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Research Article

Interval Oscillation Criteria of Second Order Mixed Nonlinear Impulsive Differential Equations with Delay

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We study the following second order mixed nonlinear impulsive differential equations with delay $(r(t)\Phi_\alpha(x'(t)))'+p_0(t)\Phi_\alpha(x(t))+\sum_{i=1}^{n} p_i(t)\Phi_{\beta_i}(x(t-\sigma)) = e(t)$, $t \geq t_0$, $t \neq \tau_k$, $x(\tau_k^+) = a_k x(\tau_k)$, $x'(\tau_k^+) = b_k x'(\tau_k)$, $k = 1,2,\ldots$, where $\Phi_*(u) = |u|^{*-1}u$, σ is a nonnegative constant, $\{\tau_k\}$ denotes the impulsive moments sequence, and $\tau_{k+1} - \tau_k > \sigma$. Some sufficient conditions for the interval oscillation criteria of the equations are obtained. The results obtained generalize and improve earlier ones. Two examples are considered to illustrate the main results.

1. Introduction

We consider the following second order impulsive differential equations with delay

$$\begin{gathered}
\left(r(t)\Phi_\alpha\left(x'(t)\right)\right)' + p_0(t)\Phi_\alpha(x(t)) + \sum_{i=1}^{n} p_i(t)\Phi_{\beta_i}(x(t-\sigma)) = e(t), \quad t \geq t_0,\ t \neq \tau_k, \\
x\left(\tau_k^+\right) = a_k x(\tau_k), \qquad x'\left(\tau_k^+\right) = b_k x'(\tau_k), \quad k = 1,2,\ldots,
\end{gathered} \tag{1.1}$$

where $\Phi_*(u) = |u|^{*-1}u$, σ is a nonnegative constant, $\{\tau_k\}$ denotes the impulsive moments sequence, and $\tau_{k+1} - \tau_k > \sigma$, for all $k \in \mathbb{N}$.

Let $J \subset \mathbb{R}$ be an interval, and we define

$$\begin{aligned}\mathrm{PLC}(J,\mathbb{R}) := \{y : J \longrightarrow \mathbb{R} \mid y \text{ is continuous everywhere except each } \tau_k \text{ at which } y(\tau_k^+) \\ \text{and } y(\tau_k^-) \text{ exist and } y(\tau_k^-) = y(\tau_k),\ k \in \mathbb{N}\}.\end{aligned} \tag{1.2}$$

For given t_0 and $\phi \in \mathrm{PLC}([t_0-\sigma,t_0],\mathbb{R})$, we say $x \in \mathrm{PLC}([t_0-\sigma,\infty),\mathbb{R})$ is a solution of (1.1) with initial value ϕ if $x(t)$ satisfies (1.1) for $t \geq t_0$ and $x(t) = \phi(t)$ for $t \in [t_0-\sigma,t_0]$.

A solution of (1.1) is said to be nonoscillatory if it is eventually positive or eventually negative. Otherwise, this solution is said to be oscillatory.

Impulsive differential equation is an adequate mathematical apparatus for the simulation of processes and phenomena observed in control theory, physics, chemistry, population dynamics, biotechnologies, industrial robotics, economics, and so forth. Because it has more richer theory than its corresponding without impulsive differential equation, much research has been done on the qualitative behavior of certain impulsive differential equations (see [1, 2]).

In the last decades, there is constant interest in obtaining new sufficient conditions for oscillation or nonoscillation of the solutions of various impulsive differential equations, see, for example, [1–9] and the references cited therein.

In recent years, interval oscillation of impulsive differential equations was also arousing the interest of many researchers. In 2007, Özbekler and Zafer [10] investigated the following equations:

$$\begin{aligned}&(m(t)\varphi_\alpha(y'))' + q(t)\varphi_\beta(y) = f(t), \quad t \neq \theta_i,\\ &\Delta(m(t)\varphi_\alpha(y')) + q_i\varphi_\beta(y) = f_i, \quad t = \theta_i,\ (i \in \mathbb{N}),\end{aligned} \tag{1.3}$$

$$\begin{aligned}&(m(t)y')' + s(t)y' + q(t)\varphi_\beta(y) = f(t), \quad t \neq \theta_i,\\ &\Delta(m(t)y') + q_i\varphi_\beta(y) = f_i, \quad t = \theta_i,\ (\beta \geq 1),\end{aligned} \tag{1.4}$$

where $\varphi_*(u) = |u|^{*-1}u$, $\beta \geq \alpha$, $\{q_i\}$ and $\{f_i\}$ are sequences of real numbers. In 2009, they further gave a research [11] for equations of the form

$$\begin{aligned}&(r(t)\varphi_\alpha(x'))' + p(t)\varphi_\alpha(x') + q(t)\varphi_\beta(x) = e(t), \quad t \neq \theta_i,\\ &\Delta(r(t)\varphi_\alpha(x')) + q_i\varphi_\beta(x) = e_i, \quad t = \theta_i,\end{aligned} \tag{1.5}$$

and obtained some interval oscillation results which improved and extended the earlier ones for the equations without impulses.

For the mixed type Emden-Fowler equations

$$\begin{aligned}&(r(t)x'(t))' + p(t)x(t) + \sum_{i=1}^{n} p_i(t)|x(t)|^{\alpha_i-1}x(t) = e(t), \quad t \neq \tau_k,\\ &x(\tau_k^+) = a_k x(\tau_k), \qquad x'(\tau_k^+) = b_k x'(\tau_k), \quad k \in \mathbb{N},\end{aligned} \tag{1.6}$$

Liu and Xu [12] established some interval oscillation results. Recently, Özbekler and Zafer [13] investigated the more general cases

$$\begin{aligned} &\left(r(t)\Phi_\alpha\left(x'(t)\right)\right)' + q(t)\Phi_\alpha(x(t)) + \sum_{k=1}^{n} q_k(t)\Phi_{\beta_k}(x(t)) = e(t), \quad t \neq \theta_i, \\ &x\left(\theta_i^+\right) = a_i x(\theta_i), \qquad x'\left(\theta_i^+\right) = b_i x'(\theta_i), \end{aligned} \tag{1.7}$$

where $\beta_1 > \cdots > \beta_m > \alpha > \beta_{m+1} > \cdots > \beta_n > 0$.

However, for the impulsive equations, almost all of interval oscillation results in the existing literature were established only for the case of "without delay." In other words, for the case of "with delay" the study on the interval oscillation is very scarce. To the best of our knowledge, Huang and Feng [14] gave the first research in this direction recently. They considered second order delay differential equations with impulses

$$\begin{aligned} &x''(t) + p(t) f(x(t-\tau)) = e(t), \quad t \geq t_0, \ t \neq t_k, \ k = 1,2,\ldots, \\ &x\left(t_k^+\right) = a_k x(t_k), \qquad x'\left(t_k^+\right) = b_k x'(t_k), \quad k = 1,2,\ldots \end{aligned} \tag{1.8}$$

and established some interval oscillation criteria which developed some known results for the equations without delay or impulses [15–17].

Motivated mainly by [13, 14], in this paper, we study the interval oscillation of the delay impulsive (1.1). By using some inequalities, Riccati transformation and $\mathscr{H}$ functions (introduced first by Philos [18]), we establish some interval oscillation criteria which generalize and improve some known results. Moreover, examples are considered to illustrate the main results.

2. Main Results

Throughout the paper, we always assume that the following conditions hold:

(A_1) the exponents satisfy that $\beta_1 > \cdots > \beta_m > \alpha > \beta_{m+1} > \cdots > \beta_n > 0$;

(A_2) $r(t) \in C([t_0,\infty),(0,\infty))$ is nondecreasing, $e(t)$, $p_i(t) \in \mathrm{PLC}([t_0,\infty),\mathbb{R})$, $i = 0,1,\ldots,n$;

(A_3) $\{a_k\}$ and $\{b_k\}$ are real constant sequences such that $b_k \geq a_k > 0$, $k \in \mathbb{N}$.

It is clear that all solutions of (1.1) are oscillatory if there exists a subsequence $\{k_i\}$ of $\{k\}$ such that $a_{k_i} \leq 0$ for all $i \in \mathbb{N}$. So, we assume $a_k > 0$ for all $k \in \mathbb{N}$ in condition (A_3).

In this section, intervals $[c_1, d_1]$ and $[c_2, d_2]$ are considered to establish oscillation criteria. For convenience, we introduce the following notations (see [12]). Let

$$\begin{aligned} &k(s) = \max\{i : t_0 < \tau_i < s\}, \qquad r_j = \max\left\{r(t) : t \in \left[c_j, d_j\right]\right\}, \\ &\Omega\left(c_j, d_j\right) = \left\{w_j \in C^1\left[c_j, d_j\right] : w_j(t) \not\equiv 0, \ w_j\left(c_j\right) = w_j\left(d_j\right) = 0\right\}, \quad j = 1,2. \end{aligned} \tag{2.1}$$

For two constants $c, d \notin \{\tau_k\}$ with $c < d$ and $k(c) < k(d)$ and a function $\varphi \in C([c,d],\mathbb{R})$, we define an operator $Q : C([c,d],\mathbb{R}) \to \mathbb{R}$ by

$$Q_c^d[\varphi] = \varphi(\tau_{k(c)+1})\frac{b_{k(c)+1}^{\alpha} - a_{k(c)+1}^{\alpha}}{a_{k(c)+1}^{\alpha}(\tau_{k(c)+1} - c)^{\alpha}} + \sum_{i=k(c)+2}^{k(d)} \varphi(\tau_i)\frac{b_i^{\alpha} - a_i^{\alpha}}{a_i^{\alpha}(\tau_i - \tau_{i-1})^{\alpha}}, \tag{2.2}$$

where $\sum_s^t = 0$ if $s > t$.

