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Research Article

A System of Generalized Mixed Equilibrium Problems, Maximal Monotone Operators, and Fixed Point Problems with Application to Optimization Problems

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We introduce a new iterative algorithm for finding a common element of the set of solutions of a system of generalized mixed equilibrium problems, zero set of the sum of a maximal monotone operators and inverse-strongly monotone mappings, and the set of common fixed points of an infinite family of nonexpansive mappings with infinite real number. Furthermore, we prove under some mild conditions that the proposed iterative algorithm converges strongly to a common element of the above four sets, which is a solution of the optimization problem related to a strongly positive bounded linear operator. The results presented in the paper improve and extend the recent ones announced by many others.

1. Introduction

Throughout this paper, we denoted by $\mathbb{N}$ and $\mathbb{R}^+$ the set of all positive integers and all positive real numbers, respectively. We always assume that H is a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, respectively, and C is a nonempty closed convex subset of H. Let $\varphi : C \to \mathbb{R}$ be a real-valued function, $\Theta : C \times C \to \mathbb{R}$ be an equilibrium bifunction, and $\Psi, \Phi : C \to H$ be two nonlinear mappings. The *generalized mixed equilibrium problem with perturbed mapping* is to find $x^* \in C$ such that

$$\Theta(x^*, y) + \varphi(y) - \varphi(x^*) + \langle (\Psi + \Phi)x^*, y - x^* \rangle \geq 0, \quad \forall y \in C. \tag{1.1}$$

The set of solutions of the problem (1.1) is denoted by $\mathrm{GMEP}(\Theta, \varphi, \Psi + \Phi)$. As special cases of the problem (1.1), we have the following.

(1) If $\Phi = 0$, then the problem (1.1) reduces to the *generalized mixed equilibrium problem* of finding $x^* \in C$ such that

$$\Theta(x^*, y) + \varphi(y) - \varphi(x^*) + \langle \Psi x^*, y - x^* \rangle \geq 0, \quad \forall y \in C, \tag{1.2}$$

which was introduced and studied by Peng and Yao [1]. The set of solutions of the problem (1.2) is denoted by $\mathrm{GMEP}(\Theta, \varphi, \Psi)$.

(2) If $\Psi = \Phi = 0$, then the problem (1.1) reduces to the *mixed equilibrium problem* of finding $x^* \in C$ such that

$$\Theta(x^*, y) + \varphi(y) - \varphi(x^*) \geq 0, \quad \forall y \in C, \tag{1.3}$$

which was consider by Ceng and Yao [2]. The set of solutions of the problem (1.3) is denoted by $\mathrm{MEP}(\Theta)$.

(3) If $\Phi = \varphi = 0$, then the problem (1.1) reduces to the *generalized equilibrium problem* of finding $x^* \in C$ such that

$$\Theta(x^*, y) + \langle \Psi x^*, y - x^* \rangle \geq 0, \quad \forall y \in C, \tag{1.4}$$

which was consider by S. Takahashi and W. Takahashi [3]. The set of solutions of the problem (1.4) is denoted by $\mathrm{GEP}(\Theta, \Psi)$.

(4) If $\Psi = \Phi = \varphi = 0$, then the problem (1.1) reduces to the *equilibrium problem* of finding $x^* \in C$ such that

$$\Theta(x^*, y) \geq 0, \quad \forall y \in C. \tag{1.5}$$

The set of solutions of the problem (1.5) is denoted by $\mathrm{EP}(\Theta)$.

(5) If $\Theta = \Phi = \varphi = 0$, then the problem (1.1) reduces to the *classical variational inequality problem* of finding $x^* \in C$ such that

$$\langle \Psi x^*, y - x^* \rangle \geq 0, \quad \forall y \in C. \tag{1.6}$$

The set of solutions of the problem (1.6) is denoted by $\mathrm{VI}(C, \Psi)$. It is known that $x^* \in C$ is a solution of the problem (1.6) if and only if x^* is a fixed point of the mapping $P_C(I - \lambda\Psi)$, where $\lambda > 0$ is a constant and I is the identity mapping.

The generalized mixed equilibrium problems with perturbation is very general in the sense that it includes fixed point problems, optimization problems, variational inequality problems, Nash equilibrium problems, and equilibrium problems as special cases (see, e.g., [4, 5]). Numerous problems in physics, optimization, and economics reduce to find a solution of problem (1.2). Several methods have been proposed to solve the fixed point problems, variational inequality problems, and equilibrium problems in the literature (see, e.g., [6–34]).

Let A be a strongly positive bounded linear operator on H; that is, there exists a constant $\overline{\gamma} > 0$ such that

$$\langle Ax, x\rangle \geq \overline{\gamma}\|x\|^2, \quad \forall x \in H. \tag{1.7}$$

Recall that a mapping $f : C \to C$ is said to be *contractive* if there exists a constant $\alpha \in (0,1)$ such that

$$\|f(x) - f(y)\| \leq \alpha\|x - y\|, \quad \forall x, y \in C. \tag{1.8}$$

A mapping $T : C \to C$ is said to be

(1) *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C, \tag{1.9}$$

(2) *firmly nonexpansive* if

$$\|Tx - Ty\|^2 \leq \langle Tx - Ty, x - y\rangle, \quad \forall x, y \in C, \tag{1.10}$$

(3) k-strictly pseudocontractive if there exists a constant $k \in [0,1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in C. \tag{1.11}$$

We denote by $F(T)$ the set of fixed points of T, that is, $F(T) = \{x \in C : x = Tx\}$.

Recall the following definitions of a nonlinear mapping $B : C \to H$; the following is mentioned.

Definition 1.1. The nonlinear mapping $B : C \to H$ is said to be

(i) *monotone* if

$$\langle Bx - By, x - y\rangle \geq 0, \quad \forall x, y \in C, \tag{1.12}$$

(ii) *β-strongly monotone* if there exists a constant $\beta > 0$ such that

$$\langle Bx - By, x - y\rangle \geq \beta\|x - y\|^2, \quad \forall x, y \in C, \tag{1.13}$$

(iii) ν*-inverse-strongly monotone* if there exists a constant $\nu > 0$ such that

$$\langle Bx - By, x - y\rangle \geq \nu \|Bx - By\|^2, \quad \forall x, y \in C. \tag{1.14}$$

Let $W : H \to 2^H$ be a set-valued mapping. The set $D(W)$ defined by $D(W) = \{x \in H : Wx \neq \emptyset\}$ is said to be the domain of W. The set $R(W)$ defined by $R(W) = \bigcup_{x\in H} Wx$ is said to be the range of W. The set $G(W)$ defined by $G(W) = \{(x, y) \in H \times H : x \in D(W), y \in R(W)\}$ is said to be the graph of W.

