag{2.3}$$

where $N \in B(E)$ and $y \in E$. Let $L^p([0,b];E)$ be the space of E-valued Bochner function on $[0,b]$ with the norm

$$\|x\|_{L^p} = \left(\int_0^b |y(s)|^p ds\right)^{1/p}, \quad 1 \le p < \infty. \tag{2.4}$$

In order to define an integral solution of problem (1.1), we will introduce the set of functions

$$\begin{aligned} \mathrm{PC} = \Big\{ y : J \longrightarrow \overline{D(A)},\ &\text{is continuous except for } t = t_k,\ k = 1,2,\dots,m, \\ &\text{there exist } y(t_k^-) \text{ and } y(t_k^+) \text{ such that } y(t_k^-) = y(t_k) \Big\}. \end{aligned} \tag{2.5}$$

Endowed with the norm $\|y\|_{\mathrm{PC}} = \sup_{t\in J}|y(t)|$, $(\mathrm{PC}, \|\cdot\|_{\mathrm{PC}})$ is a Banach space.

Seting

$$\Omega = \left\{ y : [-\tau,b] \longrightarrow \overline{D(A)} : y \in \mathfrak{D} \cap \mathrm{PC} \right\}, \tag{2.6}$$

then Ω is a Banach space with the norm

$$\|y\|_\Omega = \max\{\|y\|_{\mathfrak{D}}, \|y\|_{\mathrm{PC}}\}. \tag{2.7}$$

Definition 2.1 (see [27]). Letting E be a Banach space, an integrated semigroup is a family of operators $(S(t))_{t\ge 0}$ of bounded linear operators $S(t)$ on E with the following properties:

(i) $S(0) = 0$;

(ii) $t \to S(t)$ is strongly continuous;

(iii) $S(s)S(t) = \int_0^s (S(t+r) - S(r))dr$ for all $t, s \ge 0$.

Definition 2.2 (see [28]). An operator A is called a generator of an integrated semigroup, if there exists $\omega \in \mathbf{R}$ such that $(\omega, +\infty) \subset \rho(A)$, and there exists a strongly continuous exponentially bounded family $(S(t))_{t\geq 0}$ of linear bounded operators such that $S(0) = 0$ and $(\lambda I - A)^{-1} = \lambda \int_0^\infty e^{-\lambda t} S(t)dt$ for all $\lambda > \omega$.

Proposition 2.3 (see [27]). *Let A be the generator of an integrated semigroup $(S(t))_{t\geq 0}$. Then for all $x \in E$ and $t \geq 0$,*

$$\int_0^t S(s)xds \in D(A), \qquad S(t)x = A\int_0^t S(s)xds + tx. \tag{2.8}$$

Definition 2.4 (see [29]). We say that a linear operator A satisfies the Hille-Yosida condition if there exists $M \geq 0$ and $\omega \in \mathbf{R}$ such that $(\omega, +\infty) \subset \rho(A)$ and

$$\sup\left\{(\lambda - \omega)^n \left\|(\lambda I - A)^{-n}\right\|,\ n \in \mathbf{N}, \lambda > \omega\right\} \leq M. \tag{2.9}$$

Here and hereafter, we assume that A satisfies the Hille-Yosida condition. Let us introduce the part A_0 of A in $\overline{D(A)}$: $A_0 = A$ on $D(A_0) = \{x \in D(A); Ax \in \overline{D(A)}\}$. Let $(S(t))_{t\geq 0}$ be the integrated semigroup generated by A. We note that $(S'(t))_{t\geq 0}$ is a C_0-semigroup on $\overline{D(A)}$ generated by A_0 and $\|S'(t)\| \leq Me^{\omega t}$, $t \geq 0$, where M and ω are the constants considered in the Hille-Yosida condition [28, 30].

Let $B(\lambda, A) := \lambda(\lambda I - A)^{-1}$; then for all $x \in \overline{D(A)}$, $B(\lambda, A)x \to x$ as $\lambda \to \infty$. Also from the Hille-Yosida condition it is easy to see that $\lim_{\lambda\to\infty}|B(\lambda, A)x| \leq M|x|$.

For more properties on integral semigroup theory the interested reader may refer to [18, 30].

Definition 2.5 (see [3]). The Riemann-Liouville fractional integral of order $q \in \mathbf{R}^+$ of a function $h : (0, b] \to E$ is defined by

$$I_0^q h(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} h(s)ds, \tag{2.10}$$

provided the right-hand side is pointwise defined on $(0, b]$ and where Γ is the gamma function.