In the discussion of the impulse moments of $x(t)$ and $x(t-\sigma)$, we need to consider the following cases for $k(c_j) < k(d_j)$

(S1) $\tau_{k(c_j)} + \sigma < c_j$ and $\tau_{k(d_j)} + \sigma > d_j$,

(S2) $\tau_{k(c_j)} + \sigma < c_j$ and $\tau_{k(d_j)} + \sigma < d_j$,

(S3) $\tau_{k(c_j)} + \sigma > c_j$ and $\tau_{k(d_j)} + \sigma > d_j$,

(S4) $\tau_{k(c_j)} + \sigma > c_j$ and $\tau_{k(d_j)} + \sigma < d_j$,

and the cases for $k(c_j) = k(d_j)$

($\bar{\text{S}}$1) $\tau_{k(c_j)} + \sigma < c_j$,

($\bar{\text{S}}$2) $c_j < \tau_{k(c_j)} + \sigma < d_j$,

($\bar{\text{S}}$3) $\tau_{k(c_j)} + \sigma > d_j$.

Combining (S$*$) with ($\bar{\text{S}}*$), we can get 12 cases. In order to save space, throughout the paper, we study (1.1) under the case of combination of (S1) with ($\bar{\text{S}}$1) only. The discussions for other cases are similar and omitted.

The following preparatory lemmas will be useful to prove our theorems. The first is derived from [19] and second is from [20].

Lemma 2.1. *For any given n-tuple $\{\beta_1, \beta_2, \ldots, \beta_n\}$ satisfying (A_1), then there exists an n-tuple $\{\eta_1, \eta_2, \ldots, \eta_n\}$ such that*

$$\sum_{i=1}^{n} \beta_i \eta_i = \alpha, \qquad \sum_{i=1}^{n} \eta_i = \lambda, \quad 0 < \eta_i < 1, \tag{2.3}$$

where $\lambda \in (0,1]$.

Lemma 2.2. *Suppose X and Y are nonnegative, then*

$$\lambda X Y^{\lambda-1} - X^{\lambda} \le (\lambda - 1)Y^{\lambda}, \quad \lambda > 1, \tag{2.4}$$

where equality holds if and if $X = Y$.

Let $\alpha > 0$, $B \ge 0$, $A > 0$, and $y \ge 0$. Put

$$\lambda = 1 + \frac{1}{\alpha}, \qquad X = A^{\alpha/(\alpha+1)} y, \qquad Y = \left(\frac{\alpha}{\alpha+1}\right)^{\alpha} B^{\alpha} A^{-\alpha^2/(\alpha+1)}. \tag{2.5}$$

It follows from Lemma 2.2 that

$$By - Ay^{(\alpha+1)/\alpha} \leq \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}} \frac{B^{\alpha+1}}{A^{\alpha}}. \tag{2.6}$$

Theorem 2.3. *Assume that for any $T \geq t_0$, there exist $c_j, d_j \notin \{\tau_k\}$, $j = 1,2$, such that $T < c_1 < d_1 \leq c_2 < d_2$ and for $j = 1,2$*

$$\begin{aligned} &p_i(t) \geq 0, \quad t \in [c_j - \sigma, d_j] \setminus \{\tau_k\}, \quad i = 0,1,2,\ldots,n; \\ &(-1)^j e(t) \geq 0, \quad t \in [c_j - \sigma, d_j] \setminus \{\tau_k\}. \end{aligned} \tag{2.7}$$

If there exist $w_j(t) \in \Omega(c_j, d_j)$ and $\rho(t) \in C^1([c_j, d_j], (0,\infty))$ such that, for $k(c_j) < k(d_j)$, $j = 1,2$,

$$\begin{aligned} &\int_{c_j}^{\tau_{k(c_j)+1}} W_j(t) \frac{\left(t - \tau_{k(c_j)} - \sigma\right)^{\alpha}}{\left(t - \tau_{k(c_j)}\right)^{\alpha}} dt \\ &\quad + \sum_{i=k(c_j)+1}^{k(d_j)-1} \left[\int_{\tau_i}^{\tau_i+\sigma} W_j(t) \frac{(t-\tau_i)^{\alpha}}{b_i^{\alpha}(t+\sigma-\tau_i)^{\alpha}} dt + \int_{\tau_i+\sigma}^{\tau_{i+1}} W_j(t) \frac{(t-\tau_i-\sigma)^{\alpha}}{(t-\tau_i)^{\alpha}} dt \right] \\ &\quad + \int_{\tau_{k(d_j)}}^{d_j} W_j(t) \frac{\left(t - \tau_{k(d_j)}\right)^{\alpha}}{b_{k(d_j)}^{\alpha}\left(t + \sigma - \tau_{k(d_j)}\right)^{\alpha}} dt \\ &\quad + \int_{c_j}^{d_j} \rho(t) \left(p_0(t) |w_j(t)|^{\alpha+1} - r(t) \left(|w_j'(t)| + \frac{|\rho'(t)||w_j(t)|}{(\alpha+1)\rho(t)} \right)^{\alpha+1} \right) dt \geq \rho_j r_j Q_{c_j}^{d_j}\left[|w_j|^{\alpha+1} \right], \end{aligned} \tag{2.8}$$

where ρ_j is maximum value of $\rho(t)$ on $[c_j, d_j]$ and, for $k(c_j) = k(d_j)$, $j = 1,2$,

$$\int_{c_i}^{d_i} \left(W_j(t) \frac{(t-c_i)^{\alpha}}{(t-c_i+\sigma)^{\alpha}} + \rho(t) \left(p_0(t) |w_j(t)|^{\alpha+1} - r(t) \left(|w_j'(t)| + \frac{|\rho'(t)||w_j(t)|}{(\alpha+1)\rho(t)} \right)^{\alpha+1} \right) \right) dt \geq 0, \tag{2.9}$$

where $W_j(t) = \eta_0^{-\eta_0} |e(t)|^{\eta_0} \prod_{i=1}^{n} \eta_i^{-\eta_i} (p_i(t))^{\eta_i} |w_j(t)|^{\alpha+1}$ with $\eta_0 = 1 - \sum_{i=1}^{n} \eta_i$ and $\eta_1, \eta_2, \ldots, \eta_n$ are positive constants satisfying conditions of Lemma 2.1, then (1.1) is oscillatory.

Proof. Assume, to the contrary, that $x(t)$ is a nonoscillatory solution of (1.1). Without loss of generality, we assume that $x(t) > 0$ and $x(t-\sigma) > 0$ for $t \geq t_0$. In this case the interval of t selected for the following discussion is $[c_1, d_1]$. Define

$$u(t) = \rho(t) \frac{r(t)\Phi_{\alpha}(x'(t))}{\Phi_{\alpha}(x(t))}, \quad t \in [c_1, d_1]. \tag{2.10}$$

It follows, for $t \neq \tau_k$, that

$$u'(t) = -\rho(t)p_0(t) - \rho(t)\left[\sum_{i=1}^{n} p_i(t)\Phi_{\beta_i-\alpha}(x(t-\sigma)) + \frac{|e(t)|}{\Phi_\alpha(x(t-\sigma))}\right]\frac{\Phi_\alpha(x(t-\sigma))}{\Phi_\alpha(x(t))} + \frac{\rho'(t)}{\rho(t)}u(t) - \frac{\alpha}{(\rho(t)r(t))^{1/\alpha}}|u(t)|^{(\alpha+1)/\alpha}. \tag{2.11}$$

Now, let

$$v_0 = \eta_0^{-1}\frac{|e(t)|}{\Phi_\alpha(x(t-\sigma))}, \qquad v_i = \eta_i^{-1}p_i(t)\Phi_{\beta_i-\alpha}(x(t-\sigma)), \quad i = 1,2,\ldots,n, \tag{2.12}$$

where $\eta_1, \eta_2, \ldots, \eta_n$ are positive constants satisfying conditions of Lemma 2.1 and $\eta_0 = 1 - \sum_{i=1}^{n}\eta_i$. Employing in (2.11) the arithmetic-geometric mean inequality (see [20])

$$\sum_{i=0}^{n}\eta_i v_i \geq \prod_{i=0}^{n} v_i^{\eta_i} \tag{2.13}$$

and in view of (2.3), we have that

$$u'(t) \leq -\rho(t)p_0(t) - \rho(t)\varphi(t)\frac{\Phi_\alpha(x(t-\sigma))}{\Phi_\alpha(x(t))} - \frac{\alpha}{(\rho(t)r(t))^{1/\alpha}}|u(t)|^{(\alpha+1)/\alpha} + \frac{\rho'(t)}{\rho(t)}u(t), \tag{2.14}$$

where

$$\varphi(t) = \eta_0^{-\eta_0}|e(t)|^{\eta_0}\prod_{i=1}^{n}\eta_i^{-\eta_i}(p_i(t))^{\eta_i}. \tag{2.15}$$

First, we consider the case $k(c_1) < k(d_1)$.

In this case, we assume impulsive moments in $[c_1, d_1]$ are $\tau_{k(c_1)+1}, \tau_{k(c_1)+2}, \ldots, \tau_{k(d_1)}$. Choosing $w_1(t) \in \Omega(c_1, d_1)$, multiplying both sides of (2.14) by $|w_1(t)|^{\alpha+1}$ and then integrating it from c_1 to d_1, we obtain

$$\begin{aligned}
&\left(\int_{c_1}^{\tau_{k(c_1)+1}} + \sum_{i=k(c_1)+1}^{k(d_1)-1}\int_{\tau_i}^{\tau_{i+1}} + \int_{\tau_{k(d_1)}}^{d_1}\right)u'(t)|w_1(t)|^{\alpha+1}dt \\
&\quad\leq \left(\int_{c_1}^{\tau_{k(c_1)+1}} + \sum_{i=k(c_1)+1}^{k(d_1)-1}\int_{\tau_i}^{\tau_{i+1}} + \int_{\tau_{k(d_1)}}^{d_1}\right)\left(\frac{\rho'(t)}{\rho(t)}u(t) - \frac{\alpha}{(\rho(t)r(t))^{1/\alpha}}|u(t)|^{(1+\alpha)/\alpha}\right)|w_1(t)|^{\alpha+1}dt \\
&\qquad - \left(\int_{c_1}^{\tau_{k(c_1)+1}} + \sum_{i=k(c_1)+1}^{k(d_1)-1}\left[\int_{\tau_i}^{\tau_i+\sigma} + \int_{\tau_i+\sigma}^{\tau_{i+1}}\right] + \int_{\tau_{k(d_1)}}^{d_1}\right)\frac{x^\alpha(t-\sigma)}{x^\alpha(t)}W_1(t)dt \\
&\qquad - \int_{c_1}^{d_1}\rho(t)p_0(t)|w_1(t)|^{\alpha+1}dt,
\end{aligned} \tag{2.16}$$

