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Research Article

On Uniformly Bazilevic and Related Functions

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We introduce a new class of functions analytic in the open unit disc, which contains the class of Bazilevic functions and also generalizes the concept of uniform convexity. We establish univalence criterion for the functions in this class and investigate rate of growth of coefficients, arc length problem, inclusion results, and distortion bounds. Some interesting results are derived as special cases.

1. Introduction

Let A be the class functions analytic in the open unit disc $E = \{z : |z| < 1\}$ and satisfying the conditions $f(0) = 0, f'(0) = 1$. Let $S \subset A$ be the class of functions which are univalent, and also let $S^*(r), C(r)$ be the subclasses of S which consists of starlike and convex functions of order γ $(0 \leq \gamma < 1)$, respectively.

Kanas and Wisniowska [1, 2] studied the classes of k-uniformly convex functions, denoted by k-UCV and the corresponding class UST related with the Alexander-type relation. In [3], the domain $\Omega_k, k \in [0, \infty)$ is defined as follows:

$$\Omega_k = \left\{u + iv : u > k\sqrt{(u-1)^2 + v^2}\right\}. \tag{1.1}$$

For fixed k, Ω_k represents the conic region bounded, successively, by the imaginary axis ($k = 0$), the right branch of hyperbola ($0 < k < 1$), a parabola ($k = 1$), and an ellipse ($k > 1$). Also, we note that, for no choice of $k(k > 1), \Omega_k$ reduces to a disc.

In this paper, we will choose $k \in [0, 1]$. Related with Ω_k, we define the domain $\Omega_{k,\gamma}$, see [4], as follows:

$$\Omega_{k,\gamma} = (1-\gamma)\Omega_k + \gamma, \quad (0 \leq \gamma < 1). \tag{1.2}$$

For $k \in [0,1]$, the following functions denoted by $p_{k,\gamma}(z)$ are univalent in E, continuous as regards to k and γ, have real coefficients, and map E onto $\Omega_{k,\gamma}$, such that $p_{k,\gamma}(0) = 1$, $p'_{k,\gamma}(0) > 0$:

$$p_{k,\gamma}(z) = \begin{cases} \dfrac{1+(1-2\gamma)z}{(1-z)}, & (k=0), \\ 1 + \dfrac{2(1-\gamma)}{1-k^2}\sinh^2\left[\left(\dfrac{2}{\pi}\operatorname{arc}\cos k\right)\operatorname{arc}\tanh\sqrt{z}\right], & (0<k<1), \\ 1 + \dfrac{2(1-\gamma)}{\pi^2}\left(\log\dfrac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2, & (k=1). \end{cases} \tag{1.3}$$

Let P denote the class of the Caratheodory functions of positive real part. We define a subclass of P as follows.

Definition 1.1. Let $P(p_{k,\gamma}) \subset P$ be the class consisting of functions $p(z)$ which are analytic in E with $p(0) = 1$ and which are subordinate to $p_{k,\gamma}(z)$ in E. We write $p \in P(p_{k,\gamma})$ implies $p \prec p_{k,\gamma}$, where $p_{k,\gamma}(z)$ is the function, given by (1.3), and maps E onto $\Omega_{k,\gamma}$, $k \in [0,1], \gamma \in [0,1)$. That is $p(E) \subset p_{k,\gamma}(E)$. We note that $P(p_{0,0}) = P$ and $p \in P(p_{0,\gamma}) = P(\gamma)$ implies that $\operatorname{Re} p(z) > \gamma, z \in E$. It is easy to verify that $P(p_{k,\gamma})$ is a convex set, and $P(p_{k,\gamma}) \subset P(\gamma_1)$, $\gamma_1 = (k+\gamma)/(1+k)$.

The class $P(p_{k,\gamma})$ is extended as follows.

Definition 1.2. Let $p(z)$ be analytic in E with $p(0) = 1$. Then, $p \in P_m(p_{k,\gamma})$ if and only if, for $m \geq 2$, $0 \leq \gamma < 1$, $k \in [0,1], z \in E$, we have

$$p(z) = \left(\frac{m}{4} + \frac{1}{2}\right)p_1(z) - \left(\frac{m}{4} - \frac{1}{2}\right)p_2(z), \quad p_1, p_2 \in P(p_{k,\gamma}). \tag{1.4}$$

When $k = 0$, we obtain the class $P_m(\gamma)$ which reduces to the class P_m with $\gamma = 0$, introduced and studied in [5]. Also $P_2(p_{k,\gamma}) = P(p_{k,\gamma})$.

We now define the following.

Definition 1.3. Let $f \in A$ with $f(z)f'(z)/z \neq 0$ in E. Let, for α real and $\beta \in [-1/2, 1)$,

$$J(\alpha,\beta,f(z)) = (1-\alpha)(1-\beta)\frac{zf'(z)}{f(z)} + \alpha\left\{1-\beta+\frac{zf''(z)}{f'(z)}\right\}. \tag{1.5}$$

Then $f \in k - UB_m(\alpha,\beta,\gamma)$ if and only if

$$J(\alpha,\beta,f(z)) \in P_m(p_{k,\gamma}), \quad \text{for } z \in E. \tag{1.6}$$

For any real number α and $\beta \in [-1/2,\ 1)$, we note that the identity function belongs to $k - UB_m(\alpha,\beta,\gamma)$ so that $k - UB_m(\alpha,\beta,\gamma)$ is not empty.