Remark 2.6. According to [10], $I_0^q I_0^\beta = I_0^{q+\beta}$ holds for all $q, \beta \geq 0$.

Definition 2.7 (see [3]). The Caputo fractional derivative of order $0 < q < 1$ of a function $f \in C^1([0, \infty), E)$ is defined by

$$D^q f(t) = \frac{1}{\Gamma(1-q)} \int_0^t (t-s)^{-q} f'(s)ds, \quad t > 0. \tag{2.11}$$

3. Main Results

In this section we will establish the existence and uniqueness of integral solution for problem (1.1).

Definition 3.1. A function $y \in \Omega$ is said to be an integral solutions of (1.1) if

(i) $\int_{t_k}^{t}(t-s)^{q-1}y(s)ds \in D(A)$ for $t \in (t_k, t_{k+1}]$, $k = 0, 1, \ldots, m$,

(ii) $y(t) = \phi(t)$, $t \in [-\tau, 0]$,

(iii)

$$y(t) = \begin{cases} \phi(0) + \dfrac{1}{\Gamma(q)} A \displaystyle\int_0^t (t-s)^{q-1} y(s)ds + \dfrac{1}{\Gamma(q)} \displaystyle\int_0^t (t-s)^{q-1} f(s, y_s)ds, & t \in (0, t_1], \\ y(t_1^-) + I_1(y(t_1^-)) + \dfrac{1}{\Gamma(q)} A \displaystyle\int_{t_1}^t (t-s)^{q-1} y(s)ds & \\ \quad + \dfrac{1}{\Gamma(q)} \displaystyle\int_{t_1}^t (t-s)^{q-1} f(s, y_s)ds, & t \in (t_1, t_2], \\ \vdots & \\ y(t_m^-) + I_m(y(t_m^-)) + \dfrac{1}{\Gamma(q)} A \displaystyle\int_{t_m}^t (t-s)^{q-1} y(s)ds & \\ \quad + \dfrac{1}{\Gamma(q)} \displaystyle\int_{t_m}^t (t-s)^{q-1} f(s, y_s)ds, & t \in (t_m, b]. \end{cases} \tag{3.1}$$

Lemma 3.2. *If y is an integral solution of* (1.1), *then for all $t \in [0, b]$, $y(t) \in \overline{D(A)}$. In particular, $\phi(0)$, $y(t_1^-) + I_1(y(t_1^-))$,$\ldots$,$y(t_m^-) + I_1(y(t_m^-))$ belong to $\overline{D(A)}$.*

Proof. Using Remark 2.6, for each $t \in (t_k, t_{k+1}]$, $I_{t_k}^1 y(t) = I_{t_k}^{1-q} I_{t_k}^q y(t) \in D(A)$ since $I_{t_k}^q y(t) \in D(A)$. Consequently, for $h > 0$ such that $t + h \in (t_k, t_{k+1}]$, $(1/h)\int_t^{t+h} y(s)ds \in D(A)$ because $I_{t_k}^1 y(t) = \int_{t_k}^t y(s)ds \in D(A)$. Hence, we deduce that $y(t) = \lim_{h\to 0}(1/h)\int_t^{t+h} y(s)ds \in \overline{D(A)}$. The proof is completed. □

Lemma 3.3 (see [31]). *Let $\Psi_q(\theta) = (1/\pi)\sum_{n=1}^{\infty}(-1)^{n-1}\theta^{-qn-1}(\Gamma(nq+1)/n!)\sin(n\pi q)$, $\theta \in \mathbf{R}^+$; then $\Psi_q(\theta)$ is a one-sided stable probability density function, and its Laplace transform is given by*

$$\int_0^{\infty} e^{-p\theta}\Psi_q(\theta)d\theta = e^{-p^q}, \quad q \in (0,1),\ p > 0. \tag{3.2}$$