where $W_1(t) = \rho(t)\psi(t)|w_1(t)|^{\alpha+1}$. Using the integration by parts formula in the left-hand side of above inequality and noting the condition $w_1(c_1) = w_1(d_1) = 0$, we obtain

$$\begin{aligned}
&\sum_{i=k(c_1)+1}^{k(d_1)} |w_1(\tau_i)|^{\alpha+1}\left[u(\tau_i) - u(\tau_i^+)\right] \\
&\quad\le \left(\int_{c_1}^{\tau_{k(c_1)+1}} + \sum_{i=k(c_1)+1}^{k(d_1)-1}\int_{\tau_i}^{\tau_{i+1}} + \int_{\tau_{k(d_1)}}^{d_1}\right) \\
&\qquad\times\left[(\alpha+1)\Phi_\alpha(w_1(t))w_1'(t)u(t) + \frac{\rho'(t)}{\rho(t)}u(t)|w_1(t)|^{\alpha+1}\right. \\
&\qquad\quad\left.-\frac{\alpha}{(\rho(t)r(t))^{1/\alpha}}|u(t)|^{(1+\alpha)/\alpha}|w_1(t)|^{\alpha+1}\right]dt \\
&\qquad-\left(\int_{c_1}^{\tau_{k(c_1)+1}} + \sum_{i=k(c_1)+1}^{k(d_1)-1}\left[\int_{\tau_i}^{\tau_i+\sigma} + \int_{\tau_i+\sigma}^{\tau_{i+1}}\right] + \int_{\tau_{k(d_1)}}^{d_1}\right)\frac{x^\alpha(t-\sigma)}{x^\alpha(t)}W_1(t)dt \\
&\qquad-\int_{c_1}^{d_1}\rho(t)p_0(t)|w_1(t)|^{\alpha+1}dt, \\
&\quad\le \left(\int_{c_1}^{\tau_{k(c_1)+1}} + \sum_{i=k(c_1)+1}^{k(d_1)-1}\int_{\tau_i}^{\tau_{i+1}} + \int_{\tau_{k(d_1)}}^{d_1}\right) \\
&\qquad\times\left[(\alpha+1)|w_1(t)|^\alpha|u(t)|\left(\left|w_1'(t)\right| + \frac{\left|\rho'(t)\right||w_1(t)|}{(\alpha+1)\rho(t)}\right)\right. \\
&\qquad\quad\left.-\frac{\alpha}{(\rho(t)r(t))^{1/\alpha}}|u(t)|^{(1+\alpha)/\alpha}|w_1(t)|^{\alpha+1}\right]dt \\
&\qquad-\left(\int_{c_1}^{\tau_{k(c_1)+1}} + \sum_{i=k(c_1)+1}^{k(d_1)-1}\left[\int_{\tau_i}^{\tau_i+\sigma} + \int_{\tau_i+\sigma}^{\tau_{i+1}}\right] + \int_{\tau_{k(d_1)}}^{d_1}\right)\frac{x^\alpha(t-\sigma)}{x^\alpha(t)}W_1(t)dt \\
&\qquad-\int_{c_1}^{d_1}\rho(t)p_0(t)|w_1(t)|^{\alpha+1}dt.
\end{aligned} \tag{2.17}$$

Letting $y = |w_1(t)|^\alpha|u(t)|$, $B = (\alpha+1)(|w_1'(t)| + |\rho'(t)||w_1(t)|/(\alpha+1)|\rho(t)|)$, $A = \alpha/(\rho(t)r(t))^{1/\alpha}$ and using (2.6), we have for the integrand function in above inequality that

$$\begin{aligned}
&(\alpha+1)|w_1(t)|^\alpha\left|w_1'(t)\right||u(t)| - \frac{\alpha}{r^{1/\alpha}(t)}|u(t)|^{(1+\alpha)/\alpha}|w_1(t)|^{\alpha+1} \\
&\quad\le \rho(t)r(t)\left(\left|w_1'(t)\right| + \frac{\left|\rho'(t)\right||w_1(t)|}{(\alpha+1)\rho(t)}\right)^{\alpha+1}.
\end{aligned} \tag{2.18}$$

In view of the impulse condition in (1.1) and the definition of u we have, for $t = \tau_k$, $k = 1, 2, \ldots$, that

$$u\left(\tau_k^+\right) = \frac{b_k^\alpha}{a_k^\alpha} u(\tau_k). \tag{2.19}$$

From (2.19), we have

$$\sum_{i=k(c_1)+1}^{k(d_1)} |w_1(\tau_i)|^{\alpha+1}\left[u(\tau_i) - u\left(\tau_i^+\right)\right] = \sum_{i=k(c_1)+1}^{k(d_1)} \left(1 - \frac{b_i^\alpha}{a_i^\alpha}\right) |w_1(\tau_i)|^{\alpha+1} u(\tau_i). \tag{2.20}$$

Therefore, we get

$$\begin{aligned}
\sum_{i=k(c_1)+1}^{k(d_1)} &\left(1 - \frac{b_i^\alpha}{a_i^\alpha}\right) |w_1(\tau_i)|^{\alpha+1} u(\tau_i) \\
&\leq \int_{c_1}^{d_1} \rho(t) r(t) \left(\left|w_1'(t)\right| + \frac{\left|\rho'(t)\right| |w_1(t)|}{(\alpha+1)\rho(t)} \right)^{\alpha+1} dt \\
&\quad - \left(\int_{c_1}^{\tau_{k(c_1)+1}} + \sum_{i=k(c_1)+1}^{k(d_1)-1} \left[\int_{\tau_i}^{\tau_i+\sigma} + \int_{\tau_i+\sigma}^{\tau_{i+1}} \right] + \int_{\tau_{k(d_1)}}^{d_1} \right) \frac{x^\alpha(t-\sigma)}{x^\alpha(t)} W_1(t) dt \\
&\quad - \int_{c_1}^{d_1} \rho(t) p_0(t) |w_1(t)|^{\alpha+1} dt.
\end{aligned} \tag{2.21}$$

On the other hand, for $t \in [c_1, d_1] \setminus \{\tau_i\}$,

$$\left(r(t)\Phi_\alpha(x'(t))\right)' = e(t) - p_0(t)\Phi_\alpha(x(t)) - \sum_{i=1}^{n} p_i(t)\Phi_{\beta_i}(x(t-\sigma)) \leq 0. \tag{2.22}$$

Hence $r(t)\Phi_\alpha(x'(t))$ is nonincreasing on $[c_1, d_1] \setminus \{\tau_k\}$.

Because there are different integration intervals in (2.21), we will estimate $x(t-\sigma)/x(t)$ in each interval of t as follows.

Case 1. $t \in (\tau_i, \tau_{i+1}] \subset [c_1, d_1]$, for $i = k(c_1)+1, \ldots, k(d_1)-1$.

Subcase 1. If $\tau_i + \sigma < t \leq \tau_{i+1}$, then $(t-\sigma, t) \subset (\tau_i, \tau_{i+1}]$. Thus there is no impulsive moment in $(t-\sigma, t)$. For any $s \in (t-\sigma, t)$, we have

$$x(s) - x\left(\tau_i^+\right) = x'(\xi_1)(s - \tau_i), \qquad \xi_1 \in (\tau_i, s). \tag{2.23}$$

Since $x(\tau_i^+) > 0$, $r(s)$ is nondecreasing, function $\Phi_\alpha(\cdot)$ is an increasing function and $r(t)\Phi_\alpha(x'(t))$ is nonincreasing on (τ_i, τ_{i+1}), we have

$$\begin{aligned}\Phi_\alpha(x(s)) &\geq \frac{r(\xi_1)}{r(s)}\Phi_\alpha(x(s)) > \frac{r(\xi_1)}{r(s)}\Phi_\alpha\left(x'(\xi_1)(s-\tau_i)\right) = \frac{r(\xi_1)\Phi_\alpha(x'(\xi_1))}{r(s)}(s-\tau_i)^\alpha \\ &\geq \frac{r(s)\Phi_\alpha(x'(s))}{r(s)}(s-\tau_i)^\alpha = \Phi_\alpha\left(x'(s)(s-\tau_i)\right), \quad \xi_1 \in (\tau_i, s).\end{aligned} \tag{2.24}$$

Therefore,

$$\frac{x'(s)}{x(s)} < \frac{1}{s-\tau_i}. \tag{2.25}$$

Integrating both sides of the above inequality from $t-\sigma$ to t, we obtain

$$\frac{x(t-\sigma)}{x(t)} > \frac{t-\tau_i-\sigma}{t-\tau_i}, \quad t \in (\tau_i+\sigma, \tau_{i+1}]. \tag{2.26}$$

Subcase 2. If $\tau_i < t < \tau_i+\sigma$, then $\tau_i-\sigma < t-\sigma < \tau_i < t < \tau_i+\sigma$. There is an impulsive moment τ_i in $(t-\sigma, t)$. For any $t \in (\tau_i, \tau_i+\sigma)$, we have

$$x(t) - x\left(\tau_i^+\right) = x'(\xi_2)(t-\tau_i), \quad \xi_2 \in (\tau_i, t). \tag{2.27}$$

Using the impulsive condition of (1.1) and the monotone properties of $r(t)$, $\Phi_\alpha(\cdot)$ and $r(t)\Phi_\alpha(x'(t))$, we get

$$\begin{aligned}\Phi_\alpha(x(t)-a_i x(\tau_i)) &= \frac{r(\xi_2)\Phi_\alpha(x'(\xi_2))}{r(\xi_2)}(t-\tau_i)^\alpha \leq \frac{r(\tau_i)\Phi_\alpha\left(x'\left(\tau_i^+\right)\right)}{r(\xi_2)}(t-\tau_i)^\alpha \\ &= \frac{r(\tau_i)\Phi_\alpha(b_i x'(\tau_i)(t-\tau_i))}{r(\xi_2)}.\end{aligned} \tag{2.28}$$