Throughout this paper, we assume that $k \in [0,1], \gamma \in [0,1), m \geq 2$, $\alpha, \beta \in \mathbb{R}$, $\beta \in [-1/2, 1), z \in E$, unless otherwise specified.

We note the following special cases.

(i) For $m = 2,\ 0 \le \beta < 1,\ k = 0$, we have a subclass of a class introduced by Mocanu [6]. Also see [7, 8].

(ii) It is well known that $0 - UB_2(\alpha, \beta, 0) = B_2(\alpha, \beta)$ contains the Bazilevic functions with $\alpha > 0$ and $B_2(\alpha, \beta) \subset S$.

(iii) $k - UB_m(1, 0, \gamma) = k - V_m(\gamma) \subset V_m((k+\gamma)/(1+k)) \subset V_m$, where V_m is the well-known class of functions with bounded boundary rotation, see [9].

(iv) $k - UB_m(0, 0, \gamma) = k - UR_m(\gamma) \subset R_m((k+\gamma)/(1+k)) \subset R_m$, where R_m denotes the class of functions with bounded radius rotation, see [9].

(v) $k - UB_2(1, 0, \gamma) = k - UCV(\gamma)$ is the class of uniformly convex functions of order $\gamma,\ 0 - UCV(\gamma) = C(\gamma)$, and $1 - UCV(\gamma) = UCV(\gamma)$.

Also $k - UB_2(0, 0, \gamma) = UST(\gamma)$, and $UST(\gamma) \subset S^*(\gamma_1),\ \gamma_1 = (k+\gamma)/(1+k)$.

Remark 1.4.

(i) From Definition 1.3, it can easily be seen that $f \in k - UB_m(\alpha, \beta, \gamma)$ if and only if, for $\alpha \neq 0$, there exists a function $g \in k - UR_m(\gamma)$ such that

$$f(z) = \left[m_1 \int_0^z t^{m_1 - 1} \left(\frac{g(t)}{t} \right)^{(1-\beta)/\alpha} dt \right]^{1/m_1} = z + \cdots, \tag{1.7}$$

where

$$m_1 = 1 + \frac{(1-\alpha)(1-\beta)}{\alpha}. \tag{1.8}$$

A simple computation shows that (1.7) can be written as

$$(1-\alpha)\frac{zf'(z)}{f(z)} + \frac{\alpha}{1-\beta}\left(1 - \beta + \frac{zf''(z)}{f'(z)}\right) = \frac{zg'(z)}{g(z)}, \tag{1.9}$$

with $g \in k - UR_m(\gamma)$.

(ii) Also, for $f \in k - UB_m(\alpha, \beta, \gamma)$, it can be verified from (1.5) that $z(f(z)/z)^{(1-\alpha)}(f'(z))^{\alpha/(1-\beta)}$ belongs to $k - UR_m(\gamma)$ for all $z \in E$.

2. Preliminary Results

The following lemma is an easy generalization of a result due to Kanas [3].

Lemma 2.1 (see [10]). *Let $k \in [0, \infty)$, and let σ, δ_1 be any complex numbers with $\sigma \neq 0$ and $0 \le \gamma < \operatorname{Re}(\sigma k/(k+1) + \delta_1)$. If $h(z)$ is analytic in E, $h(0) = 1$ and satisfies*

$$\left(h(z) + \frac{zh'(z)}{\sigma h(z) + \delta_1} \right) \prec p_{k,\gamma}(z), \tag{2.1}$$

and $q_{k,\gamma}(z)$ *is an analytic solution of*

$$q_{k,\gamma}(z) + \frac{zq'_{k,\gamma}(z)}{\sigma q_{k,\gamma}(z) + \delta_1} = p_{k,\gamma}(z), \tag{2.2}$$

then $q_{k,\gamma}(z)$ *is univalent,*

$$h(z) \prec q_{k,\gamma}(z) \prec p_{k,\gamma}(z), \tag{2.3}$$

and $q_{k,\gamma}(z)$ *is the best dominant of* (2.1)*, where* $q_{k,\gamma}(z)$ *is given as*

$$q_{k,\gamma}(z) = \left\{ \left[\left(\int_0^1 \exp \int_t^{tz} \frac{p_{k,\gamma}(u) - 1}{u} du \right) dt \right]^{-1} + \frac{\delta_1}{\sigma} \right\}. \tag{2.4}$$

Lemma 2.2 (see [11]). *Let* $u = u_1 + iu_2$ *and* $v = v_1 + iv_2$*, and let* $\psi(u,v)$ *be a complex-valued function satisfying the following conditions:*

(i) $\psi(u,v)$ *is continuous in* $D \subset \mathbb{C}^2$,

(ii) $(1,0) \in D$ *and* $\operatorname{Re}\psi(1,0) > 0$,

(iii) $\operatorname{Re}\psi(iu_2, v_1) \leq 0$*, whenever* $(iu_2, v_1) \in D$ *and* $v_1 \leq -(1+u_2^2)/2$.