Recall that W is said to be monotone if

$$\langle x - y, f - g\rangle > 0, \quad \forall (x, f), (y, g) \in G(W). \tag{1.15}$$

W is said to be *maximal monotone* if it is not properly contained in any other monotone operator. Equivalently, W is maximal monotone if $R(I + rW) = H$ for all $r > 0$. For a maximal monotone operator M on H and $r > 0$, we may define the single-valued resolvent $J_r = (I + rW)^{-1} : H \to D(W)$. It is known that J_r is firmly nonexpansive $W^{-1}(0) = F(J_r)$, where $F(J_r)$ denotes the fixed point set of J_r.

We discuss the following algorithms for solving the solutions of variational inequality problems and fixed point problems for a nonexpansive mapping (see, e.g., [29, 35–43]).

In 2010, Chantarangsi et al. [44] introduced a new viscosity hybrid steepest descent method for solving the generalized mixed equilibrium problems (1.2), variational inequality problems, and fixed point problems of nonexpansive mappings in a real Hilbert space. More precisely, they proved the following theorem.

Theorem CCK [see [44]]

Let C be a nonempty closed and convex subset of a real Hilbert space H. Let Θ_1, Θ_2 be two bifunctions satisfying condition (H1)–(H5), let Ψ_1, Ψ_2 be ξ-inverse-strongly monotone mapping and β-inverse-strongly monotone mapping, respectively, and let $T : C \to C$ be a nonexpansive mapping. Let B be an ω-Lipschitz continuous and relaxed (v, ν) cocoercive mapping, $f : C \to C$ a contraction mapping with coefficient $\alpha \in (0, 1)$, and A a strongly positive linear bounded self-adjoint operator with coefficient $\overline{\gamma} > 0$ and $0 < \gamma < \overline{\gamma}/\alpha$. Suppose that $\Omega := F(T) \cap \mathrm{GMEP}(\Theta_1, \varphi, \Psi_1) \cap \mathrm{GMEP}(\Theta_2, \varphi, \Psi_2) \cap \mathrm{VI}(C, B)$. Let $\{z_n\}$, $\{u_n\}$, $\{y_n\}$, and $\{x_n\}$ be generated by

$$\begin{aligned} u_n &= V_{r_n}^{(\Theta_2,\varphi_2)}(x_n - r_n\Psi_2 x_n), \\ v_n &= V_{\mu_n}^{(\Theta_1,\varphi_1)}(u_n - \mu_n\Psi_1 u_n), \\ z_n &= P_C(v_n - \alpha_n BT v_n), \\ y_n &= \epsilon_n\gamma f(x_n) + \beta_n x_n + \left((1 - \beta_n)I - \epsilon_n A\right)z_n, \\ x_{n+1} &= \gamma_n x_n + (1 - \gamma_n)y_n, \quad \forall n \geq 1, \end{aligned} \tag{1.16}$$

where $\{\gamma_n\} \subset [a, b] \subset [0, 2\xi]$, $\{s_n\} \subset [c, d] \subset [0, 2\beta]$, $\{\gamma_n\} \subset [h, j] \subset (0, 1)$, $\{\gamma_n\}$, $\{\epsilon_n\}$, and $\{\beta_n\}$ are three sequences in $(0, 1)$ satisfying the following conditions:

(C1) $\lim_{n\to\infty}\epsilon_n = 0$ and $\sum_{n=1}^{\infty}\epsilon_n = \infty$,

(C2) $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$,

(C3) $0 < \liminf_{n\to\infty}\mu_n \le \limsup_{n\to\infty}\mu_n < 2\beta$ and $\lim_{n\to\infty}|\mu_{n+1} - \mu_n| = 0$,

(C4) $0 < \liminf_{n\to\infty} r_n \le \limsup_{n\to\infty} r_n < 2\xi$ and $\lim_{n\to\infty}|r_{n+1} - r_n| = 0$,

(C5) $\{\alpha_n\} \subset [e,g] \subset (0, 2(\nu - v\omega^2)/\omega^2)$, $\nu > v\omega^2$ and $\lim_{n\to\infty}|\alpha_{n+1} - \alpha_n| = 0$.

Then, $\{x_n\}$ converges strongly to $x^* = P_\Omega(\gamma f + (I - A))(x^*)$.

Very recently, Yu and Liang [45] proved the following convergence theorem of finding a common element in the fixed point set of a strict pseudocontraction and in the zero set of a nonlinear mapping which is the sum of a maximal monotone operator and inverse strongly monotone mapping in a real Hilbert space.

Theorem YL [see [45]]

Let H be a real Hilbert space and C a nonempty close and convex subset of H. Let $W_1 : H \to 2^H$ and $W_2 : H \to 2^H$ be two maximal monotone operators such that $D(W_1) \subset C$ and $D(W_2) \subset C$, respectively. Let $S : C \to C$ be a k-strict pseudocontraction mapping, $A : C \to H$ an α-inverse-strongly monotone mapping, and $B : C \to H$ an β-inverse-strongly monotone mapping. Assume that $\Omega := F(S) \cap (A + W_1)^{-1}(0) \cap (B + W_2)^{-1}(0) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by

$$\begin{gathered} x_1 \in C, \\ y_n = J_{t_n}(x_n - t_n B x_n), \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n \left[\delta_n J_{s_n}(y_n - s_n A y_n) + (1 - \delta_n) J_{s_n}(y_n - s_n A y_n)\right], \quad \forall n \ge 1, \end{gathered} \tag{1.17}$$

where $u \in C$ is a fixed element, $J_{s_n} = (I + s_n W_1)^{-1}$, $J_{t_n} = (I + t_n W_2)^{-1}$, $\{s_n\}$ is a sequence in $(0, 2\alpha)$, $\{t_n\}$ is a sequence in $(0, 2\beta)$, and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\delta_n\}$ are sequences in $[0,1]$ satisfying the following conditions:

(C1) $\lim_{n\to\infty}\alpha_n = 0$ and $\sum_{n=1}^{\infty}\alpha_n = \infty$,

(C2) $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$,

(C3) $0 < a \le s_n \le b < 2\alpha$ and $\lim_{n\to\infty}(s_{n+1} - s_n) = 0$,

(C4) $0 < c \le t_n \le d < 2\beta$ and $\lim_{n\to\infty}(t_{n+1} - t_n) = 0$,

(C5) $0 < c \le k \le \delta_n < e < 1$ and $\lim_{n\to\infty}(\delta_{n+1} - \delta_n) = 0$.