Since $x(\tau_i) > 0$, we have

$$\Phi_\alpha\left(\frac{x(t)}{x(\tau_i)} - a_i\right) \leq \frac{r(\tau_i)}{r(\xi_2)}\Phi_\alpha\left(b_i \frac{x'(\tau_i)}{x(\tau_i)}(t-\tau_i)\right). \tag{2.29}$$

In addition,

$$x(\tau_i) > x(\tau_i) - x(\tau_i-\sigma) = x'(\xi_3)\sigma, \quad \xi_3 \in (\tau_i-\sigma, \tau_i). \tag{2.30}$$

Using the same analysis as (2.24) and (2.25), we have

$$\frac{x'(\tau_i)}{x(\tau_i)} < \frac{1}{\sigma}. \tag{2.31}$$

From (2.29) and (2.31) and note that the monotone properties of $\Phi_\alpha(\cdot)$ and $r(t)$, we get

$$\Phi_\alpha\left(\frac{x(t)}{x(\tau_i)} - a_i\right) < \frac{r(\tau_i)}{r(\xi_2)}\Phi_\alpha\left(\frac{b_i}{\sigma}(t-\tau_i)\right) \le \Phi_\alpha\left(\frac{b_i}{\sigma}(t-\tau_i)\right). \tag{2.32}$$

Then,

$$\frac{x(t)}{x(\tau_i)} < a_i + \frac{b_i}{\sigma}(t-\tau_i). \tag{2.33}$$

In view of (A_3), we have

$$\frac{x(\tau_i)}{x(t)} > \frac{\sigma}{\sigma a_i + b_i(t-\tau_i)} \ge \frac{\sigma}{b_i(t+\sigma-\tau_i)} > 0. \tag{2.34}$$

On the other hand, similar to the above analysis, we get

$$\frac{x'(s)}{x(s)} < \frac{1}{s-\tau_i+\sigma}, \quad s \in (\tau_i - \sigma, \tau_i). \tag{2.35}$$

Integrating (2.35) from $t-\sigma$ to τ_i, where $t \in (\tau_i, \tau_i + \sigma)$, we have

$$\frac{x(t-\sigma)}{x(\tau_i)} > \frac{t-\tau_i}{\sigma} \ge 0. \tag{2.36}$$

From (2.34) and (2.36), we obtain

$$\frac{x(t-\sigma)}{x(t)} > \frac{t-\tau_i}{b_i(t+\sigma-\tau_i)}, \quad t \in (\tau_i, \tau_i+\sigma). \tag{2.37}$$

Case 2 ($t \in [c_1, \tau_{k(c_1)+1}]$). Since $\tau_{k(c_1)} + \sigma < c_1$, then $t-\sigma \in [c_1-\sigma, \tau_{k(c_1)+1}-\sigma] \subset (\tau_{k(c_1)}, \tau_{k(c_1)+1}-\sigma]$. So, there is no impulsive moment in $(t-\sigma, t)$. Similar to (2.26) of Subcase 1, we have

$$\frac{x(t-\sigma)}{x(t)} > \frac{t-\tau_{k(c_1)}-\sigma}{t-\tau_{k(c_1)}}, \quad t \in \left[c_1, \tau_{k(c_1)+1}\right]. \tag{2.38}$$

Case 3 ($t \in (\tau_{k(d_1)}, d_1]$). Since $\tau_{k(d_1)} + \sigma > d_1$, then $t-\sigma \in (\tau_{k(d_1)}-\sigma, d_1-\sigma] \subset (\tau_{k(d_1)}-\sigma, \tau_{k(d_1)})$. Hence, there is an impulsive moment $\tau_{k(d_1)}$ in $(t-\sigma, t)$. Making a similar analysis of Subcase 2, we obtain

$$\frac{x(t-\sigma)}{x(t)} > \frac{t-\tau_{k(d_1)}}{b_{k(d_1)}\left(t+\sigma-\tau_{k(d_1)}\right)} \ge 0, \quad t \in \left(\tau_{k(d_1)}, d_1\right]. \tag{2.39}$$

From (2.21), (2.26), (2.37), (2.38), and (2.39) we get

$$\begin{aligned}\sum_{i=k(c_1)+1}^{k(d_1)}\left(1-\frac{b_i^{\alpha}}{a_i^{\alpha}}\right)|w_1(\tau_i)|^{\alpha+1}u(\tau_i)\\ &< \int_{c_1}^{d_1}\rho(t)r(t)\left(|w_1'(t)|+\frac{|\rho'(t)||w_1(t)|}{(\alpha+1)\rho(t)}\right)^{\alpha+1}dt-\int_{c_1}^{\tau_{k(c_1)+1}}W_1(t)\frac{\left(t-\tau_{k(c_1)}-\sigma\right)^{\alpha}}{\left(t-\tau_{k(c_1)}\right)^{\alpha}}dt\\ &\quad-\sum_{i=k(c_1)+1}^{k(d_1)-1}\left[\int_{\tau_i}^{\tau_i+\sigma}W_1(t)\frac{(t-\tau_i)^{\alpha}}{b_i^{\alpha}(t+\sigma-\tau_i)^{\alpha}}dt-\int_{\tau_i+\sigma}^{\tau_{i+1}}W_1(t)\frac{(t-\tau_i-\sigma)^{\alpha}}{(t-\tau_i)^{\alpha}}dt\right]\\ &\quad-\int_{\tau_{k(d_1)}}^{d_1}W_1(t)\frac{\left(t-\tau_{k(d_1)}\right)^{\alpha}}{b_{k(d_1)}^{\alpha}\left(t+\sigma-\tau_{k(d_1)}\right)^{\alpha}}dt-\int_{c_1}^{d_1}\rho(t)p_0(t)|w_1(t)|^{\alpha+1}dt.\end{aligned} \tag{2.40}$$

On the other hand, for $t\in(\tau_{i-1},\tau_i]\subset[c_1,d_1]$, $i=k(c_1)+2,\dots,k(d_1)$, we have

$$x(t)-x(\tau_{i-1})=x'(\xi)(t-\tau_{i-1}),\quad \xi\in(\tau_{i-1},t). \tag{2.41}$$

In view of $x(\tau_{i-1})>0$ and the monotone properties of $\Phi_\alpha(\cdot)$, $r(t)\Phi_\alpha(x'(t))$ and $r(t)$, we obtain

$$\Phi_\alpha(x(t))>\Phi_\alpha\left(x'(\xi)\right)\Phi_\alpha(t-\tau_{i-1})\geq\frac{r(t)}{r(\xi)}\Phi_\alpha\left(x'(t)\right)\Phi_\alpha(t-\tau_{i-1}). \tag{2.42}$$

This is

$$\rho(t)\frac{r(t)\Phi_\alpha(x'(t))}{\Phi_\alpha(x(t))}<\frac{\rho_1 r(\xi)}{(t-\tau_{i-1})^{\alpha}}. \tag{2.43}$$

Letting $t\to\tau_i^-$, we have

$$u(\tau_i)=\rho(t)\frac{r(\tau_i)\Phi_\alpha(x'(\tau_i))}{\Phi_\alpha(x(\tau_i))}<\frac{\rho_1 r_1}{(\tau_i-\tau_{i-1})^{\alpha}},\qquad i=k(c_1)+2,\dots,k(d_1). \tag{2.44}$$

Using similar analysis on $(c_1,\tau_{k(c_1)+1}]$, we get

$$u\left(\tau_{k(c_1)+1}\right)<\frac{\rho_1 r_1}{\left(\tau_{k(c_1)+1}-c_1\right)^{\alpha}}. \tag{2.45}$$

Then from (2.44), (2.45), and (A_3), we have

$$\begin{aligned}\sum_{i=k(c_1)+1}^{k(d_1)}\left(\frac{b_i^{\alpha}}{a_i^{\alpha}}-1\right)|w_1(\tau_i)|^{\alpha+1}u(\tau_i)&<\rho_1 r_1\left[\left|w_1\left(\tau_{k(c_1)+1}\right)\right|^{\alpha+1}\theta(c_1)+\sum_{i=k(c_1)+2}^{k(d_1)}|w_1(\tau_i)|^{\alpha+1}\zeta(\tau_i)\right]\\ &=\rho_1 r_1 Q_{c_1}^{d_1}\left[|w_1|^{\alpha+1}\right],\end{aligned} \tag{2.46}$$

where $\theta(c_1)=(b_{k(c_1)+1}^{\alpha}-a_{k(c_1)+1}^{\alpha})/a_{k(c_1)+1}^{\alpha}(\tau_{k(c_1)+1}-c_1)^{\alpha}$ and $\zeta(\tau_i)=(b_i^{\alpha}-a_i^{\alpha})/a_i^{\alpha}(\tau_i-\tau_{i-1})^{\alpha}$.

From (2.40) and (2.46), we obtain

$$\begin{aligned}
&\int_{c_1}^{\tau_{k(c_1)+1}} W_1(t)\frac{\left(t-\tau_{k(c_1)}-\sigma\right)^{\alpha}}{\left(t-\tau_{k(c_1)}\right)^{\alpha}}dt \\
&\quad+\sum_{i=k(c_1)+1}^{k(d_1)-1}\left[\int_{\tau_i}^{\tau_i+\sigma} W_1(t)\frac{(t-\tau_i)^{\alpha}}{b_i^{\alpha}(t+\sigma-\tau_i)^{\alpha}}dt+\int_{\tau_i+\sigma}^{\tau_{i+1}} W_1(t)\frac{(t-\tau_i-\sigma)^{\alpha}}{(t-\tau_i)^{\alpha}}dt\right] \\
&\quad+\int_{\tau_{k(d_1)}}^{d_1} W_1(t)\frac{\left(t-\tau_{k(d_1)}\right)^{\alpha}}{b_{k(d_1)}^{\alpha}\left(t+\sigma-\tau_{k(d_1)}\right)^{\alpha}}dt \\
&\quad+\int_{c_1}^{d_1}\rho(t)p_0(t)|w_1(t)|^{\alpha+1}dt-\int_{c_1}^{d_1}\rho(t)r(t)\left(\left|w_1'(t)\right|+\frac{\left|\rho'(t)\right||w_1(t)|}{(\alpha+1)\rho(t)}\right)^{\alpha+1}dt \\
&< \rho_1 r_1 Q_{c_1}^{d_1}\left[|w_1|^{\alpha+1}\right].
\end{aligned} \tag{2.47}$$

This contradicts (2.8).