If $h(z)$ *is a function analytic in* E *such that* $(h(z), zh'(z)) \in D$ *and* $\operatorname{Re}\psi\{h(z), zh'(z)\} > 0$ *for* $z \in E$*, then* $\operatorname{Re} h(z) > 0$ *in* E.

Lemma 2.3 (see [12]). *Let* $f \in A$ *with* $f(z)f'(z)/z \neq 0$ *in* E*. Then* $f(z)$ *is a Bazilevic function (hence univalent) in* E *if and only if, for* $0 \leq \theta_1 < \theta_2 \leq 2\pi$, $0 < r < 1$, $z = re^{i\theta}$*, one has*

$$\int_{\theta_1}^{\theta_2} \left[\operatorname{Re}\left\{ 1 + \frac{zf''(z)}{f'(z)} + (\alpha_1 - 1)\frac{zf'(z)}{f(z)} \right\} - \beta_1 \operatorname{Im} \frac{zf'(z)}{f(z)} \right] d\theta > -\pi, \tag{2.5}$$

where $\alpha_1 > 0$, β_1 *real.*

3. Main Results

In the following, we establish the criterion of univalence for the class $k - UB_m(\alpha,\beta,\gamma)$ with certain restriction on the upper bound of the value of m.

Theorem 3.1. *Let* $f \in k - UB_m(\alpha,\beta,\gamma)$, $\alpha > 0$, $\beta \in [0,1)$*. Then* $f \in S$ *for* $m \leq 2\{\alpha(1+2\beta)(k+1)/(1-\beta)(1-\gamma) + 1\}$ *in* E.

Proof. Since $f \in k - UB_m(\alpha,\beta,\gamma) \subset B_m(\alpha,\beta,\gamma_1)$, $\gamma_1 = (k+\gamma)/(1+k)$, we note that, in (1.9), $g \in R_m(\gamma_1)$. It is known [13] that there exists $g_1 \in R_m$ such that, for $z \in E$,

$$g(z) = z\left(\frac{g_1(z)}{z}\right)^{(1-\gamma_1)}. \tag{3.1}$$

Now from (1.9) and (3.1), we have, for $z = re^{i\theta}$, $0 < r < 1$, $0 \le \theta_1 < \theta_2 \le 2\pi$,

$$\int_{\theta_1}^{\theta_2} \operatorname{Re}\left\{ \frac{(1-\alpha)(1-\beta)}{\alpha} \frac{zf'(z)}{f(z)} + \left(1 + \frac{zf''(z)}{f'(z)}\right) \right\} d\theta > -\pi \left\{ \frac{(1-\beta)}{\alpha} \left(\frac{1-\gamma}{1+k}\right) \left(\frac{m}{2} - 1\right) - 2\beta \right\}. \tag{3.2}$$

We use Lemma 2.3 with $\alpha_1 = 1/\alpha + \beta(1 - 1/\alpha)$, $\beta_1 = 0$ to have the required result. □

As special cases, we note that

(i) for $k = 0$, $\beta = 0$, $\alpha = 1$, $f(z)$ is univalent for $m \le 2(1/(1-\gamma) + 1)$. We observe that, when $\gamma = 0$, we obtain a well-known result that the class V_m of functions with bounded boundary rotation contains univalent functions for $2 \le m \le 4$, see [9];

(ii) for $k = 1$, $\beta = 0$, $\alpha = 1$, $\gamma = 0$, $f \in 1 - UV_m$ is univalent for $m \le 6$.

Let Γ denote the Gamma function, and let $F(a, b, c; z)$ be the hypergeometric function which is analytic in E and is defined by

$$F(a, b, c; z) = \frac{\Gamma(c)}{\Gamma(c)\Gamma(c-b)} \int_0^1 u^{b-1} (1-u)^{c-b-1} (1-zu)^{-a} du, \tag{3.3}$$

where $\operatorname{Re} b > 0$, $\operatorname{Re}(c - b) > 0$.

Define

$$K(m_1, k_1, r) = r\left[F\left(k_1\left(\frac{m}{2}+1\right), m_1, \left\{k_1\left(\frac{m}{2}-1\right) + m_1 + 1\right\}, r\right)\right]^{1/m_1}, \tag{3.4}$$

where

$$\begin{aligned} m_1 &= 1 + \frac{(1-\alpha)(1-\beta)}{\alpha}, \\ k_1 &= \frac{(1-\beta)}{\alpha}\left(\frac{1-\gamma}{1+k}\right), \quad \alpha_1 \ne 0. \end{aligned} \tag{3.5}$$

Also, for $0 \le \theta \le 2\pi$, let

$$f_\theta(m_1, k_1, z) = \left[m_1 \int_0^z \xi^{m_1-1} \left(1 + e^{i\theta}\xi\right)^{-k_1((m/2)+1)} \left(1 - e^{i\theta}\xi\right)^{k_1((m/2)-1)} d\xi\right]^{1/m_1}. \tag{3.6}$$

We now consider the distortion problem.