Lemma 3.4. *For $t \in (0, b]$, the integral solution in Definition 3.1 is given by*

$$y(t) = \begin{cases} \mathfrak{S}(t)\phi(0) + \lim\limits_{\lambda\to\infty} \displaystyle\int_0^t (t-s)^{q-1}\mathfrak{T}(t-s)B(\lambda, A)f(s, y_s)ds, & t \in (0, t_1], \\ \mathfrak{S}(t-t_1)(y(t_1^-) + I_1(y(t_1^-))) & \\ \quad + \lim\limits_{\lambda\to\infty} \displaystyle\int_{t_1}^t (t-s)^{q-1}\mathfrak{T}(t-s)B(\lambda, A)f(s, y_s)ds, & t \in (t_1, t_2], \\ \vdots & \\ \mathfrak{S}(t-t_m)(y(t_m^-) + I_m(y(t_m^-))) & \\ \quad + \lim\limits_{\lambda\to\infty} \displaystyle\int_{t_m}^t (t-s)^{q-1}\mathfrak{T}(t-s)B(\lambda, A)f(s, y_s)ds, & t \in (t_m, b], \end{cases} \tag{3.3}$$

where

$$\mathfrak{S}(t) = \int_0^\infty h_q(\theta) S'(t^q\theta)d\theta, \quad \mathfrak{T}(t) = q\int_0^\infty \theta h_q(\theta) S'(t^q\theta)d\theta, \tag{3.4}$$

where $h_q(\theta) = (1/q)\theta^{-1-1/q}\Psi_q(\theta^{-1/q})$ *is the probability density function defined on* $\mathbf{R}^+$.

Proof. From the definition, for $t \in (0, t_1]$ we have

$$y(t) = \phi(0) + \frac{1}{\Gamma(q)} A\int_0^t (t-s)^{q-1} y(s)ds + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} f(s, y_s)ds, \quad t \in [0, b]. \tag{3.5}$$

Consider the Laplace transform

$$v(p) = \int_0^\infty e^{-pt} B(\lambda, A) y(t)dt, \qquad w(p) = \int_0^\infty e^{-pt} B(\lambda, A) f(t, y_t)dt, \quad p > 0. \tag{3.6}$$

Note that for each $0 < t \leq t_1$, $B_\lambda y(t)$, $B(\lambda, A) f(t, y_t) \in D(A)$, then we have $v(p), w(p) \in \overline{D(A)}$. Applying (3.6) to (3.5) yields

$$\begin{aligned} v(p) &= \frac{1}{p} B(\lambda, A)\phi(0) + \frac{1}{p^q} A v(p) + \frac{1}{p^q} w(p) \\ &= p^{q-1}(p^q I - A)^{-1} B(\lambda, A)\phi(0) + (p^q I - A)^{-1} w(p) \\ &= p^{q-1}\int_0^\infty e^{-p^q s} S'(s) B(\lambda, A)\phi(0) ds + \int_0^\infty e^{-p^q s} S'(s) w(p) ds, \end{aligned} \tag{3.7}$$

where I is the identity operator defined on E.

From (3.2), we get

$$\begin{aligned} & p^{q-1}\int_0^\infty e^{-p^q s} S'(s) B(\lambda, A)\phi(0) ds \\ &= \int_0^\infty p^{q-1} e^{-(pt)^q} S'(t^q) B(\lambda, A)\phi(0) q t^{q-1} dt \\ &= \int_0^\infty -\frac{1}{p}\frac{d}{dt}\left(e^{-(pt)^q}\right) S'(t^q) B(\lambda, A)\phi(0) dt \\ &= \iint_0^\infty \left(\theta\Psi_q(\theta) e^{-pt\theta} S'(t^q) B(\lambda, A)\phi(0)\right) d\theta\, dt \\ &= \iint_0^\infty \left(\Psi_q(\theta) e^{-ps} S'\left(\left(\frac{s}{\theta}\right)^q\right) B(\lambda, A)\phi(0)\right) d\theta\, ds \\ &= \int_0^\infty e^{-pt}\left(\int_0^\infty \Psi_q(\theta) S'\left(\left(\frac{t}{\theta}\right)^q\right) B(\lambda, A)\phi(0) d\theta\right) dt, \end{aligned}$$