Next we consider the case $k(c_1)=k(d_1)$. From the condition ($\overline{\text{S}}1$) we know that there is no impulsive moment in $[c_1,d_1]$. Multiplying both sides of (2.47) by $|w_1(t)|^{\alpha+1}$ and integrating it from c_1 to d_1, we obtain

$$\begin{aligned}
\int_{c_1}^{d_1} u'(t)|w_1(t)|^{\alpha+1}dt \le &-\int_{c_1}^{d_1}\frac{\alpha}{(\rho(t)r(t))^{1/\alpha}}|u(t)|^{(\alpha+1)/\alpha}|w_1(t)|^{\alpha+1}dt \\
&-\int_{c_1}^{d_1}\frac{x^{\alpha}(t-\sigma)}{x^{\alpha}(t)}W_1(t)dt-\int_{c_1}^{d_1}\rho(t)p_0(t)|w_1(t)|^{\alpha+1}dt.
\end{aligned} \tag{2.48}$$

Similar to the proof of (2.21), we have

$$\int_{c_1}^{d_1}\left[\frac{x^{\alpha}(t-\sigma)}{x^{\alpha}(t)}W_1(t)+\rho(t)p_0(t)\left|w_j(t)\right|^{\alpha+1}-\rho(t)r(t)\left(\left|w_j'(t)\right|+\frac{\left|\rho'(t)\right|\left|w_j(t)\right|}{(\alpha+1)\rho(t)}\right)^{\alpha+1}\right]dt\le 0. \tag{2.49}$$

Using same way as Subcase 1, we get

$$\frac{x(t-\sigma)}{x(t)}>\frac{t-c_1}{t-c_1+\sigma},\qquad t\in[c_1,d_1]. \tag{2.50}$$

From (2.49) and (2.50) we obtain

$$\int_{c_1}^{d_1}\left[W_1(t)\frac{(t-c_1)^{\alpha}}{(t-c_1+\sigma)^{\alpha}}+\rho(t)p_0(t)\left|w_j(t)\right|^{\alpha+1}-\rho(t)r(t)\left(\left|w_j'(t)\right|+\frac{\left|\rho'(t)\right|\left|w_j(t)\right|}{(\alpha+1)\rho(t)}\right)^{\alpha+1}\right]dt<0. \tag{2.51}$$

This contradicts condition (2.9).

When $x(t) < 0$, we can choose interval $[c_2, d_2]$ to study (1.1). The proof is similar and will be omitted. Therefore, we complete the proof.

□

Remark 2.4. In article [14], the authors obtained the following inequalities:

$$\frac{x(t)}{x(t_j)} > a_j \frac{t_j + \tau - t}{\tau} \geq 0, \quad t \in (t_j, t_j + \tau), \tag{2.52}$$

See [14, equation (2.9)],

$$\frac{x(t) - \tau}{x(t_j)} > \frac{t - t_j}{\tau} \geq 0, \quad t \in (t_j, t_j + \tau). \tag{2.53}$$

See [14, equation (2.10)].

Dividing [14, equation (2.10)] by [14, equation (2.9)], they obtained

$$\frac{x(t) - \tau}{x(t)} > \frac{t - t_j}{a_j (t_j + \tau - t)} \geq 0, \quad t \in (t_j, t_j + \tau). \tag{2.54}$$

See [14, equation (2.11)]

This is an error. Moreover, similar errors appeared many times in the later arguments, for example, in inequalities (2.15), (2.19), and (2.20) in [14]. Moreover, the above substitution can lead to some divergent integrals, for example, the integrals in (2.22), (2.24) in [14]. Therefore, the conditions of their Theorems 2.1–2.5 must be defective. In the proof of our Theorem 2.3, this error is remedied.

Remark 2.5. When $\sigma = 0$, that is, the delay disappears, (1.1) reduces to (1.7) studied by Özbekler and Zafer [13]. In this case, our result with $\rho(t) = 1$ is Theorem 2.1 of [13].

Remark 2.6. When $\sigma = 0$, that is, the delay disappears in (1.1) and $\alpha = 1$, our result reduces to Theorem 2.1 of [12].

Remark 2.7. When $a_k = b_k = 1$ for all $k = 1, 2, \ldots$ and $\sigma = 0$, that is, both impulses and delay disappear in (1.1), our result with $\alpha = 1$ and $\rho(t) = 1$ reduces to Theorem 1 of [21].

In the following we will establish a Kong-type interval oscillation criteria for (1.1) by the ideas of Philos [18] and Kong [22].

Let $D = \{(t, s) : t_0 \leq s \leq t\}$, $H_1, H_2 \in C^1(D, \mathbb{R})$, then a pair function H_1, H_2 is said to belong to a function set $\mathscr{H}$, defined by $(H_1, H_2) \in \mathscr{H}$, if there exist $h_1, h_2 \in L_{\text{loc}}(D, \mathbb{R})$ satisfying the following conditions:

(C_1) $H_1(t, t) = H_2(t, t) = 0$, $H_1(t, s) > 0, H_2(t, s) > 0$ for $t > s$;

(C_2) $(\partial/\partial t) H_1(t, s) = h_1(t, s) H_1(t, s)$, $(\partial/\partial s) H_2(t, s) = h_2(t, s) H_2(t, s)$.

We assume that there exist $c_j, d_j, \delta_j \notin \{\tau_k, k = 1, 2, \ldots.\} (j = 1, 2)$ such that $T < c_1 < \delta_1 < d_1 \leq c_2 < \delta_2 < d_2$ for any $T \geq t_0$. Noticing whether or not there are impulsive moments of $x(t)$ in $[c_j, \delta_j]$ and $[\delta_j, d_j]$, we should consider the following four cases, namely, (S5) $k(c_j) < k(\delta_j) < k(d_j)$; (S6) $k(c_j) = k(\delta_j) < k(d_j)$; (S7) $k(c_j) < k(\delta_j) = k(d_j)$ and (S8) $k(c_j) = k(\delta_j) = k(d_j)$. Moreover, in the discussion of the impulse moments of $x(t-\sigma)$, it is necessary to consider the following two cases: ($\bar{\text{S}}5$) $\tau_{k(\delta_j)} + \sigma > \delta_j$ and ($\bar{\text{S}}6$) $\tau_{k(\delta_j)} + \sigma \le \delta_j$. In the following theorem, we only consider the case of combination of (S5) with ($\bar{\text{S}}5$). For the other cases, similar conclusions can be given and the proofs will be omitted here.

For convenience in the expression below, we define, for $j = 1, 2$,

$$
\begin{aligned}
\Pi_{1,j} =: &\frac{1}{H_1(\delta_j, c_j)} \\
&\times \Bigg\{ \int_{c_j}^{\tau_{k(c_j)+1}} \widetilde{H}_1(t, c_j) \frac{\left(t - \tau_{k(c_j)} - \sigma\right)^{\alpha}}{\left(t - \tau_{k(c_j)}\right)^{\alpha}} dt \\
&+ \sum_{i=k(c_j)+1}^{k(\delta_j)-1} \left[\int_{\tau_i}^{\tau_i+\sigma} \widetilde{H}_1(t, c_j) \frac{(t - \tau_i)^{\alpha}}{b_i^{\alpha} (t + \sigma - \tau_i)^{\alpha}} dt + \int_{\tau_i+\sigma}^{\tau_{i+1}} \widetilde{H}_1(t, c_j) \frac{(t - \tau_i - \sigma)^{\alpha}}{(t - \tau_i)^{\alpha}} dt \right] \\
&+ \int_{\tau_{k(\delta_j)}}^{\delta_j} \widetilde{H}_1(t, c_j) \frac{\left(t - \tau_{k(\delta_j)}\right)^{\alpha}}{b_{k(\delta_j)}^{\alpha} \left(t + \sigma - \tau_{k(\delta_j)}\right)^{\alpha}} dt + \int_{c_j}^{\delta_j} \rho(t) p_0(t) H_1(t, c_j) dt \\
&- \frac{1}{(\alpha+1)^{\alpha+1}} \int_{c_j}^{\delta_j} \rho(t) r(t) H_1(t, c_j) \left| h_1(t, c_j) + \frac{\rho'(t)}{\rho(t)} \right|^{\alpha+1} dt \Bigg\}, \\
\Pi_{2,j} =: &\frac{1}{H_2(d_j, \delta_j)} \Bigg\{ \int_{\delta_j}^{\tau_{k(\delta_j)}+\sigma} \widetilde{H}_2(d_j, t) \frac{\left(t - \tau_{k(\delta_j)}\right)^{\alpha}}{b_{k(\delta_j)}^{\alpha} \left(t + \sigma - \tau_{k(\delta_j)}\right)^{\alpha}} dt \\
&+ \int_{\tau_{k(\delta_j)}+\sigma}^{\tau_{k(\delta_j)+1}} \widetilde{H}_2(d_j, t) \frac{\left(t - \tau_{k(\delta_j)} - \sigma\right)^{\alpha}}{\left(t - \tau_{k(\delta_j)}\right)^{\alpha}} dt \\
&+ \sum_{i=k(\delta_j)+1}^{k(d_j)-1} \left[\int_{\tau_i}^{\tau_i+\sigma} \widetilde{H}_2(d_j, t) \frac{(t - \tau_i)^{\alpha}}{b_i^{\alpha} (t + \sigma - \tau_i)^{\alpha}} dt + \int_{\tau_i+\sigma}^{\tau_{i+1}} \widetilde{H}_2(d_j, t) \frac{(t - \tau_i - \sigma)^{\alpha}}{(t - \tau_i)^{\alpha}} dt \right] \\
&+ \int_{\tau_{k(d_j)}}^{d_j} \widetilde{H}_2(d_j, t) \frac{\left(t - \tau_{k(d_j)}\right)^{\alpha}}{b_{k(d_j)}^{\alpha} \left(t + \sigma - \tau_{k(d_j)}\right)^{\alpha}} dt \\
&- \frac{1}{(\alpha+1)^{\alpha+1}} \int_{\delta_j}^{d_j} \rho(t) r(t) H_2(d_j, t) \left| h_2(d_j, t) + \frac{\rho'(t)}{\rho(t)} \right|^{\alpha+1} dt \\
&+ \int_{\delta_j}^{d_j} \rho(t) p_0(t) H_2(d_j, t) dt \Bigg\},
\end{aligned}
\tag{2.55}
$$

where $\widetilde{H}_1(t,c_j)= H_1(t,c_j)\psi(t)$, $\widetilde{H}_2(d_j,t) = H_2(d_j,t)\psi(t)$ and $\psi(t) = \eta_0^{-\eta_0}|e(t)|^{\eta_0}\prod_{i=1}^{n}\eta_i^{-\eta_i}(p_i(t))^{\eta_i}$ with $\eta_0 = 1 - \sum_{i=1}^{n}\eta_i$ and $\eta_1,\eta_2,\dots,\eta_n$ are positive constants satisfying conditions of Lemma 2.1.