Then, the sequence $\{x_n\}$ converges strongly to $x^* = P_\Omega x^*$.

On the other hand, the following optimization problem has been studied extensively by many authors:

$$\min_{x\in\overline{\Omega}} \frac{\mu}{2}\langle Ax, x\rangle + \frac{1}{2}\|x - u\|^2 - h(x), \tag{1.18}$$

where $\overline{\Omega} = \bigcap_{n=1}^{\infty} C_n, C_1, C_2, \ldots$ are infinitely many closed convex subsets of H such that $\bigcap_{n=1}^{\infty} C_n \neq \emptyset$, $u \in H$, $\mu \ge 0$ is a real number, A is a strongly positive linear bounded operator on H, and h is a potential function for γf (i.e., $h'(x) = \gamma f(x)$ for all $x \in H$). This kind of optimization problem has been studied extensively by many authors (see, e.g., [5, 46–52]) for when $\overline{\Omega} = \bigcap_{n=1}^{\infty} C_n$ and $h(x) = \langle x, b\rangle$, where b is a given point in H.

The following questions naturally arise in connection with above the results.

Question 1. Could we weaken the control conditions of Theorems CCK and YL in (C3) and (C4)?

Question 2. Can Theorem YL be extended to finding a common element of the set of solutions of a system generalized mixed equilibrium problems and the set of common fixed points of infinite family of nonexpansive mappings?

The purpose of this paper is to give the affirmative answers to these questions mentioned above. Motivated by the iterative process (1.16) and (1.17), we introduce a new iterative algorithm (3.2) below, for finding a common element of the set of solutions of a system of generalized mixed equilibrium problems, zero set of the sum of a maximal monotone operators and inverse-strongly monotone mappings, and the set of common fixed points of an infinite family of nonexpansive mappings with infinite real number. Then, we prove the strong convergence theorem of these iterative process in a real Hilbert space. The results presented in the paper improve and extend the recent ones announced by many others.

2. Preliminaries

Definition 2.1 (see [53]). Let C be a nonempty convex subset of a real Hilbert space H. Let T_i, $i = 1, 2, \ldots$, be mappings of C into itself. For each $j = 1, 2, \ldots$, let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, where $I = [0, 1]$ and $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$. For every $n \in \mathbb{N}$, we define the mapping $S_n : C \to C$ as follows:

$$
\begin{aligned}
U_{n,n+1} &= I, \\
U_{n,n} &= \alpha_1^n T_n U_{n,n+1} + \alpha_2^n U_{n,n+1} + \alpha_3^n I, \\
U_{n,n-1} &= \alpha_1^{n-1} T_{n-1} U_{n,n} + \alpha_2^{n-1} U_{n,n} + \alpha_3^{n-1} I, \\
&\vdots \\
U_{n,k+1} &= \alpha_1^{k+1} T_{k+1} U_{n,k+2} + \alpha_2^{k+1} U_{n,k+2} + \alpha_3^{k+1} I, \\
U_{n,k} &= \alpha_1^{k} T_{k} U_{n,k+1} + \alpha_2^{k} U_{n,k+1} + \alpha_3^{k} I, \\
&\vdots \\
U_{n,2} &= \alpha_1^{2} T_{2} U_{n,3} + \alpha_2^{2} U_{n,3} + \alpha_3^{2} I, \\
S_n = U_{n,1} &= \alpha_1^{1} T_{1} U_{n,2} + \alpha_2^{1} U_{n,2} + \alpha_3^{1} I.
\end{aligned}
\tag{2.1}
$$

Such a mapping S_n is nonexpansive from C into itself, and it is called *S-mapping* generated by $T_n, T_{n-1}, \ldots, T_1$ and $\alpha_n, \alpha_{n-1}, \ldots, \alpha_1$.

Lemma 2.2 (see [53]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{T_i\}_{i=1}^{\infty}$ be nonexpansive mappings of C into itself with $F(T_i) \neq \emptyset$ and let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, where $I = [0,1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j + \alpha_2^j \leq b < 1$, and $\alpha_1^j, \alpha_2^j, \alpha_3^j \in (0,1)$ for all $j = 1, 2, \ldots$. For all $n \in \mathbb{N}$, let S_n and S be S-mappings generated by $T_n, T_{n-1}, \ldots, T_1$ and $\alpha_n, \alpha_{n-1}, \ldots, \alpha_1$ and $T_n, T_{n-1}, \ldots$ and $\alpha_n, \alpha_{n-1}, \ldots$, respectively. Then,*

(i) *S_n is nonexpansive and $F(S_n) = \bigcap_{i=1}^{n} F(T_i)$, for all $n \geq 1$,*

(ii) *for all $x \in C$ and for all positive integer k, the $\lim_{n\to\infty} U_{n,k}$ exists,*

(iii) *the mapping $S : C \to C$ defined by*

$$Sx := \lim_{n\to\infty} S_n x = \lim_{n\to\infty} U_{n,1} x, \quad \forall x \in C \tag{2.2}$$

is a nonexpansive mapping such that $F(S) = \bigcap_{i=1}^{\infty} F(T_i)$, and it is called the S-mapping generated by $T_n, T_{n-1}, \ldots$ and $\alpha_n, \alpha_{n-1}, \ldots$,