$$
\begin{aligned}
&\int_0^\infty e^{-p^q s} S'(s) w(p)\, ds \\
&\quad = \iint_0^\infty e^{-p^q s} e^{-pt} S'(s) B(\lambda, A) f(t, y_t)\, dt\, ds \\
&\quad = \iint_0^\infty q s^{q-1} e^{-(ps)^q} e^{-pt} S'(s^q) B(\lambda, A) f(t, y_t)\, dt\, ds \\
&\quad = \iiint_0^\infty q \Psi_q(\theta) e^{-ps\theta} e^{-pt} S'(s^q) B(\lambda, A) f(t, y_t)\, d\theta\, dt\, ds \\
&\quad = \iiint_0^\infty q \Psi_q(\theta) e^{-p(s+t)} \frac{s^{q-1}}{\theta^q} S'\left(\left(\frac{s}{\theta}\right)^q\right) B(\lambda, A) f(t, y_t)\, d\theta\, dt\, ds \\
&\quad = \int_0^\infty e^{-ps} q \int_0^s \int_0^\infty \Psi_q(\theta) \frac{(s-t)^{q-1}}{\theta^q} S'\left(\frac{(s-t)^q}{\theta^q}\right) B(\lambda, A) f(t, y_t)\, d\theta\, dt\, ds \\
&\quad = \int_0^\infty e^{-pt} q \int_0^t \int_0^\infty \Psi_q(\theta) \frac{(t-s)^{q-1}}{\theta^q} S'\left(\frac{(t-s)^q}{\theta^q}\right) B(\lambda, A) f(s, y_s)\, d\theta\, ds\, dt.
\end{aligned}
\tag{3.8}
$$

According to (3.7) and (3.8), we have

$$
\begin{aligned}
v(p) = &\int_0^\infty e^{-pt} \left(\int_0^\infty \Psi_q(\theta) S'\left(\left(\frac{t}{\theta}\right)^q\right) B(\lambda, A) \phi(0)\, d\theta \right) dt \\
&+ \int_0^\infty e^{-pt} q \int_0^t \int_0^\infty \Psi_q(\theta) \frac{(t-s)^{q-1}}{\theta^q} S'\left(\frac{(t-s)^q}{\theta^q}\right) B(\lambda, A) f(s, y_s)\, d\theta\, ds\, dt.
\end{aligned}
\tag{3.9}
$$

Inverting the last Laplace transform, we obtain

$$
\begin{aligned}
B(\lambda, A) y(t) = &\int_0^\infty \Psi_q(\theta) S'\left(\left(\frac{t}{\theta}\right)^q\right) B(\lambda, A) \phi(0)\, d\theta \\
&+ q \int_0^t \int_0^\infty \Psi_q(\theta) \frac{(t-s)^{q-1}}{\theta^q} S'\left(\frac{(t-s)^q}{\theta^q}\right) B(\lambda, A) f(s, y_s)\, d\theta\, ds \\
= &\int_0^\infty h_q(\theta) S'(t^q \theta) B(\lambda, A) \phi(0)\, d\theta \\
&+ q \int_0^t \int_0^\infty \theta (t-s)^{q-1} h_q(\theta) S'((t-s)^q \theta) B(\lambda, A) f(s, y_s)\, d\theta\, ds.
\end{aligned}
\tag{3.10}
$$

In view of $\lim_{\lambda\to\infty} B(\lambda, A)x = x$ for $x \in \overline{D(A)}$ and Lemma 3.2, we have

$$\begin{aligned} y(t) &= \int_0^\infty h_q(\theta) S'(t^q\theta)\phi(0)d\theta + \lim_{\lambda\to\infty} q\int_0^t\int_0^\infty \theta(t-s)^{q-1}h_q(\theta)S'\left((t-s)^q\theta\right)B(\lambda, A) \\ &\quad \times f\left(s, y_s\right)d\theta\, ds \\ &= \mathfrak{S}(t)\phi(0) + \lim_{\lambda\to\infty}\int_0^t (t-s)^{q-1}\mathfrak{T}(t-s)B(\lambda, A)f\left(s, y_s\right)ds. \end{aligned} \tag{3.11}$$

For $t \in (t_k, t_{k+1}]$, $k = 1, 2, \ldots, m$, we can prove the results by the similar methods used previously. The proof is completed. □

Remark 3.5. According to [31], one can easily check that

$$\int_0^\infty \theta h_q(\theta)d\theta = \int_0^\infty \frac{1}{\theta^q}\Psi_q(\theta)d\theta = \frac{1}{\Gamma(1+q)}. \tag{3.12}$$

We are now in a position to state and prove our main results for the existence and uniqueness of solutions of problem (1.1).

Let us list the following hypotheses.

(H1) A satisfies the Hille-Yosida condition, and assume that $\overline{M} := \sup\{\|S'(t)\| : t \in [0, +\infty]\} < \infty$.