Theorem 3.2. *Let $f \in k - UB_m(\alpha, \beta, \gamma)$, $\alpha \ne 0$, $\beta \in [0, 1)$. Then, for $|z| = r$,*

(i) $-K(m_1, k_1, -r) \le |f(z)| \le K(m_1, k_1, r)$, *for $\alpha > 0$,*

(ii) $K(m_1, k_1, r) \le |f(z)| \le -K(m_1, k_1, -r)$, *for $\alpha < 0$.*

Proof. Suppose z_0 is a point on the circumference $|z| = r$ such that $|f(z_0)| = \min_{0\leq\theta\leq 2\pi}|f(re^{i\theta})|$, and let τ denote the preimage under f on the segment $[0, f(z_0)]$.

Consider first case $\alpha > 0$.

Distortion results for $g_1 \in R_m$ are given as follows:

$$\frac{(1-|\xi|)^{m/2-1}}{(1+|\xi|)^{m/2+1}} \leq \left|\frac{g_1(\xi)}{\xi}\right| \leq \frac{(1+|\xi|)^{m/2-1}}{(1-|\xi|)^{m/2+1}}, \tag{3.7}$$

see [9]. In view of (1.7), (3.1), and (3.7), we have

$$\begin{aligned} \left|f(z_0)\right|^{m_1} &= m_1 \int_\tau \xi^{m_1-1}\left|\frac{g(\xi)}{\xi}\right|^{(1-\beta)/\alpha} |d\xi|, \\ &\geq m_1 \int_0^r t^{m_1-1}(1-t)^{k_1(m/2-1)}(1+t)^{-k_1(m/2+1)}dt, \\ &= m_1 r^{m_1}\int_0^1 u^{m_1-1}(1+ru)^{-k_1(m/2+1)}(1-u)^{k_1(m/2-1)}du. \end{aligned} \tag{3.8}$$

Hence

$$\left|f(z)\right| \geq \left|f(z_0)\right| \geq -K(m_1, k_1, -r), \tag{3.9}$$

where $K(m_1, k_1, r)$ is defined in (3.4).

The proof of (i) for the upper bound of $|f(z)|$ can be obtained in the similar manner.

The proof of (ii) for the case $\alpha < 0$ is analogous. □

Remark 3.3. The bounds in Theorem 3.2 are sharp for $f \in B_m(\alpha, \beta, \gamma_1)$, $\gamma_1 = (k+\gamma)/(1+k)$, and the equality occurs for the function $f_\theta(m_1, k_1, z)$ given by (3.6) with θ suitably chosen.

As an application of Theorem 3.2, we derive bound for initial Taylor coefficient of $f(z)$ as follows.

Corollary 3.4. *Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ satisfy the conditions of Theorem 3.2. Then*

$$|a_2| \leq \frac{k_1(m/2+1)}{\{k_1(m/2-1)+m_1+1\}}, \tag{3.10}$$

where k_1 and m_1 are given in (3.4).

Proof. First consider the case $\alpha > 0$ and assume a_2 to be real.

We find

$$\begin{gathered} K(m_1, k_1, r) = r + \frac{k_1(m/2+1)}{\{k_1(m/2-1)+m_1+1\}}r^2 + O\left(r^3\right), \\ \left|f(r)\right| = r + a_2 r^2 + O\left(r^3\right). \end{gathered} \tag{3.11}$$

In view of Theorem 3.2(i), we have

$$a_2 \le \frac{k_1(m/2+1)}{\{k_1(m/2-1)+m_1+1\}}. \tag{3.12}$$

For $\alpha < 0$, we proceed in a similar manner and use Theorem 3.2(ii) to complete the proof. □

We note that $-K(m_1,k_1,-r)$ and $K(m_1,k_1,r)$ are increasing functions of r. Thus letting $r \to 1$ in the left hand side of (i) and (ii) in Theorem 3.2, we have the following.

Corollary 3.5. *Let f satisfy the conditions of Theorem 3.2. Then*

$$\bigcap_{f\in k-\cup B_m(\alpha,\beta,\gamma)} f(E) = \{w : |w| < r(m_1,k_1)\}, \tag{3.13}$$

where

$$r(m_1,k_1) = \begin{cases} F\Big(k_1\Big(\dfrac{m}{2}+1\Big), m_1, \Big\{k_1\Big(\dfrac{m}{2}-1\Big)+m_1+1\Big\}; 1\Big), & \text{for } \alpha > 0, \\ F\Big(k_1\Big(\dfrac{m}{2}+1\Big), m_1, \Big\{k_1\Big(\dfrac{m}{2}-1\Big)+m_1+1\Big\}; -1\Big), & \text{for } \alpha < 0. \end{cases} \tag{3.14}$$

Using Theorem 3.1 and Corollary 3.4, we have the following covering result.

Corollary 3.6. *Let $f \in k-UB_m(\alpha,\beta,\gamma)$, $\alpha > 0$, $\beta \in [1,0)$, and $m \le 2\{\alpha(1+2\beta)(k+1)/(1-\beta)(1-\gamma)+1\}$. If D is the boundary of the image of E under f, then every point of D is at distance at least*

$$\left[\frac{k_1(m/2-1)+m_1+1}{2[k_1(m/2-1)+m_1+1]+k_1(m/2+1)}\right] \tag{3.15}$$

from the origin.