(iv) *if K is any bounded subset of C, then*

$$\lim_{n\to\infty} \sup_{x\in K} \|S_n x - Sx\| = 0. \tag{2.3}$$

Lemma 2.3 (see [54]). *Let $\{x_n\}$ and $\{l_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$. Suppose that $x_{n+1} = (1-\beta_n)l_n + \beta_n x_n$ for all integers $n \geq 0$ and $\limsup_{n\to\infty}(\|l_{n+1} - l_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then, $\lim_{n\to\infty}\|l_n - x_n\| = 0$.*

Lemma 2.4 (see [55]). *Let H be a real Hilbert space. Then, the following inequalities hold:*

(i) *$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2$, $\forall x, y \in H$ and $\lambda \in [0,1]$,*

(ii) *$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle$, $\forall x, y \in H$.*

Lemma 2.5 (see [56]). *Let C be a nonempty closed convex subset of a real Hilbert space H, $A : C \to H$ a mapping, and $W : H \to 2^H$ a maximal monotone mapping. Then,*

$$F(J_r(I - rA)) = (A + W)^{-1}(0), \quad \forall r > 0. \tag{2.4}$$

Lemma 2.6 (see [57]). *Let H be a real Hilbert space and let M be a maximal monotone operator on H. For $r > 0$ and $x \in H$, define the resolvent $J_r x$. Then, the following holds:*

$$\frac{s-t}{s}\langle J_s x - J_t x, J_s x - x\rangle \geq \|J_s x - J_t x\|^2 \tag{2.5}$$

for all $s, t > 0$ and $x \in H$.

For solving the equilibrium problem for bifunction $\Theta : C \times C \to \mathbb{R}$, let us assume that Θ satisfies the following conditions:

(H1) $\Theta(x,x) = 0$ for all $x \in C$,

(H2) Θ is monotone; that is, $\Theta(x,y) + \Theta(y,x) \leq 0$ for all $x, y \in C$,

(H3) for each $y \in C$, $x \mapsto \Theta(x,y)$ is concave and upper semicontinuous,

(H4) for each $y \in C$, $x \mapsto \Theta(x,y)$ is convex,

(H5) for each $y \in C$, $x \mapsto \Theta(x,y)$ is lower semicontinuous.

Definition 2.7. A differentiable function $K : C \to \mathbb{R}$ on a convex set C is called

(i) *convex* [2] if

$$K(y) - K(x) \geq \langle K'(x), y - x \rangle, \quad \forall x, y \in C, \tag{2.6}$$

where $K'(x)$ is the *Fréchet* differentiable of K at x;

(ii) *strongly convex* [2] if there exists a constant $\sigma > 0$ such that

$$K(y) - K(x) - \langle K'(x), x - y \rangle \geq \left(\frac{\sigma}{2}\right) \|x - y\|^2, \quad \forall x, y \in C. \tag{2.7}$$

It is easy to see that if $K : C \to \mathbb{R}$ is a differentiable strongly convex function with constant $\sigma > 0$, then $K' : C \to H$ is strongly monotone with constant $\sigma > 0$.

Let $\Theta : C \times C \to \mathbb{R}$ be an equilibrium bifunction satisfying the conditions (H1)–(H5). Let r be any given positive number. For a given point $x \in C$, consider the *auxiliary mixed equilibrium problem* to finding $y \in C$ such that

$$\Theta(y,z) + \varphi(z) - \varphi(y) + \frac{1}{r}\langle K'(y) - K'(x), z - y \rangle \geq 0, \quad \forall z \in C, \tag{2.8}$$

where $K'(x)$ is the *Fréchet* differentiable of K at x. Let $V_r^{(\Theta,\varphi)} : C \to C$ be the mapping such that for each $x \in C$, $V_r^{(\Theta,\varphi)}(x)$ is the set of solutions of MEP(x,y), that is,

$$V_r^{(\Theta,\varphi)}(x) = \left\{ y \in C : \Theta(y,z) + \varphi(z) - \varphi(y) + \frac{1}{r}\langle K'(y) - K'(x), z - y \rangle \geq 0, \ \forall z \in C \right\}. \tag{2.9}$$

Then, the following conclusion holds.

Lemma 2.8 (see [58]). *Let C be a nonempty closed convex subset of a real Hilbert space H, and let $\varphi : C \to \mathbb{R}$ be a lower semicontinuous and convex functional. Let $\Theta : C \times C \to \mathbb{R}$ be a bifunction satisfying the conditions (H1)–(H5). Assume that*

(i) *$K : C \to \mathbb{R}$ is strongly convex with constant $\sigma > 0$ and the function $x \mapsto \langle y - x, K'(x) \rangle$ is weakly upper semicontinuous for each $y \in C$;*

(ii) *for each $x \in C$, there exist a bounded subset $D_x \subset C$ and z_x such that for all $y \notin D_x$,*

$$\Theta(y, z_x) + \varphi(z_x) - \varphi(y) + \frac{1}{r}\langle K'(y) - K'(x), z_x - y \rangle < 0. \tag{2.10}$$

Then, the following holds:

(a) $V_r^{(\Theta,\varphi)}$ *is single-valued mapping;*

(b) $V_r^{(\Theta,\varphi)}$ *is nonexpansive if* K' *is Lipschitz continuous with constant* $\nu > 0$ *and*

$$\langle K'(x_1) - K'(x_2), u_1 - u_2\rangle \geq \langle K'(u_1) - K'(u_2), u_1 - u_2\rangle, \quad \forall x_1, x_2 \in C, \tag{2.11}$$

where $u_i = V_r^{(\Theta,\varphi)}(x_i)$ *for* $i = 1, 2$*;*

(c) $F(V_r^{(\Theta,\varphi)}) = \mathrm{MEP}(\Theta, \varphi)$*;*

(d) $\mathrm{MEP}(\Theta, \varphi)$ *is closed and convex.*

In particular, whenever $\Theta : C \times C \to \mathbb{R}$ is a bifunction satisfying the conditions (H1)–(H5) and $K(x) = \|x\|^2/2$, for all $x \in C$, then $V_r^{(\Theta,\varphi)}$ is firmly nonexpansive; that is, for any $x, y \in C$,

$$\left\|V_r^{(\Theta,\varphi)}x - V_r^{(\Theta,\varphi)}y\right\|^2 \leq \left\langle V_r^{(\Theta,\varphi)}x - V_r^{(\Theta,\varphi)}y, x - y\right\rangle. \tag{2.12}$$

In this case, $V_r^{(\Theta,\varphi)}$ is rewritten as $T_r^{(\Theta,\varphi)}$. If, in addition, $\varphi \equiv 0$, then $T_r^{(\Theta,\varphi)}$ is rewritten as T_r^{Θ} (see [59, Lemma 2.1] for more details).