Theorem 2.8. *Assume* (2.7) *holds. If there exists a pair of* $(H_1,H_2)\in\mathcal{H}$ *such that*

$$\Pi_{1,j}+\Pi_{2,j} > \frac{\rho_j r_j}{H_1(\delta_j,c_j)}Q_{c_j}^{\delta_j}[H_1(\cdot,c_j)] + \frac{\rho_j r_j}{H_2(d_j,\delta_j)}Q_{\delta_j}^{d_j}[H_2(d_j,\cdot)], \quad j=1,2, \tag{2.56}$$

then (1.1) *is oscillatory.*

Proof. Assume, to the contrary, that $x(t)$ is a nonoscillatory solution of (1.1). Without loss of generality, we assume that $x(t)>0$ and $x(t-\sigma)>0$ for $t\ge t_0$. In this case the interval of t selected for the following discussion is $[c_1,d_1]$. Similar to the proof of Theorem 2.3, we can get (2.14) and (2.19). Multiplying both sides of (2.14) by $H_1(t,c_1)$ and integrating it from c_1 to δ_1, we have

$$\begin{aligned}\int_{c_1}^{\delta_1}H_1(t,c_1)u'(t)dt \le{}& \int_{c_1}^{\delta_1}H_1(t,c_1)\left(\frac{\rho'(t)}{\rho(t)}u(t)-\frac{\alpha}{(\rho(t)r(t))^{1/\alpha}}|u(t)|^{(1+\alpha)/\alpha}\right)dt\\ &-\int_{c_1}^{\delta_1}\widetilde{H}_1(t,c_1)\frac{x^\alpha(t-\sigma)}{x^\alpha(t)}dt-\int_{c_1}^{\delta_1}H_1(t,c_1)\rho(t)p_0(t)dt,\end{aligned} \tag{2.57}$$

where $\widetilde{H}_1(t,c_1) = H_1(t,c_1)\rho(t)\psi(t)$, $\psi(t) = \eta_0^{-\eta_0}|e(t)|^{\eta_0}\prod_{i=1}^{n}\eta_i^{-\eta_i}(p_i(t))^{\eta_i}$ with $\eta_0 = 1-\sum_{i=1}^{n}\eta_i$ and $\eta_1,\eta_2,\dots,\eta_n$ are positive constants satisfying conditions of Lemma 2.1.

Noticing impulsive moments $\tau_{k(c_1)+1},\tau_{k(c_1)+2},\dots,\tau_{k(\delta_1)}$ are in $[c_1,\delta_1]$ and using the integration by parts formula on the left-hand side of above inequality, we obtain

$$\begin{aligned}\int_{c_1}^{\delta_1}H_1(t,c_1)u'(t)dt ={}& \left(\int_{c_1}^{\tau_{k(c_1)+1}}+\int_{\tau_{k(c_1)+1}}^{\tau_{k(c_1)+2}}+\cdots+\int_{\tau_{k(\delta_1)}}^{\delta_1}\right)H_1(t,c_1)du(t)\\ ={}& \sum_{i=k(c_1)+1}^{k(\delta_1)}\left(1-\frac{b_i^\alpha}{a_i^\alpha}\right)H_1(\tau_i,c_1)u(\tau_i)+H_1(\delta_1,c_1)u(\delta_1)\\ &-\left(\int_{c_1}^{\tau_{k(c_1)+1}}+\int_{\tau_{k(c_1)+1}}^{\tau_{k(c_1)+2}}+\cdots+\int_{\tau_{k(\delta_1)}}^{\delta_1}\right)H_1(t,c_1)h_1(t,c_1)u(t)dt.\end{aligned} \tag{2.58}$$

Substituting (2.58) into (2.57), we obtain

$$\begin{aligned}\int_{c_1}^{\delta_1}\widetilde{H}_1(t,c_1)\frac{x^\alpha(t-\sigma)}{x^\alpha(t)}dt \le{}& \sum_{i=k(c_1)+1}^{k(\delta_1)}\left(\frac{b_i^\alpha}{a_i^\alpha}-1\right)H_1(\tau_i,c_1)u(\tau_i)-H_1(\delta_1,c_1)u(\delta_1)\\ &+\left(\int_{c_1}^{\tau_{k(c_1)+1}}+\int_{\tau_{k(c_1)+1}}^{\tau_{k(c_1)+2}}+\cdots+\int_{\tau_{k(\delta_1)}}^{\delta_1}\right)H_1(t,c_1)\\ &\times\left[\left|h_1(t,c_1)+\frac{\rho'(t)}{\rho(t)}\right||u(t)|-\frac{\alpha}{(\rho(t)r(t))^{1/\alpha}}|u(t)|^{(\alpha+1)/\alpha}\right]dt\\ &-\int_{c_1}^{\delta_1}\rho(t)p_0(t)H_1(t,c_1)dt.\end{aligned} \tag{2.59}$$

Letting $A=\alpha/(\rho(t)r(t))^{1/\alpha}$, $B=|h_1(t,c_1)+\rho'(t)/\rho(t)|$, $y=|u(t)|$ and using (2.6) to the right-hand side of above inequality, we have

$$\begin{aligned}\int_{c_1}^{\delta_1}\widetilde{H}_1(t,c_1)\frac{x^\alpha(t-\sigma)}{x^\alpha(t)}dt \le{}& \sum_{i=k(c_1)+1}^{k(\delta_1)}\left(\frac{b_i^\alpha}{a_i^\alpha}-1\right)H_1(\tau_i,c_1)u(\tau_i)-H_1(\delta_1,c_1)u(\delta_1)\\ &+\frac{1}{(\alpha+1)^{\alpha+1}}\int_{c_1}^{\delta_1}\rho(t)r(t)H_1(t,c_1)\left|h_1(t,c_1)+\frac{\rho'(t)}{\rho(t)}\right|^{\alpha+1}dt\\ &-\int_{c_1}^{\delta_1}\rho(t)p_0(t)H_1(t,c_1)dt.\end{aligned} \tag{2.60}$$

Similar to the proof of Theorem 2.3, we need to divide the integration interval $[c_1,\delta_1]$ into several subintervals for estimating the function $x(t-\sigma)/x(t)$. Using the methods of (2.26), (2.37), (2.38), and (2.39) we estimate the left-hand side of above inequality as follows:

$$\begin{aligned}&\int_{c_1}^{\delta_1}\widetilde{H}_1(t,c_1)\frac{x^\alpha(t-\sigma)}{x^\alpha(t)}dt\\ &\quad>\int_{c_1}^{\tau_{k(c_1)+1}}\widetilde{H}_1(t,c_1)\frac{\left(t-\tau_{k(c_1)}-\sigma\right)^\alpha}{\left(t-\tau_{k(c_1)}\right)^\alpha}dt\\ &\qquad+\sum_{i=k(c_1)+1}^{k(\delta_1)-1}\left[\int_{\tau_i}^{\tau_i+\sigma}\widetilde{H}_1(t,c_1)\frac{(t-\tau_i)^\alpha}{b_i^\alpha(t+\sigma-\tau_i)^\alpha}dt+\int_{\tau_i+\sigma}^{\tau_{i+1}}\widetilde{H}_1(t,c_1)\frac{(t-\tau_i-\sigma)^\alpha}{(t-\tau_i)^\alpha}dt\right]\\ &\qquad+\int_{\tau_{k(\delta_1)}}^{\delta_1}\widetilde{H}_1(t,c_1)\frac{\left(t-\tau_{k(\delta_1)}\right)^\alpha}{b_{k(\delta_1)}^\alpha\left(t+\sigma-\tau_{k(\delta_1)}\right)^\alpha}dt.\end{aligned} \tag{2.61}$$

From (2.60) and (2.61), we have

$$\begin{aligned}
&\int_{c_1}^{\tau_{k(c_1)+1}} \widetilde{H}_1(t,c_1)\frac{\left(t-\tau_{k(c_1)}-\sigma\right)^{\alpha}}{\left(t-\tau_{k(c_1)}\right)^{\alpha}}dt \\
&\quad+\sum_{i=k(c_1)+1}^{k(\delta_1)-1}\left[\int_{\tau_i}^{\tau_i+\sigma}\widetilde{H}_1(t,c_1)\frac{(t-\tau_i)^{\alpha}}{b_i^{\alpha}(t+\sigma-\tau_i)^{\alpha}}dt+\int_{\tau_i+\sigma}^{\tau_{i+1}}\widetilde{H}_1(t,c_1)\frac{(t-\tau_i-\sigma)^{\alpha}}{(t-\tau_i)^{\alpha}}dt\right] \\
&\quad+\int_{\tau_{k(\delta_1)}}^{\delta_1}\widetilde{H}_1(t,c_1)\frac{\left(t-\tau_{k(\delta_1)}\right)^{\alpha}}{b_{k(\delta_1)}^{\alpha}\left(t+\sigma-\tau_{k(\delta_1)}\right)^{\alpha}}dt \\
&\quad-\frac{1}{(\alpha+1)^{\alpha+1}}\int_{c_1}^{\delta_1}\rho(t)r(t)H_1(t,c_1)\left|h_1(t,c_1)+\frac{\rho'(t)}{\rho(t)}\right|^{\alpha+1}dt+\int_{c_1}^{\delta_1}\rho(t)p_0(t)H_1(t,c_1)dt \\
&\quad<\sum_{i=k(c_1)+1}^{k(\delta_1)}\left(\frac{b_i^{\alpha}}{a_i^{\alpha}}-1\right)H_1(\tau_i,c_1)u(\tau_i)-H_1(\delta_1,c_1)u(\delta_1).
\end{aligned} \tag{2.62}$$