(H2) For $u \in \mathfrak{D}$, $f(\cdot, u) : [0, b] \to E$ is strongly measurable.

(H3) There exists a constant $q_1 \in (0, q)$ and $l \in L^{1/q_1}([0, b]; \mathbf{R}^+)$ such that

$$\left|f(t, u)\right| \le l(t), \text{ a.e. } t \in J, \text{ and each } u \in \mathfrak{D}. \tag{3.13}$$

(H4) There exists $\rho > 0$ such that

$$|I_k(u) - I_k(v)| \le \rho|u - v| \quad \forall u, v \in E,\ k = 1, \ldots, m. \tag{3.14}$$

(H5) There exists a constant κ such that

$$\left|f(t, u) - f(t, v)\right| \le \kappa\|u - v\|_{\mathfrak{D}}, \quad \text{for } t \in J \text{ and every } u, v \in \mathfrak{D}. \tag{3.15}$$

Theorem 3.6. *Assuming that hypotheses* (H1)–(H5) *hold, then problem* (1.1) *has a unique integral solution $y \in \Omega$ provided that* $\overline{M}(1 + \rho) + (\overline{M}M\kappa b^q/\Gamma(1 + q)) < 1$.

Proof. Define $Q : \Omega \to \Omega$ by

$$(Qy)(t) = \phi(t), \quad t \in [-\tau, 0], \tag{3.16}$$

and for $t \in J$,

$$(Qy)(t) = \begin{cases} \mathfrak{S}(t)\phi(0) + \lim\limits_{\lambda\to\infty}\displaystyle\int_0^t (t-s)^{q-1}\mathfrak{T}(t-s) \times B(\lambda, A) f(s, y_s)\,ds, & t \in (0, t_1], \\ \mathfrak{S}(t-t_1)\left(y(t_1^-) + I_1(y(t_1^-))\right) & \\ \qquad + \lim\limits_{\lambda\to\infty}\displaystyle\int_{t_1}^t (t-s)^{q-1}\mathfrak{T}(t-s) B(\lambda, A) f(s, y_s)\,ds, & t \in (t_1, t_2], \\ \vdots & \\ \mathfrak{S}(t-t_m)\left(y(t_m^-) + I_m(y(t_m^-))\right) & \\ \qquad + \lim\limits_{\lambda\to\infty}\displaystyle\int_{t_m}^t (t-s)^{q-1}\mathfrak{T}(t-s) B(\lambda, A) f(s, y_s)\,ds, & t \in (t_m, b]. \end{cases} \tag{3.17}$$

Firstly we check that Q is well defined on Ω.

For each $y \in \Omega$, take $t \in (0, t_1]$. It is obvious that $\mathfrak{S}(t)\phi(0)$ is well defined. Direct calculation shows that $(t-s)^{q-1} \in L^{(1/(1-q_1))}[0, t]$, for $t \in [0, t_1]$ and $q_1 \in (0, q)$. Let

$$a = \frac{q-1}{1-q_1} \in (-1, 0), \qquad M_1 = \|l\|_{L^{1/q_1}[0,b]}. \tag{3.18}$$

Then for $t \in [0, t_1]$, we have

$$\begin{aligned} \int_0^t \left|(t-s)^{q-1} f(s, y_s)\right| ds &\le \left(\int_0^t (t-s)^{((q-1)/(1-q_1))}\,ds\right)^{1-q_1} \|l\|_{L^{1/q_1}[0,t]} \\ &\le \frac{M_1}{(1+a)^{1-q_1}} b^{(1+a)(1-q_1)}. \end{aligned} \tag{3.19}$$

From (H1), (3.12), (3.19), and the fact that $\|B(\lambda, A)\| \le M$, we get

$$\begin{aligned} &\int_0^t \left|\int_0^\infty \theta(t-s)^{q-1} h_q(\theta) S'((t-s)^q\theta) B(\lambda, A) f(s, y_s)\,d\theta\right| ds \\ &\quad \le M\overline{M}\int_0^t\int_0^\infty \theta h_q(\theta)\left|(t-s)^{q-1} f(s, y_s)\right| d\theta\,ds \\ &\quad \le \frac{MM_0}{\Gamma(1+q)}\int_0^t \left|(t-s)^{q-1} f(s, y_s)\right| ds \\ &\quad \le \frac{MM_0M_1}{\Gamma(1+q)(1+a)^{1-q_1}} b^{(1+a)(1-q_1)}, \quad \text{for } t \in [0, t_1]. \end{aligned} \tag{3.20}$$