Remark 2.9. We remark that Lemma 2.8 is not a consequence of [2, Lemma 3.1] because the condition of the sequential continuity from the weak topology to the strong topology for the derivative K' of the function $K : C \to \mathbb{R}$ does not cover the case $K(x) = \|x\|^2/2$.

Lemma 2.10. *Let* C, H, Θ, φ*, and* $V_r^{(\Theta,\varphi)}$ *be as in Lemma 2.8. Then, the following holds:*

$$\begin{aligned}
&\left\langle K'\left(V_s^{(\Theta,\varphi)}x\right) - K'\left(V_t^{(\Theta,\varphi)}x\right), V_s^{(\Theta,\varphi)}x - V_t^{(\Theta,\varphi)}x\right\rangle \\
&\quad \leq \frac{s-t}{s}\left\langle K'\left(V_s^{(\Theta,\varphi)}x\right) - K'(x), V_s^{(\Theta,\varphi)}x - V_t^{(\Theta,\varphi)}x\right\rangle,
\end{aligned} \tag{2.13}$$

for all $s, t > 0$ *and* $x \in C$.

Proof. By similar argument as in the proof of Proposition 1 in [58], for all $s, t > 0$ and $x \in C$, let $u = V_s^{(\Theta,\varphi)}x$ and $v = V_t^{(\Theta,\varphi)}x$; we have

$$\Theta(u, y) + \varphi(y) - \varphi(u) + \frac{1}{s}\langle K'(u) - K'(x), y - u\rangle \geq 0, \quad \forall x \in C, \tag{2.14}$$

$$\Theta(v, y) + \varphi(y) - \varphi(v) + \frac{1}{t}\langle K'(v) - K'(x), y - v\rangle \geq 0, \quad \forall x \in C. \tag{2.15}$$

Let $y = v$ in (2.14) and $y = u$ in (2.15); we have

$$\Theta(u,v) + \varphi(v) - \varphi(u) + \frac{1}{s}\langle K'(u) - K'(x), v - u\rangle \geq 0,$$
$$\Theta(v,u) + \varphi(u) - \varphi(v) + \frac{1}{t}\langle K'(v) - K'(x), u - v\rangle \geq 0. \tag{2.16}$$

Adding up the last two inequalities and from the monotonicity of Θ, we obtain that

$$\frac{1}{s}\langle K'(u) - K'(x), v - u\rangle + \frac{1}{t}\langle K'(v) - K'(x), u - v\rangle \geq 0. \tag{2.17}$$

It follows that

$$\left\langle \frac{K'(v) - K'(x)}{t} - \frac{K'(u) - K'(x)}{s}, u - v \right\rangle \geq 0. \tag{2.18}$$

We derive from (2.18) that

$$\begin{aligned} 0 &\leq \left\langle K'(v) - K'(x) - \frac{t}{s}(K'(u) - K'(x)), u - v \right\rangle \\ &= \left\langle K'(v) - K'(u) + K'(u) - K'(x) - \frac{t}{s}(K'(u) - K'(x)), u - v \right\rangle \\ &= \left\langle K'(v) - K'(u) + \left(1 - \frac{t}{s}\right)(K'(u) - K'(x)), u - v \right\rangle. \end{aligned} \tag{2.19}$$

Hence, we obtain that

$$\langle K'(u) - K'(v), u - v\rangle \leq \frac{s-t}{s}\langle K'(u) - K'(x), u - v\rangle. \tag{2.20}$$

□

The following lemma can be found in [60, 61] (see also [62, Lemma 2.2]).

Lemma 2.11. *Let C be a nonempty closed convex subset of a real Hilbert space H and $g : C \to \mathbb{R} \cup \{+\infty\}$ a proper lower semicontinuous differentiable convex function differentiable convex function. If x^* is a solution to the minimization problem*

$$g(x^*) = \inf_{x \in C} g(x), \tag{2.21}$$

then

$$\langle g'(x), x - x^*\rangle \geq 0, \quad x \in C. \tag{2.22}$$

In particular, if x^* solves the optimization problem

$$\min_{x\in\Omega} \frac{\mu}{2}\langle Ax, x\rangle + \frac{1}{2}\|x-u\|^2 - h(x), \tag{2.23}$$

then

$$\langle u + (\gamma f - (I+\mu A))x^*, x - x^*\rangle \le 0, \quad x \in C, \tag{2.24}$$

where h is a potential function for γf.

Lemma 2.12 (see [49]). *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \le (1-\sigma_n)a_n + \delta_n, \tag{2.25}$$

where $\{\sigma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that

(i) $\sum_{n=0}^{\infty}\sigma_n = \infty$,

(ii) $\limsup_{n\to\infty}\delta_n/\sigma_n \le 0$ *or* $\sum_{n=0}^{\infty}|\delta_n| < \infty$.

Then, $\lim_{n\to\infty}a_n = 0$.

3. Main Results

Now, we give our main results in this paper.

Theorem 3.1. *Let C be a nonempty closed convex subset of a Hilbert space H. Let $\varphi_1, \varphi_2 : C \to \mathbb{R}$ be two lower semicontinuous and convex functionals, and let $\Theta_1, \Theta_2 : C\times C \to \mathbb{R}$ be two bifunctions satisfying conditions (H1)–(H5). Let $\Psi_1, \Psi_2, \Phi_1, \Phi_2 : C \to H$ be $\widetilde{\beta}_1$-inverse-strongly monotone mapping, $\widetilde{\beta}_2$-inverse-strongly monotone mapping, $\widetilde{\gamma}_1$-inverse-strongly monotone mapping, and $\widetilde{\gamma}_2$-inverse-strongly monotone mapping, respectively, and let $B_1, B_2 : C \to H$ be $\widetilde{\xi}_1$-inverse-strongly monotone mapping and $\widetilde{\xi}_2$-inverse-strongly monotone mapping, respectively. Let $\{T_i\}_{i=1}^{\infty} : C \to C$ be an infinite family of nonexpansive mappings, and let $\alpha_i = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I\times I\times I$, where $I = [0,1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j + \alpha_2^j \le b < 1$, and $\alpha_1^j, \alpha_2^j, \alpha_3^j \in (0,1)$ for all $j = 1, 2, \ldots$. For all $n \in \mathbb{N}$, let S_n and S be S-mappings generated by $T_n, T_{n-1}, \ldots, T_1$ and $\alpha_n, \alpha_{n-1}, \ldots, \alpha_1$ and $T_n, T_{n-1}, \ldots$ and $\alpha_n, \alpha_{n-1}, \ldots$, respectively. Let $W_1, W_2 : H \to 2^H$ be two maximal monotone operators such that $D(W_1) \subset C$ and $D(W_2) \subset C$, respectively. Assume that*