Proof. Let $f(z) \ne w_0$, $w_0 \ne 0$. Then f_1 given by $f_1(z) = w_0 f(z)/(w_0 - f(z))$ is univalent, since f is univalent by Theorem 3.1. Writing $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, we have

$$f_1(z) = \frac{w_0 f(z)}{w_0 - f(z)} = z + \left(a_2 + \frac{1}{w_0}\right)z^2 + \cdots, \tag{3.16}$$

and since $f_1 \in S$, it follows that

$$\left|a_2 + \frac{1}{w_0}\right| \le 2. \tag{3.17}$$

Now using Corollary 3.4,

$$\left|\frac{1}{w_0}\right| \le 2 + \frac{k_1((m/2)+1)}{\{k_1((m/2)-1)+m_1+1\}}, \tag{3.18}$$

and this proves the result. □

As a special case, for $\alpha = 1,\ \beta = 0,\ \gamma = 0,\ k = 0$, and $m = 2,\ f$ is convex, and in this case, $|w_0| > 1/4$.

We now proceed to investigate some inclusion properties.

Theorem 3.7. *Let* $f \in k - UB_m(\alpha,\beta,\gamma),\ \alpha > 0,\ \beta \in [1,0)$. *Then* $f \in k - UR_m(\gamma)$ *for* $0 \le \gamma < \{(1-\beta+\alpha\beta)k/\alpha(k+1)\},\ z \in E$. *In particular, for* $\alpha > 0,\ \beta \in [1,0)$, *one has*

$$k - UB_m(\alpha,\beta) \subset k - UR_m. \tag{3.19}$$

Proof. Since $f \in k - UB_m(\alpha,\beta,\gamma)$, we can write from (1.7)

$$(1-\alpha)(1-\beta)\frac{zf'(z)}{f(z)} + \alpha\left(1-\beta+\frac{zf''(z)}{f'(z)}\right) = (1-\beta)\frac{zg'(z)}{g(z)}, \tag{3.20}$$

or

$$(1-\alpha)\frac{zf'(z)}{f(z)} + \frac{\alpha}{1-\beta}\left(1-\beta+\frac{zf''(z)}{f'(z)}\right) = \frac{zg'(z)}{g(z)} \in P_m(p_{k,\gamma}). \tag{3.21}$$

Let $zf'(z)/f(z) = p(z)$. Then $p(z)$ is analytic in E with $p(0) = 1$. Then we can write (3.21) as

$$\frac{(1-\beta+\alpha\beta)}{1-\beta}\left\{p(z) + \frac{\alpha}{1-\beta+\alpha\beta}\frac{zp'(z)}{p(z)} - \frac{\alpha\beta}{1-\beta+\alpha\beta}\right\} = h_0(z) = \frac{zg'(z)}{g(z)}, \tag{3.22}$$

or

$$\left\{p(z) + \frac{(1/m_1)zp'(z)}{p(z)}\right\}\frac{(1-\beta)}{\alpha m_1}h_0(z) + \frac{\beta}{m_1}, \tag{3.23}$$

where $m_1 = 1 + (1-\alpha)(1-\beta)/\alpha$.

Since $h_0 \in P_m(p_{k,\gamma})$ and $P_m(p_{k,\gamma})$ is a convex set, see [4], it follows that $\{((1-\beta)/\alpha m_1)h_0(z) + (\beta/m_1)p_0(z)\}$, with $p_0(z) = 1$, belong to $P_m(p_{k,\gamma})$ in E.

Define

$$\phi_{m_1}(z) = \frac{z}{(1-z)^{1/m_1}}\left\{\frac{(1-(z/2))}{(1-z)^2}\right\}. \tag{3.24}$$

Writing

$$p(z) = \left(\frac{m}{4}+\frac{1}{2}\right)p_1(z) - \left(\frac{m}{4}-\frac{1}{2}\right)p_2(z), \tag{3.25}$$

we use convolution technique, see [4], to have

$$\left(p(z) * \frac{\phi_{m_1}(z)}{z}\right) = p(z) + \frac{(1/m_1)zp'(z)}{p(z)}, \tag{3.26}$$

where $*$ denotes convolution (Hadamard product).

Therefore, from (3.23), (3.25), and (3.26), we have for $i = 1, 2$

$$\left\{ p_i(z) + \frac{zq_i'(z)}{m_1 p(z)} \right\} \in P(p_{k,\gamma}). \tag{3.27}$$

We now use Lemma 2.1 with $\sigma = m_1$, $\delta_1 = 0$, $\operatorname{Re}(m_1 k/(k+1)) > \gamma$ to obtain $p_i(z) \prec p_{k,\gamma}(z)$, $i = 1, 2$, and consequently $p \in P_m(p_{k,\gamma})$ in E. This proves that $f \in k - UR_m(\gamma)$ in E. □

For $k = 0$, we can improve this result by restricting α, β suitably as follows.