It means that $\left|\int_0^\infty \theta(t-s)^{q-1} h_q(\theta) S'((t-s)^q\theta) B(\lambda, A) f(s, y_s)\,d\theta\right|$ is Lebesgue integrable with respect to $s \in [0, t]$ for all $t \in [0, t_1]$. Therefore $\int_0^\infty \theta(t-s)^{q-1} h_q(\theta) S'((t-s)^q\theta) B(\lambda, A) f(s, y_s)\,d\theta$ is Bochner integrable with respect to $s \in [0, t]$ for all $t \in [0, t_1]$.

From [19], we know $\lim_{\lambda\to\infty}\int_0^t (t-s)^{q-1}S'((t-s)^q\theta)B(\lambda,A)f(s,y_s)ds$ exists; then

$$\begin{aligned}
&\lim_{\lambda\to\infty}\int_0^t (t-s)^{q-1}\mathfrak{T}(t-s)B(\lambda,A)f(s,y_s)ds\\
&\quad=\lim_{\lambda\to\infty} q\int_0^t\int_0^\infty \theta(t-s)^{q-1}h_q(\theta)S'\left((t-s)^q\theta\right)B(\lambda,A)f(s,y_s)d\theta\, ds\\
&\quad=\lim_{\lambda\to\infty} q\int_0^\infty \theta h_q(\theta)\int_0^t (t-s)^{q-1}S'\left((t-s)^q\theta\right)B(\lambda,A)f(s,y_s)ds\, d\theta\\
&\quad=q\int_0^\infty \theta h_q(\theta)\lim_{\lambda\to\infty}\int_0^t (t-s)^{q-1}S'\left((t-s)^q\theta\right)B(\lambda,A)f(s,y_s)ds\, d\theta
\end{aligned}\tag{3.21}$$

exists. Therefore we get $(Qy)(\cdot)$ which is well defined on $[0,t_1]$.

For $t\in(t_k,t_{k+1}]$, $k=1,2,\dots,m$, similar discussion could obtain $(Qy)(\cdot)$ is well defined. Hence, Q is well defined on Ω.

Secondly, we will prove operator Q is a contraction.

For $t\in(0,t_1]$ and $y,z\in\Omega$, by the hypotheses and $\|B(\lambda,A)\|\le M$, we get

$$\begin{aligned}
&\left|(Qy)(t)-(Qz)(t)\right|\\
&\quad=\left|\lim_{\lambda\to\infty}\int_0^t (t-s)^{q-1}\mathfrak{T}(t-s)B(\lambda,A)\left(f(s,y_s)-f(s,z_s)\right)ds\right|\\
&\quad=\left|\lim_{\lambda\to\infty} q\int_0^t\int_0^\infty \theta(t-s)^{q-1}h_q(\theta)S'\left((t-s)^q\theta\right)B(\lambda,A)\left(f(s,y_s)-f(s,z_s)\right)d\theta\, ds\right|\\
&\quad\le\frac{\overline{M}M}{\Gamma(1+q)}\int_0^t q(t-s)^{q-1}\left|f(s,y_s)-f(s,z_s)\right|ds\\
&\quad\le\frac{\overline{M}M\kappa}{\Gamma(1+q)}\int_0^t q(t-s)^{q-1}\|y_s-z_s\|_{\mathfrak{D}}ds\\
&\quad\le\frac{\overline{M}M\kappa b^q}{\Gamma(1+q)}\|y-z\|_{\Omega}.
\end{aligned}\tag{3.22}$$

Now take $t\in(t_k,t_{k+1}]$, $k=1,2,\dots,m$ and $y,z\in\Omega$:

$$\begin{aligned}
&\left|(Qy)(t)-(Qz)(t)\right|\\
&\quad\le\left|\mathfrak{S}(t-t_k)\left[y(t_k^-)+I_k\left(y(t_k^-)\right)-z(t_k^-)-I_k\left(z(t_k^-)\right)\right]\right|\\
&\qquad+\left|\lim_{\lambda\to\infty}\int_{t_k}^t (t-s)^{q-1}\mathfrak{T}(t-s)B(\lambda,A)\left(f(s,y_s)-f(s,z_s)\right)ds\right|
\end{aligned}$$

$$\leq \overline{M}(1+\rho)\|y-z\|_{\Omega} + \frac{\overline{M}M\kappa b^q}{\Gamma(1+q)}\|y-z\|_{\Omega}$$
$$\leq \left(\overline{M}(1+\rho) + \frac{\overline{M}M\kappa b^q}{\Gamma(1+q)}\right)\|y-z\|_{\Omega}. \tag{3.23}$$