$$\begin{aligned}\Omega := \bigcap_{i=1}^{\infty} F(T_i) \cap \mathrm{GMEP}(\Theta_1, \varphi_1, \Psi_1 + \Phi_1) \\ \cap\, \mathrm{GMEP}(\Theta_2, \varphi_2, \Psi_2 + \Phi_2) \cap (B_1 + W_1)^{-1}(0) \cap (B_2 + W_2)^{-1}(0) \ne \emptyset.\end{aligned} \tag{3.1}$$

Let $f : C \to H$ be a contraction mapping with a coefficient $\alpha \in (0,1)$, and let $A : C \to H$ be a strongly positive linear bounded operator with a coefficient $\overline{\gamma} \in (0,1)$. Let $\mu > 0$ and $\gamma > 0$ be two constants such that $0 < \gamma < (1+\mu)\overline{\gamma}/\alpha$. Let $\{x_n\}$ be a sequence defined by $u, x_1 \in H$ and

$$\begin{aligned} u_n &= V_{r_n}^{(\Theta_2,\varphi_2)}(x_n - r_n(\Psi_2 + \Phi_2)x_n), \\ v_n &= V_{\mu_n}^{(\Theta_1,\varphi_1)}(u_n - \mu_n(\Psi_1 + \Phi_1)u_n), \\ y_n &= J_{t_n}(v_n - t_n B_2 v_n), \\ x_{n+1} &= \epsilon_n(u + \gamma f(S_n v_n)) + \beta_n x_n + ((1-\beta_n)I - \epsilon_n(I + \mu A))S_n J_{s_n}(y_n - s_n B_1 y_n), \quad \forall n \geq 1, \end{aligned} \tag{3.2}$$

where $J_{s_n} = (I + s_n W_1)^{-1}$, $J_{t_n} = (I + t_n W_2)^{-1}$, $\{s_n\} \subset (0, 2\widetilde{\xi}_1)$, $\{t_n\} \subset (0, 2\widetilde{\xi}_2)$, $\{\mu_n\} \subset (0, \min\{\widetilde{\beta}_1, \widetilde{\gamma}_1\})$, $\{r_n\} \subset (0, \min\{\widetilde{\beta}_2, \widetilde{\gamma}_2\})$, $\{\epsilon_n\}$ and $\{\beta_n\} \subset (0,1)$. Assume that the following conditions are satisfied:

(C1) *for all $i = 1,2$, $K_i : C \to \mathbb{R}$ is strongly convex with constant $\sigma_i > 0$ and its derivative K_i' is Lipschitz continuous with constant $\nu_i > 0$ such that the function $x \mapsto \langle y - x, K_i'(x)\rangle$ is weakly upper semicontinuous for each $y \in C$,*

(C2) *for all $i = 1,2$ and for each $x \in C$, there exist a bounded subset $D_x \subset C$ and $z_x \in C$ such that, for all $y \notin D_x$,*

$$\Theta_i(y, z_x) + \varphi_i(z_x) - \varphi_i(y) + \frac{1}{r_i}\langle K_i'(y) - K_i'(x), z_x - y\rangle < 0, \tag{3.3}$$

(C3) $\lim_{n\to\infty}\alpha_n^1 = 0$,

(C4) $\lim_{n\to\infty}\epsilon_n = 0$ *and* $\sum_{n=1}^{\infty}\epsilon_n = \infty$,

(C5) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$,

(C6) $0 < \liminf_{n\to\infty}\mu_n \leq \limsup_{n\to\infty}\mu_n < \min\{\widetilde{\beta}_1, \widetilde{\gamma}_1\}$ *and* $\lim_{n\to\infty}\mu_n/\mu_{n+1} = 1$,

(C7) $0 < \liminf_{n\to\infty}r_n \leq \limsup_{n\to\infty}r_n < \min\{\widetilde{\beta}_2, \widetilde{\gamma}_2\}$ *and* $\lim_{n\to\infty}r_n/r_{n+1} = 1$,

(C8) $0 < \liminf_{n\to\infty}s_n \leq \limsup_{n\to\infty}s_n < 2\widetilde{\xi}_1$ *and* $\lim_{n\to\infty}s_n/s_{n+1} = 1$,

(C9) $0 < \liminf_{n\to\infty}t_n \leq \limsup_{n\to\infty}t_n < 2\widetilde{\xi}_2$ *and* $\lim_{n\to\infty}t_n/t_{n+1} = 1$.

Then, the sequence $\{x_n\}$ defined by (3.2) converges strongly to $x^* \in \Omega$, provided $V_{r,n}^{(\Theta,\varphi)}$ is firmly nonexpansive, where x^* solves the following optimization problem:

$$\min_{x\in\Omega} \frac{\mu}{2}\langle Ax, x\rangle + \frac{1}{2}\|x - u\|^2 - h(x). \tag{3.4}$$

Proof. By the conditions (C4) and (C5), we may assume, without loss of generality, that $\epsilon_n \leq (1-\beta_n)(1+\mu\|A\|)^{-1}$ for all $n \in \mathbb{N}$. Since A is a linear bounded self-adjoint operator on C, we have

$$\|A\| = \sup\{|\langle A\widetilde{u}, \widetilde{u}\rangle| : \widetilde{u} \in C,\ \|\widetilde{u}\| = 1\}. \tag{3.5}$$