Corollary 3.8. *Let $f \in B_m(\alpha, \beta, \gamma), \alpha \in (0, 1]$, $\beta \in [0, 1)$. Then, $f \in R_m(\gamma_0)$, where*

$$\gamma_0 = \left\{ \frac{2}{(2m_1\gamma - 1) + \sqrt{(2m_1\gamma - 1)^2 + 8m_1}} \right\}. \tag{3.28}$$

Proof. Writing $p_i(z) = (1 - \gamma_0)h_i(z) + \gamma_0$ and proceeding as in Theorem 3.7, it follows from (3.25) and (3.27) that for $i = 1, 2$

$$\left\{ (1 - \gamma_0)h_i(z) + \frac{(1 - \gamma_0)zh_i'(z)}{m_1(1 - \gamma_0)h_i(z) + m_1\gamma_0} + (\gamma_0 - \gamma) \right\} \in P, \quad z \in E. \tag{3.29}$$

Constructing the functional $\psi(u, v)$ with $u = h_i(z), v = zh_i'(z)$, we note that the first two conditions of Lemma 2.2 are easily verified. For condition (iii), we proceed as follows:

$$\begin{aligned} \operatorname{Re}\psi(iu_2, v_1) &= (\gamma_0 - \gamma) + \frac{m_1\gamma_0(1 - \gamma_0)v_1}{m_1^2\gamma_0^2 + m_1^2(1 - \gamma_0)^2 u_2^2}, \quad \left(v_1 \leq -\frac{(1 + u_2^2)}{2} \right), \\ &\leq \frac{2(\gamma_0 - \gamma)\left[m_1^2\gamma_0^2 + m_1^2(1 - \gamma_0)^2 u_2^2\right] - m_1\gamma_0(1 - \gamma_0)(1 + u_2^2)}{2\left[m_1^2\gamma_0^2 + m_1^2(1 - \gamma_0)^2 u_2^2\right]}, \\ &= \frac{A + Bu_2^2}{2C}, \end{aligned} \tag{3.30}$$

where $A = 2(\gamma_0 - \gamma)(m_1^2\gamma_0^2) + m_1\gamma_0(1 - \gamma_0)$, $B = 2(\gamma_0 - \gamma)m_1^2(1 - \gamma_0)^2 - m_1\gamma_0(1 - \gamma_0), C = [m_1^2\gamma_0^2 + m_1^2(1 - \gamma_0)^2 u_2^2] > 0$.

The right-hand side of (3.30) is less than or equal to zero if $A \leq 0$ and $B \leq 0$, and condition (iii) is satisfied. Form $A \leq 0$, we obtain γ_0 as given by (3.28), and B ≤ 0 ensures that $\gamma_0 \in [0, 1)$.

We now use Lemma 2.2 to have $h_i \in P$, $i = 1, 2$, and this implies $p_i \in P(\gamma_0)$, $i = 1, 2$. Consequently $p \in P_m(\gamma_0)$ in E, and this completes the proof. □

We note that $k - UB_m(\alpha, \beta, \gamma) \subset B_m(\alpha, \beta, \gamma_1)$ with $\gamma_1 = (k + \gamma)/(1 + k)$. Thus the above result can be restated in a general form as follows.

Corollary 3.9. *Let $f \in k - UB_m(\alpha,\beta,\gamma)$, $\alpha \in (0,1]$, $\beta \in [0,1)$. Then $f \in R_m(\gamma_2)$ in E, where*

$$\gamma_2 = \left\{ \frac{2}{(2m\gamma_1 - 1) + \sqrt{(2m\gamma_1 - 1)^2 + 8m_1}} \right\}. \tag{3.31}$$

By taking $\alpha = 1$, $\beta = 0$, $k = 1$, we have the following.

Corollary 3.10. *Let $f \in 1 - UB_m(1,0,\gamma)$. Then $f \in R_m(\sigma_1)$, where $0 \le \sigma_1 < 1$ with*

$$\sigma_1 = \left[\int_0^1 \exp\left\{ \frac{2(1-\gamma)}{\pi^2} \int_t^1 \frac{1}{x} \left(\log \frac{1+\sqrt{x}}{1-\sqrt{x}} \right)^2 dx \right\} dt \right]^{-1}. \tag{3.32}$$

When $\gamma = 0$, one has $\sigma_1 = 0.73719\cdots$, and this value of σ_1 is sharp, see [14].

Proof. In Theorem 3.7, with $k = 1$, $\alpha = 1$, $\beta = 0$, we have $m_1 = 1$ and therefore, from (3.27), it follows that

$$\left(p_i(z) + \frac{z p_i'(z)}{p_i(z)} \right) \prec p_{1,\gamma}(z), \quad i = 1,2. \tag{3.33}$$

By taking $p_i(z) = z f_i'(z)/f_i(z)$, it implies that

$$\operatorname{Re}\left(1 + \frac{z f_i''(z)}{f_i'(z)} \right) > \left| \frac{z f_i''(z)}{f_i'(z)} \right| + \gamma, \quad z \in E, \ i = 1,2. \tag{3.34}$$

That is, $f_i \in UCV(\gamma)$, and it has been proved in [14] that every function in the class $UCV(\gamma)$ is starlike of order σ_1 where this order is exact and is given by (3.32).

Now, using the argument given in Theorem 3.7, we have $p(z) = (zf(z)/f(z)) \in P_m(\sigma_1), z \in E$, and the proof is complete. □

We now prove the following.