In view of $\overline{M}(1+\rho)+(\overline{M}M\kappa b^q/\Gamma(1+q)) < 1$, we have that the operator Q is a contraction. By the Banach contraction principle we have that Q has a unique fixed point $y \in \Omega$, which gives rise to a unique integral solution to the problem (1.1). The proof is finished. □

Remark 3.7. For impulsive Caputo fractional differential equations, its integral solutions (or mild solutions; see [14]) can be expressed only by using piecewise functions. Thus Definition 2.3 given in [15] is unsuitable.

4. An Example

As an application of our results we consider the following fractional differential equations of the form

$$\begin{gathered} D^q u(t,z) = \frac{\partial^2}{\partial z^2}u(t,z) + F(t,u_t(\varsigma,z)), \quad z \in [0,\pi], \quad t \in [0,1] \setminus \left\{\frac{1}{2}\right\}, \\ u\left(\frac{1}{2}^{+},z\right) - u\left(\frac{1}{2}^{-},z\right) = \rho u\left(\frac{1}{2}^{-},z\right), \quad z \in [0,\pi], \\ u(t,0) = u(t,\pi) = 0, \quad t \in [0,1], \\ u(\varsigma,z) = \phi(\varsigma,z), \quad \varsigma \in [-1,0], \quad z \in [0,\pi]. \end{gathered} \tag{4.1}$$

Consider $E = C([0,\pi];\mathbf{R})$ endowed with the supnorm and the operator $A : D(A) \subset E \to E$ defined by

$$D(A) = \left\{u \in C^2([0,\pi];\mathbf{R}) : u(t,0) = u(t,\pi) = 0\right\}, \qquad Au = \frac{\partial^2}{\partial z^2}u(t,z). \tag{4.2}$$

Now, we have $\overline{D(A)} = \{u \in E : u(t,0) = u(t,\pi) = 0\} \neq E$. As we know from [17] that A satisfies the Hille-Yosida condition with $(0,+\infty) \subseteq \rho(A)$ and $\lambda > 0$, $|R(\lambda,A)| \leq 1/\lambda$. Hence, operator A satisfies (H1) and $M = \overline{M} = 1/2$.

Then the system (4.1) can be reformulated as

$$\begin{gathered} D^q y(t) = Ay(t) + f(t,y_t), \quad t \in J := [0,b], \quad t \neq \frac{1}{2}, \\ \Delta y\big|_{t=1/2} = I\left(y\left(\frac{1}{2}^{-}\right)\right), \quad k = 1,\ldots,m, \\ y(t) = \phi(t), \quad t \in [-\tau,0], \end{gathered} \tag{4.3}$$

where $y(t)(z) = u(t,z)$, $f(t,y_t)(z) = F(t,u_t(\varsigma,z))$, $I(x) = \rho x$, $\phi(t)(z) = \phi(t,z)$.

If we take $q = 1/3$, $\rho = 1/10$, $f(t, y_t) = (1/(t+1)(t+2)) \sin y_t$. We easily get that

$$\left| f(t,u) - t(t,v) \right| \leq \frac{1}{3} \|u - v\|_{\mathfrak{D}}, \quad \text{for } t \in J \text{ and every } u, v \in \mathfrak{D}. \tag{4.4}$$

Then all conditions of Theorem 3.6 are satisfied and we deduce (4.1) has a unique integral solution.

5. Conclusions

An essence error of the formula of solutions which appeared in the recent work on the nondensely defined fractional evolution differential equations is reported in this work. A correct formula of integral solutions for nondensely defined fractional evolution equations could be obtained from the results in this paper.

In view of the complicated definitions for integral or mild solutions for impulsive fractional evolution equations, many fixed point theorems related to completely continuous operators are hard to be used to establish the existence results. As far as we know, only [14] applied Leray Schauder Alternative theorem to the existence of mild solutions of impulsive fractional differential equations. But there is a mistake in proving that $\Gamma(B_r)$ is equicontinuous (page 2009, Step 3). How to overcome this difficulty is our next work.

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