Observe that

$$\begin{aligned}\langle((1-\beta_n)I-\epsilon_n(I+\mu A))\tilde{u},\tilde{u}\rangle &= 1-\beta_n-\epsilon_n-\epsilon_n\mu\langle A\tilde{u},\tilde{u}\rangle\\ &\geq 1-\beta_n-\epsilon_n-\epsilon_n\mu\|A\|\\ &\geq 0.\end{aligned} \tag{3.6}$$

This shows that $(1-\beta_n)I-\epsilon_n(I+\mu A)$ is positive. It follows that

$$\begin{aligned}\|(1-\beta_n)I-\epsilon_n(I+\mu A)\| &= \sup\{|\langle((1-\beta_n)I-\epsilon_n(I+\mu A))\tilde{u},\tilde{u}\rangle| : \tilde{u}\in C,\ \|\tilde{u}\|=1\}\\ &= \sup\{1-\beta_n-\epsilon_n-\epsilon_n\mu\langle A\tilde{u},\tilde{u}\rangle : \tilde{u}\in C,\ \|\tilde{u}\|=1\}\\ &\leq 1-\beta_n-\epsilon_n(1+\mu\overline{\gamma})\\ &< 1-\beta_n-\epsilon_n(1+\mu)\overline{\gamma}.\end{aligned} \tag{3.7}$$

First, we show that $\{x_n\}$ is bounded. Take $\overline{x}\in\Omega$. Since $0<\mu_n<\min\{\widetilde{\beta}_1,\widetilde{\gamma}_1\}$ and by Lemma 2.4 (i), we have

$$\begin{aligned}&\|v_n-\overline{x}\|^2\\ &\quad= \left\|V_{\mu_n}^{(\Theta_1,\varphi_1)}(u_n-\mu_n(\Psi_1+\Phi_1)u_n)-V_{\mu_n}^{(\Theta_1,\varphi_1)}(\overline{x}-\mu_n(\Psi_1+\Phi_1)\overline{x})\right\|^2\\ &\quad\leq \|(u_n-\mu_n(\Psi_1+\Phi_1)u_n)-(\overline{x}-\mu_n(\Psi_1+\Phi_1)\overline{x})\|^2\\ &\quad= \|(u_n-\overline{x})-\mu_n[(\Psi_1+\Phi_1)u_n-(\Psi_1+\Phi_1)\overline{x}]\|^2\\ &\quad= \left\|\frac{1}{2}((u_n-\overline{x})-2\mu_n(\Psi_1u_n-\Psi_1\overline{x}))+\frac{1}{2}((u_n-\overline{x})-2\mu_n(\Phi_1u_n-\Phi_1\overline{x}))\right\|^2\\ &\quad\leq \frac{1}{2}\|(u_n-\overline{x})-2\mu_n(\Psi_1u_n-\Psi_1\overline{x})\|^2+\frac{1}{2}\|(u_n-\overline{x})-2\mu_n(\Phi_1u_n-\Phi_1\overline{x})\|^2\\ &\quad= \frac{1}{2}\left(\|u_n-\overline{x}\|^2-4\mu_n\langle u_n-\overline{x},\Psi_1u_n-\Psi_1\overline{x}\rangle+4\mu_n^2\|\Psi_1u_n-\Psi_1\overline{x}\|^2\right)\\ &\qquad+\frac{1}{2}\left(\|u_n-\overline{x}\|^2-4\mu_n\langle u_n-\overline{x},\Phi_1u_n-\Phi_1\overline{x}\rangle+4\mu_n^2\|\Phi_1u_n-\Phi_1\overline{x}\|^2\right)\\ &\quad\leq \frac{1}{2}\left(\|u_n-\overline{x}\|^2+4\mu_n\left(\mu_n-\widetilde{\beta}_1\right)\|\Psi_1u_n-\Psi_1\overline{x}\|^2\right)\\ &\qquad+\frac{1}{2}\left(\|u_n-\overline{x}\|^2+4\mu_n(\mu_n-\widetilde{\gamma}_1)\|\Phi_1u_n-\Phi_1\overline{x}\|^2\right)\\ &\quad= \|u_n-\overline{x}\|^2+2\mu_n\left(\mu_n-\widetilde{\beta}_1\right)\|\Psi_1u_n-\Psi_1\overline{x}\|^2+2\mu_n(\mu_n-\widetilde{\gamma}_1)\|\Phi_1u_n-\Phi_1\overline{x}\|^2\\ &\quad\leq \|u_n-\overline{x}\|^2.\end{aligned} \tag{3.8}$$

In a similar way, we can get

$$\begin{aligned}\|u_n - \overline{x}\|^2 &= \left\| V_{r_n}^{(\Theta_2,\varphi_2)}(x_n - r_n(\Psi_2 + \Phi_2)x_n) - V_{r_n}^{(\Theta_2,\varphi_2)}(\overline{x} - r_n(\Psi_2 + \Phi_1)\overline{x}) \right\|^2 \\ &\le \|x_n - \overline{x}\|^2 + 2r_n\left(r_n - \widetilde{\beta}_2\right)\|\Psi_2 x_n - \Psi_2\overline{x}\|^2 + 2r_n(r_n - \widetilde{\gamma}_2)\|\Phi_2 x_n - \Phi_2\overline{x}\|^2 \\ &\le \|x_n - \overline{x}\|^2.\end{aligned} \tag{3.9}$$

It follows from (3.8), and (3.9) that

$$\|v_n - \overline{x}\| \le \|u_n - \overline{x}\| \le \|x_n - \overline{x}\|. \tag{3.10}$$

Setting $z_n := J_{s_n}(y_n - s_n B_1 y_n)$. Since $0 < s_n < 2\widetilde{\xi}_1$, we have

$$\begin{aligned}\|z_n - \overline{x}\|^2 &= \left\| J_{s_n}(y_n - s_n B_1 y_n) - J_{s_n}(\overline{x} - s_n B_1\overline{x}) \right\|^2 \\ &\le \left\| (y_n - s_n B_1 y_n) - (\overline{x} - s_n B_1\overline{x}) \right\|^2 \\ &= \left\| (y_n - \overline{x}) - s_n(B_1 y_n - B_1\overline{x}) \right\|^2 \\ &= \left\| y_n - \overline{x} \right\|^2 - 2s_n\langle y_n - \overline{x}, B_1 y_n - B_1\overline{x}\rangle + s_n^2\left\| B_1 y_n - B_1\overline{x} \right\|^2 \\ &\le \left\| y_n - \overline{x} \right\|^2 + s_n(s_n - 2\xi_1)\left\| B_1 y_n - B_1\overline{x} \right\|^2 \\ &\le \left\| y_n - \overline{x} \right\|^2.\end{aligned} \tag{3.11}$$