Theorem 3.11. *Let $0 < \alpha_2 < \alpha_1 \le 1$, $\beta \in [0,1)$. Then*

$$k - UB_m(\alpha_1,\beta,\gamma) \subset k - UB_m(\alpha_2,\beta,\gamma), \quad z \in E. \tag{3.35}$$

Proof. Let $f \in k - UB_m(\alpha_1,\beta,\gamma)$. Then, for $z \in E$

$$(1-\alpha_1)\frac{zf'(z)}{f(z)} + \frac{\alpha_1}{1-\beta}\left(1 - \beta + \frac{zf''(z)}{f'(z)} \right) = H_1(z), \quad H_1 \in P_m(p_{k,\gamma}). \tag{3.36}$$

Also, by Theorem 3.7, we have

$$\frac{zf'(z)}{f(z)} = H_2(z), \quad H_2 \in P_m(p_{k,\gamma}). \tag{3.37}$$

Now

$$(1-\alpha_2)\frac{zf'(z)}{f(z)} + \frac{\alpha_2}{1-\beta}\left(1-\beta+\frac{zf''(z)}{f'(z)}\right) = \frac{\alpha_2}{\alpha_1}H_1(z) + \left(1-\frac{\alpha_2}{\alpha_1}\right)H_2(z) = H(z), \tag{3.38}$$

and since $P_m(p_{k,\gamma})$ is convex set, $H \in P_m(p_{k,\gamma})$ in E. This proves the result. □

Let $M(r) = \max_{0\le\theta\le 2\pi}|f(re^{i\theta})|$, $L(r)$ the length of the curve C, $C = f(re^{i\theta})$, $0 \le \theta \le 2\pi$, and $A(r)$ the area of the region bounded by C.

We will now study the arc length problem for the class $B_m(\alpha,\beta,\gamma)$ as follows.

Theorem 3.12. *For $k = 0$, let $f \in B_m(\alpha,\beta,\gamma)$. Then, for $0 < r < 1$,*

(i) $L(r) \le (\pi M(r)/\alpha(1-\beta))[m\{(1-\gamma)+(\alpha-1)(1-\gamma_0)\}+2\alpha\beta]$, $\alpha \ge 2$, *where γ_0 is given by* (3.28).

(ii) $L(r) \le (\pi M(r)/\alpha)[m(2-\alpha)+2\alpha\beta/(1-\beta)]$, $0 < \alpha < 2$.

Proof. We prove (i), and proof of (ii) follows on similar lines.

Solving (1.7) for f', we obtain a formal representation as

$$zf'(z) = [g(z)]^{(1-\beta)/\alpha} z^{\beta} [f(z)]^{-(1-\alpha)(1-\beta)/\alpha}, \quad (\alpha \ne 0), \tag{3.39}$$

with $z = re^{i\theta}$.

$$L(r) = \int_0^{2\pi} |zf'(z)| d\theta = \int_0^{2\pi} zf'(z)e^{-i\arg(zf'(z))} d\theta. \tag{3.40}$$

Integration by parts gives us

$$\begin{aligned} L(r) &= \int_0^{2\pi} f(z)e^{-i\arg(zf'(z))}\,\mathrm{Re}\left\{\frac{(zf'(z))'}{f'(z)}\right\}d\theta \\ &\le \frac{M(r)}{\alpha}\int_0^{2\pi}\left|\mathrm{Re}\,J(\alpha,\beta,f(z)) + (\alpha-1)\frac{zf'(z)}{f(z)} + \frac{\alpha\beta}{1-\beta}\right|d\theta, \end{aligned} \tag{3.41}$$

where we have used Logarithmic differentiation of (3.39).

Now, from Corollary 3.8, it follows that $zf'/f \in P_m(\gamma_0)$, where γ_0 is given by (3.28). Also, since $f \in B_m(\alpha,\beta,\gamma)$, we have

$$\int_0^{2\pi} |\mathrm{Re}\,J(\alpha,\beta,f(z))| d\theta \le m(1-\gamma)\pi. \tag{3.42}$$

Using these observations in (3.41), we prove part (i). □

Remark 3.13. In Theorem 3.12, we have, for $\alpha > 0$,

$$L(r) = O(1)M(r), \tag{3.43}$$

where $O(1)$ is a constant depending only on α, β, m, and γ. This can be improved by using the univalence criterion of $f \in B_m(\alpha, \beta, \gamma)$ as follows.

Corollary 3.14. *Let $f \in B_m(\alpha, \beta, \gamma), \alpha > 0,\ \beta \in [0, 1)$, and $m \le 2\{\alpha(1+2\beta)/(1-\beta)(1-\gamma)+1\}$. Then*

$$L(r) = O(1)\sqrt{A(r)\log\frac{1}{1-r^2}}. \tag{3.44}$$

Proof. In view of the univalence of f by Theorem 3.1, we have $M(r) \le (4/r)M(r^2)$, see [9], and

$$M\left(r^2\right) = \sum_{n=1}^{\infty}|a_n|r^{2n} = \sum_{n=1}^{\infty}\left(n^{1/2}|a_n|r^n\right)\left(n^{-1/2}r^n\right). \tag{3.45}$$

Making now use of the area theorem and the Schwarz inequality, we obtain the required result. □

Corollary 3.15. *Let $f : f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ satisfy the condition of Theorem 3.12. Then*

$$a_n = O(1)\frac{M((n-1)/n)}{n}, \quad (n \longrightarrow \infty). \tag{3.46}$$

Proof is immediately since by Cauchy's Theorem.
We can write

$$n|a_n| = \frac{1}{2\pi r^n}\left|\int_0^{2\pi} z f'(z)e^{-in\theta}d\theta\right| = \frac{1}{2\pi r^n}L(r). \tag{3.47}$$

Now taking $r = (1-1/n)$ in Theorem 3.12, we prove the result.