On the other hand, multiplying both sides of (2.14) by $H_2(d_1,t)$ and using similar analysis to the above, we can obtain

$$\begin{aligned}
&\int_{\delta_1}^{\tau_{k(\delta_1)}+\sigma}\widetilde{H}_2(d_1,t)\frac{\left(t-\tau_{k(\delta_1)}\right)^{\alpha}}{b_{k(\delta_1)}^{\alpha}\left(t+\sigma-\tau_{k(\delta_1)}\right)^{\alpha}}+\int_{\tau_{k(\delta_1)}+\sigma}^{\tau_{k(\delta_1)+1}}\widetilde{H}_2(d_1,t)\frac{\left(t-\tau_{k(\delta_1)}-\sigma\right)^{\alpha}}{\left(t-\tau_{k(\delta_1)}\right)^{\alpha}}dt \\
&\quad+\sum_{i=k(\delta_1)+1}^{k(d_1)-1}\left[\int_{\tau_i}^{\tau_i+\sigma}\widetilde{H}_2(d_1,t)\frac{(t-\tau_i)^{\alpha}}{b_i^{\alpha}(t+\sigma-\tau_i)^{\alpha}}dt+\int_{\tau_i+\sigma}^{\tau_{i+1}}\widetilde{H}_2(d_1,t)\frac{(t-\tau_i-\sigma)^{\alpha}}{(t-\tau_i)^{\alpha}}dt\right] \\
&\quad+\int_{\tau_{k(d_1)}}^{d_1}\widetilde{H}_2(d_1,t)\frac{\left(t-\tau_{k(d_1)}\right)^{\alpha}}{b_{k(d_1)}^{\alpha}\left(t+\sigma-\tau_{k(d_1)}\right)^{\alpha}}dt+\int_{\delta_1}^{d_1}\rho(t)p_0(t)H_2(d_1,t)dt \\
&\quad-\frac{1}{(\alpha+1)^{\alpha+1}}\int_{\delta_1}^{d_1}\rho(t)r(t)H_2(d_1,t)\left|h_2(d_1,t)+\frac{\rho'(t)}{\rho(t)}\right|^{\alpha+1}dt \\
&\quad<\sum_{i=k(\delta_1)+1}^{k(d_1)}\left(\frac{b_i^{\alpha}}{a_i^{\alpha}}-1\right)H_2(d_1,\tau_i)u(\tau_i)+H_2(d_1,\delta_1)u(\delta_1).
\end{aligned} \tag{2.63}$$

Dividing (2.62) and (2.63) by $H_1(\delta_1,c_1)$, and $H_2(d_1,\delta_1)$ respectively, and adding them, we get

$$\begin{aligned}
\Pi_{1,1}+\Pi_{2,1}<{}&\frac{1}{H_1(\delta_1,c_1)}\sum_{i=k(c_1)+1}^{k(\delta_1)}\left(\frac{b_i^{\alpha}}{a_i^{\alpha}}-1\right)H_1(\tau_i,c_1)u(\tau_i) \\
&+\frac{1}{H_2(d_1,\delta_1)}\sum_{i=k(\delta_1)+1}^{k(d_1)}\left(\frac{b_i^{\alpha}}{a_i^{\alpha}}-1\right)H_2(d_1,\tau_i)u(\tau_i).
\end{aligned} \tag{2.64}$$

Using the same methods as (2.46), we have

$$\sum_{i=k(c_1)+1}^{k(\delta_1)}\left(\frac{b_i^\alpha}{a_i^\alpha}-1\right)H_1(\tau_i,c_1)u(\tau_i)\le\rho_1r_1Q_{c_1}^{\delta_1}[H_1(\cdot,c_1)],$$
$$\sum_{i=k(\delta_1)+1}^{k(d_1)}\left(\frac{r_ib_i^\alpha}{a_i^\alpha}-1\right)H_2(d_1,\tau_i)u(\tau_i)\le\rho_1r_1Q_{\delta_1}^{d_1}[H_2(d_1,\cdot)]. \tag{2.65}$$

From (2.64), (2.65), we can obtain a contradiction to the condition (2.56).

When $x(t)<0$, we choose interval $[c_2,d_2]$ to study (1.1). The proof is similar and will be omitted. Therefore, we complete the proof. □

Remark 2.9. When $\sigma=0$, that is, the delay disappears and $\alpha=1$ in (1.1), our result Theorem 2.8 reduces to Theorem 2.2 of [12].

3. Examples

In this section, we give two examples to illustrate the effectiveness and nonemptiness of our results.

Example 3.1. Consider the following delay differential equation with impulse:

$$x''(t)+\mu_1\left|x\left(t-\frac{\pi}{12}\right)\right|^{3/2}x\left(t-\frac{\pi}{12}\right)+\mu_2\left|x\left(t-\frac{\pi}{12}\right)\right|^{-1/2}x\left(t-\frac{\pi}{12}\right)=-\sin(2t),\quad t\neq\tau_k,$$
$$x(\tau_k^+)=a_kx(\tau_k),\qquad x'(\tau_k^+)=b_kx'(\tau_k),\quad t=\tau_k, \tag{3.1}$$

where $\tau_k:\tau_{n,1}=2n\pi+5\pi/18,\ \tau_{n,2}=2n\pi+11\pi/18, n\in\mathbb{N}$ and μ_1,μ_2 are positive constants.

For any $T>0$, we can choose large n_0 such that $T<c_1=2n\pi+\pi/6,\ d_1=2n\pi+\pi/3,\ c_2=2n\pi+\pi/2,\ d_2=2n+2\pi/3,\ n=n_0,n_0+1,\ldots$. There are impulsive moments $\tau_{n,1}$ in $[c_1,d_1]$ and $\tau_{n,2}$ in $[c_2,d_2]$. From $\tau_{n,2}-\tau_{n,1}=\pi/3>\pi/12$ and $\tau_{n+1,1}-\tau_{n,2}=5\pi/3>\pi/12$ for all $n>n_0$, we know that condition $\tau_{k+1}-\tau_k>\sigma$ is satisfied. Moreover, we also see the conditions (S1) and (2.7) are satisfied.

We can choose $\eta_0=\eta_1=\eta_2=1/3$ such that Lemma 2.1 holds. Let $w_1(t)=w_2(t)=\sin(6t)$ and $\rho(t)=1$. It is easy to verify that $W_1(t)=3(\mu_1\mu_2)^{1/3}|\sin(2t)|^{1/3}\sin^2(6t)$. By a simple calculation, the left side of (2.8) is the following:

$$\int_{c_1}^{\tau_{k(c_1)+1}}W_1(t)\frac{(t-\tau_{k(c_1)}-\sigma)^\alpha}{(t-\tau_{k(c_1)})^\alpha}dt$$
$$+\sum_{i=k(c_1)+1}^{k(d_1)-1}\left[\int_{\tau_i}^{\tau_i+\sigma}W_1(t)\frac{(t-\tau_i)^\alpha}{b_i^\alpha(t+\sigma-\tau_i)^\alpha}dt+\int_{\tau_i+\sigma}^{\tau_{i+1}}W_1(t)\frac{(t-\tau_i-\sigma)^\alpha}{(t-\tau_i)^\alpha}dt\right]$$
$$+\int_{\tau_{k(d_1)}}^{d_1}W_1(t)\frac{(t-\tau_{k(d_1)})^\alpha}{b_{k(d_1)}^\alpha(t+\sigma-\tau_{k(d_1)})^\alpha}dt+\int_{c_1}^{d_1}\rho(t)p_0(t)|w_1(t)|^{\alpha+1}dt$$

$$
\begin{aligned}
&-\int_{c_1}^{d_1}\rho(t)r(t)\left(\left|w_1'(t)\right|+\frac{\left|\rho'(t)\right||w_1(t)|}{(\alpha+1)\rho(t)}\right)^{\alpha+1}dt\\
&=\int_{2n\pi+\pi/6}^{2n\pi+5\pi/18}W_1(t)\frac{t-2(n-1)\pi-11\pi/18-\pi/12}{t-2(n-1)\pi-11\pi/18}dt\\
&\quad+\int_{2n\pi+5\pi/18}^{2n\pi+\pi/3}W_1(t)\frac{t-2n\pi-5\pi/18}{b_{n,1}(t-2n\pi-5\pi/18+\pi/12)}dt-\int_{2n\pi+\pi/6}^{2n\pi+\pi/3}36\cos^2 6t\,dt\\
&=\int_{\pi/6}^{5\pi/18}W_1(t)\frac{t+47\pi/36}{t+25\pi/18}dt+\int_{5\pi/18}^{\pi/3}W_1(t)\frac{t-5\pi/18}{b_{n,1}(t-7\pi/36)}dt-3\pi\\
&\approx 3\sqrt[3]{\mu_1\mu_2}\left(0.199+\frac{0.715}{b_{n,1}}\right)-3\pi.
\end{aligned}
\tag{3.2}
$$

On the other hand, we have

$$
Q_{c_1}^{d_1}\left[w_1^2\right]=\frac{27}{4\pi}\frac{b_{n,1}-a_{n,1}}{a_{n,1}}. \tag{3.3}
$$

Thus if

$$
3\sqrt[3]{\mu_1\mu_2}\left(0.199+\frac{0.715}{b_{n,1}}\right)\geq 3\pi+\frac{27}{4\pi}\frac{b_{n,1}-a_{n,1}}{a_{n,1}}, \tag{3.4}
$$

the condition (2.8) is satisfied in $[c_1,d_1]$. Similarly, we can show that for $t\in[c_2,d_2]$ the condition (2.8) is satisfied if

$$
3\sqrt[3]{\mu_1\mu_2}\left(0.057+\frac{0.003}{b_{n,2}}\right)\geq 3\pi+\frac{27}{4\pi}\frac{b_{n,2}-a_{n,2}}{a_{n,2}}. \tag{3.5}
$$