We study the arc length problem and corresponding rate of growth of coefficient for the class $k - UB_m(\alpha, \beta, \gamma)$.

Theorem 3.16. *Let, for $\alpha \ge 1,\ -1/2 \le \beta < 1, f \in k - UB_m(\alpha, \beta, \gamma)$. Then for $m > 2(1-k_1)/k_1$, one has*

$$L(r) = O(1)M(r)^{1-m_1}\left(\frac{1}{1-r}\right)^{\{k_1(m/2+1)-1\}}, \quad (r \longrightarrow 1), \tag{3.48}$$

where $O(1)$ depends only on m_1, k_1, and m and m_1, k_1 are as given by (3.4).

Proof. From (1.7), we can write

$$(zf'(z))(f(z))^{(1-\alpha)(1-\beta)/\alpha} = (g(z))^{(1-\beta)/\alpha}, \quad g \in k - \bigcup R_m(\gamma). \tag{3.49}$$

Now, for $z = re^{i\theta}$, $\alpha \geq 1$,

$$\begin{aligned} L(r) &= \int_{|z|=r} |f'(z)|\, |dz| = \int_0^{2\pi} zf'(z)e^{-i\arg zf'(z)}d\theta \\ &\leq \int_0^{2\pi} \left|(g(z))^{(1-\beta)/\alpha} z^{\beta}(f(z))^{-(1-\alpha)(1-\beta)/\alpha} e^{-i\arg zf'}\right| d\theta. \end{aligned} \tag{3.50}$$

Since $g \in k - UR_m(\gamma)$, we use a result proved in [4] and write

$$g(z) = \frac{(g_1(z))^{m/4+1/2}}{(g_2(z))^{m/4-1/2}}, \quad g_1, g_2 \in k - UR_2(\gamma). \tag{3.51}$$

Since $k - UR_2(\gamma) \subset S^*((k+\gamma)/(1+k))$, see [4].

We can write

$$\begin{aligned} g_i(z) &= z\left(\frac{s_i(z)}{z}\right)^{1-((k+\gamma)/(1+k))}, \quad s_i \in S^*, \\ &= z\left(\frac{s_i(z)}{z}\right)^{(1-\gamma)/(1+k)}, \quad i = 1, 2. \end{aligned} \tag{3.52}$$

Using (3.51), (3.52), and distortion results for the class S^* of starlike functions, we obtain from (3.50)

$$\begin{aligned} L(r) &\leq C_1(M(r))^{1-m_1} \int_0^{2\pi} \left(\frac{r}{|1-re^{i\theta}|^2}\right)^{k_1(m/4+1/2)} d\theta, \\ &= O(1)(M(r))^{1-m_1}\left(\frac{1}{1-r}\right)^{k_1(m/4+1/2)-1}, \quad (r \longrightarrow 1), \end{aligned} \tag{3.53}$$

where $k_1(m/2+1) > 1$. □

Remark 3.17.

(i) For the case $0 < \alpha < 1$, we can solve the arc length problem in a similar manner. We define $\tilde{m}(r) = \min_{0\leq\theta\leq 2\pi}|f(z)|$ and proceed as before to obtain

$$L(r) = O(1)(\tilde{m}(r))^{1-m_1}\left(\frac{1}{1-r}\right)^{k_1(m/2+1)-1}, \quad (r \longrightarrow 1). \tag{3.54}$$

(ii) For $\alpha < 0$, we have

$$L(r) = O(1)(M(r))^{1-m_1}\left(\frac{1}{1-r}\right)^{k_1(m/2-1)-1}, \quad \text{for } m > \left(\frac{1}{k_1}+1\right). \tag{3.55}$$

We can derive the following result of order of growth of coefficients from Theorem 3.16 with the same method used before.

Corollary 3.18. *Let* $f : f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in k - UB_m(\alpha,\beta,\gamma), \beta \in [0,1), \gamma \in [0,1), \alpha \geq 1$. *Then*

$$a_n = O(1)M^{(1-m_1)}n^{\{k_1(m/2+1)-2\}}, \quad (n \longrightarrow \infty), \tag{3.56}$$

where $O(1)$ *depends only on* m, m_1, k_1 *and* γ.

The exponent $\{k_1(m/2+1)-2\}$ *is best possible for* $k = 0$, $\gamma = 0$. *The extremal function satisfies the equation*

$$\left(\frac{f(z)}{z}\right)^{1-\alpha}(f'(z))^{\alpha/(1-\beta)} = \left(\frac{g_0(z)}{z}\right), \tag{3.57}$$

with

$$g_0(z) = \frac{1}{(m+2)}\left\{\left(\frac{1+z}{1-z}\right)^{m/2+1} - 1\right\} \in R_m. \tag{3.58}$$

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