Hence, by Theorem 2.3, (3.1) is oscillatory, if (3.4) and (3.5) hold. Particularly, let $a_k=b_k$, for all $k\in\mathbb{N}$, condition (3.4) and (3.5) become

$$
\begin{aligned}
&\sqrt[3]{\mu_1\mu_2}\left(0.199+\frac{0.715}{b_{n,1}}\right)\geq\pi,\\
&\sqrt[3]{\mu_1\mu_2}\left(0.057+\frac{0.003}{b_{n,2}}\right)\geq\pi.
\end{aligned}
\tag{3.6}
$$

Example 3.2. Consider the following equation:

$$
\begin{aligned}
&x''(t)+\mu_1p_1(t)\left|x\left(t-\frac{2}{3}\right)\right|^{3/2}x\left(t-\frac{2}{3}\right)+\mu_2p_2(t)\left|x\left(t-\frac{2}{3}\right)\right|^{-1/2}x\left(t-\frac{2}{3}\right)=e(t),\quad t\neq\tau_k,\\
&x\left(\tau_k^+\right)=a_kx(\tau_k),\qquad x'\left(\tau_k^+\right)=b_kx'(\tau_k),\quad k=1,2,\ldots,
\end{aligned}
\tag{3.7}
$$

where μ_1, μ_2 are positive constants; $\tau_k : \tau_{n,1} = 9n + 3/2$, $\tau_{n,2} = 9n + 5/2$, $\tau_{n,3} = 9n + 15/2$, $\tau_{n,4} = 9n + 17/2$ $(n = 0, 1, 2, \ldots)$ and $\tau_{k+1} - \tau_k > \sigma = 2/3$. In addition, let

$$p_1(t) = p_2(t) = \begin{cases} (t-9n)^3, & t \in [9n, 9n+3], \\ 3^3, & t \in [9n+3, 9n+6], \\ (9n+9-t)^3, & t \in [9n+6, 9n+9], \end{cases}$$
$$e(t) = (t-9n-3)^3, \quad t \in [9n, 9n+9]. \tag{3.8}$$

For any $t_0 > 0$, we choose n large enough such that $t_0 < 9n$, and let $[c_1, d_1] = [9n+1, 9n+3]$, $[c_2, d_2] = [9n+7, 9n+9]$, $\delta_1 = 9n+2$ and $\delta_2 = 9n+8$. It is easy to see that condition (2.7) in Theorem 2.8 is satisfied. Letting $H_1(t,s) = H_2(t,s) = (t-s)^3$, we get $h_1(t,s) = -h_2(t,s) = 3/(t-s)$. By simple calculation, we have

$$\begin{aligned}
\Pi_{1,1} &= 3\sqrt[3]{\mu_1\mu_2}\left\{\int_{9n+1}^{9n+3/2}(t-9n-1)^3(9n+3-t)(t-9n)^2\frac{t-9n-1/6}{t-9n+1/2}dt\right. \\
&\quad \left. + \int_{9n+3/2}^{9n+2}(t-9n-1)^3(9n+3-t)(t-9n)^2\frac{t-9-3/2}{b_{n,1}(t-9n-5/6)}dt\right\} - \frac{9}{8} \\
&= 3\sqrt[3]{\mu_1\mu_2}\left(\int_1^{3/2}\frac{u^2(u-1)^3(3-u)(u-1/6)}{u+1/2}du + \int_{3/2}^{2}\frac{u^2(u-1)^3(3-u)(u-3/2)}{b_{n,1}(u-5/6)}du\right) - \frac{9}{8} \\
&\approx 3\sqrt[3]{\mu_1\mu_2}\left(0.32 + \frac{0.290}{b_{n,1}}\right) - \frac{9}{8}, \\
\Pi_{2,1} &= 3\sqrt[3]{\mu_1\mu_2}\left\{\int_{9n+2}^{9n+13/6}(9n+3-t)^4(t-9n)^2\frac{t-9n-3/2}{b_{n,1}(t-9n+5/6)}dt\right. \\
&\quad + \int_{9n+13/6}^{9n+5/2}(9n+3-t)^4(t-9n)^2\frac{t-9n-13/6}{t-9n-3/2}dt \\
&\quad \left. + \int_{9n+5/2}^{9n+3}(9n+3-t)^4(t-9n)^2\frac{t-9n-5/2}{b_{n,2}(t-9n-11/6)}dt\right\} - \frac{9}{8} \\
&= 3\sqrt[3]{\mu_1\mu_2}\left\{\int_2^{13/6}\frac{u^2(3-u)^4(u-3/2)}{b_{n,1}(u+5/6)}dt + \int_{13/6}^{5/2}\frac{u^2(3-u)^4(u-13/6)}{u-3/2}dt\right. \\
&\quad \left. + \int_{5/2}^{3}\frac{u^2(3-u)^4(u-5/2)}{b_{n,2}(u-11/6)}dt\right\} - \frac{9}{8} \\
&\approx 3\sqrt[3]{\mu_1\mu_2}\left(\frac{0.102}{b_{n,1}} + 0.056 + \frac{0.005}{b_{n,2}}\right) - \frac{9}{8}.
\end{aligned} \tag{3.9}$$

Then the left-hand side of the inequality (2.56) is

$$\Pi_{1,1}+\Pi_{2,1}\approx 3\sqrt[3]{\mu_1\mu_2}\left(0.376+\frac{0.392}{b_{n,1}}+\frac{0.005}{b_{n,2}}\right)-\frac{9}{4}. \tag{3.10}$$

Because $r_1 = r_2 = 1$, $\tau_{k(c_1)+1} = \tau_{k(\delta_1)} = \tau_{n,1} = 9n+3/2 \in (c_1,\delta_1)$ and $\tau_{k(\delta_1)+1} = \tau_{k(d_1)} = \tau_{n,2} = 9n+5/2 \in (\delta_1, d_1)$, it is easy to get that the right-hand side of the inequality (2.56) for $j=1$ is

$$\frac{r_1}{H_1(\delta_1,c_1)}Q_{c_1}^{\delta_1}[H_1(\cdot,c_1)]+\frac{r_1}{H_2(d_1,\delta_1)}Q_{\delta_1}^{d_1}[H_2(d_1,\cdot)]=\frac{b_{n,1}-a_{n,1}}{4a_{n,1}}+\frac{b_{n,2}-a_{n,2}}{4a_{n,2}}. \tag{3.11}$$

Thus (2.56) is satisfied with $j=1$ if

$$3\sqrt[3]{\mu_1\mu_2}\left(0.376+\frac{0.392}{b_{n,1}}+\frac{0.005}{b_{n,2}}\right)>\frac{9}{4}+\frac{b_{n,1}-a_{n,1}}{4a_{n,1}}+\frac{b_{n,2}-a_{n,2}}{4a_{n,2}}. \tag{3.12}$$

When $j=2$, with the same argument as above we get that the left-hand side of inequality (2.56) is

$$\begin{aligned}
\Pi_{1,2}&+\Pi_{2,2}\\
&=3\sqrt[3]{\mu_1\mu_2}\left\{\int_7^{15/2}\frac{(u-9)^2(u-3)(u-7)^3(u-19/6)}{u-5/2}du\right.\\
&\qquad+\int_{15/2}^{8}\frac{(u-9)^2(u-3)(u-7)^3(u-15/2)}{b_{n,3}(u-41/6)}du\\
&\qquad+\int_{8}^{49/6}\frac{(u-3)(9-u)^5(u-15/2)}{b_{n,3}(u-41/6)}du+\int_{49/6}^{17/2}\frac{(u-3)(9-u)^5(u-19/6)}{u-5/2}du\\
&\qquad\left.+\int_{17/2}^{9}\frac{(u-3)(9-u)^5(u-17/2)}{b_{n,4}(u-47/6)}du\right\}-\frac{9}{4}\\
&\approx 3\sqrt[3]{\mu_1\mu_2}\left(0.400+\frac{0.724}{b_{n,3}}+\frac{0.001}{b_{n,4}}\right)-\frac{9}{4},
\end{aligned} \tag{3.13}$$

and the right-hand side of the inequality (2.56) is

$$\frac{r_2}{H_2(\delta_2,c_2)}Q_{c_2}^{\delta_2}[H_1(\cdot,c_2)]+\frac{r_2}{H_2(d_2,\delta_2)}Q_{\delta_2}^{d_2}[H_2(d_2,\cdot)]=\frac{b_{n,3}-a_{n,3}}{4a_{n,3}}+\frac{b_{n,4}-a_{n,4}}{4a_{n,4}}. \tag{3.14}$$

Therefore, (2.51) is satisfied with $j=2$ if

$$3\sqrt[3]{\mu_1\mu_2}\left(0.400+\frac{0.724}{b_{n,3}}+\frac{0.001}{b_{n,4}}\right)>\frac{9}{4}+\frac{b_{n,3}-a_{n,3}}{4a_{n,3}}+\frac{b_{n,4}-a_{n,4}}{4a_{n,4}}. \tag{3.15}$$

Hence, by Theorem 2.8, (3.7) is oscillatory if

$$\begin{aligned} &3\sqrt[3]{\mu_1\mu_2}\left(0.376+\frac{0.392}{b_{n,1}}+\frac{0.005}{b_{n,2}}\right)>\frac{9}{4}+\frac{b_{n,1}-a_{n,1}}{4a_{n,1}}+\frac{b_{n,2}-a_{n,2}}{4a_{n,2}},\\ &3\sqrt[3]{\mu_1\mu_2}\left(0.400+\frac{0.724}{b_{n,3}}+\frac{0.001}{b_{n,4}}\right)>\frac{9}{4}+\frac{b_{n,3}-a_{n,3}}{4a_{n,3}}+\frac{b_{n,4}-a_{n,4}}{4a_{n,4}}. \end{aligned} \tag{3.16}$$

Particularly, when $a_k = b_k$, for all $k \in \mathbb{N}$, condition (3.16) becomes

$$\begin{aligned} &3\sqrt[3]{\mu_1\mu_2}\left(0.376+\frac{0.392}{b_{n,1}}+\frac{0.005}{b_{n,2}}\right)>\frac{9}{4},\\ &3\sqrt[3]{\mu_1\mu_2}\left(0.400+\frac{0.724}{b_{n,3}}+\frac{0.001}{b_{n,4}}\right)>\frac{9}{4}. \end{aligned} \tag{3.17}$$

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