In a similar way, we can get

$$\begin{aligned}\left\| y_n - \overline{x} \right\|^2 &= \|J_{t_n}(v_n - t_n B_2 v_n) - J_{t_n}(\overline{x} - t_n B_2\overline{x})\|^2 \\ &\le \|v_n - \overline{x}\|^2 + t_n(t_n - 2\xi_2)\|B_2 v_n - B_2\overline{x}\|^2 \\ &\le \|v_n - \overline{x}\|^2.\end{aligned} \tag{3.12}$$

It follows from (3.10), (3.11) and (3.12) that

$$\|z_n - \overline{x}\| \le \left\| y_n - \overline{x} \right\| \le \|v_n - \overline{x}\| \le \|u_n - \overline{x}\| \le \|x_n - \overline{x}\|. \tag{3.13}$$

Since $x_{n+1} = \epsilon_n(u + \gamma f(S_n v_n)) + \beta_n x_n + ((1-\beta_n)I - \epsilon_n(I + \mu A))S_n z_n$, it follows from (3.13) that

$$\begin{aligned}&\|x_{n+1} - \overline{x}\| \\ &\quad= \left\| \epsilon_n u + \epsilon_n(\gamma f(S_n v_n) - (I + \mu A)\overline{x}) + \beta_n(x_n - \overline{x}) + ((1-\beta_n)I - \epsilon_n(I + \mu A))(S_n z_n - \overline{x}) \right\| \\ &\quad\le \epsilon_n\|u\| + \epsilon_n\left\| \gamma f(S_n v_n) - (I + \mu A)\overline{x} \right\| + \beta_n\|x_n - \overline{x}\| + (1 - \beta_n - \epsilon_n(1 + \mu\overline{\gamma}))\|S_n z_n - \overline{x}\|\end{aligned}$$

$$
\begin{aligned}
&\leq \epsilon_n\|u\| + \epsilon_n\gamma\left\|f(S_nv_n) - f(\overline{x})\right\| + \epsilon_n\left\|\gamma f(\overline{x}) - (I+\mu A)\overline{x}\right\| + \beta_n\|x_n - \overline{x}\| \\
&\quad + \left(1-\beta_n - \epsilon_n(1+\mu)\overline{\gamma}\right)\|z_n - \overline{x}\| \\
&\leq \epsilon_n\|u\| + \epsilon_n\gamma\alpha\|S_nv_n - \overline{x}\| + \epsilon_n\left\|\gamma f(\overline{x}) - (I+\mu A)\overline{x}\right\| + \beta_n\|x_n - \overline{x}\| \\
&\quad + \left(1-\beta_n - \epsilon_n(1+\mu)\overline{\gamma}\right)\|x_n - \overline{x}\| \\
&\leq \left(1 - \left((1+\mu)\overline{\gamma} - \gamma\alpha\right)\epsilon_n\right)\|x_n - \overline{x}\| + \epsilon_n\left\|\gamma f(\overline{x}) - (I+\mu A)\overline{x}\right\| + \epsilon_n\|u\| \\
&= \left(1 - \left((1+\mu)\overline{\gamma} - \gamma\alpha\right)\epsilon_n\right)\|x_n - \overline{x}\| + \left((1+\mu)\overline{\gamma} - \gamma\alpha\right)\epsilon_n \frac{\left\|\gamma f(\overline{x}) - (I+\mu A)\overline{x}\right\| + \|u\|}{(1+\mu)\overline{\gamma} - \gamma\alpha}.
\end{aligned}
\tag{3.14}
$$

By induction, we have

$$
\|x_n - \overline{x}\| \leq \max\left\{\|x_1 - \overline{x}\|, \frac{\left\|\gamma f(\overline{x}) - (I+\mu A)\overline{x}\right\| + \|u\|}{(1+\mu)\overline{\gamma} - \gamma\alpha}\right\}, \quad \forall n \geq 1. \tag{3.15}
$$

Hence, $\{x_n\}$ is bounded, so are $\{v_n\}$, $\{y_n\}$, and $\{z_n\}$.

Next, we show that $\lim_{n\to\infty}\|x_{n+1} - x_n\|$. Since $u_n = V_{r_n}^{(\Theta_2,\varphi_2)}(x_n - r_n(\Psi_2 + \Phi_2)x_n)$ and $u_{n+1} = V_{r_{n+1}}^{(\Theta_2,\varphi_2)}(x_{n+1} - r_{n+1}(\Psi_2 + \Phi_2)x_{n+1})$, we have

$$
\begin{aligned}
&\|u_{n+1} - u_n\| \\
&\quad = \left\|V_{r_{n+1}}^{(\Theta_2,\varphi_2)}(x_{n+1} - r_{n+1}(\Psi_2+\Phi_2)x_{n+1}) - V_{r_n}^{(\Theta_2,\varphi_2)}(x_n - r_n(\Psi_2+\Phi_2)x_n)\right\| \\
&\quad \leq \left\|V_{r_{n+1}}^{(\Theta_2,\varphi_2)}(x_{n+1} - r_{n+1}(\Psi_2+\Phi_2)x_{n+1}) - V_{r_{n+1}}^{(\Theta_2,\varphi_2)}(x_n - r_n(\Psi_2+\Phi_2)x_n)\right\| \\
&\qquad + \left\|V_{r_{n+1}}^{(\Theta_2,\varphi_2)}(x_n - r_n(\Psi_2+\Phi_2)x_n) - V_{r_n}^{(\Theta_2,\varphi_2)}(x_n - r_n(\Psi_2+\Phi_2)x_n)\right\| \\
&\quad \leq \|(x_{n+1} - r_{n+1}(\Psi_2+\Phi_2)x_{n+1}) - (x_n - r_n(\Psi_2+\Phi_2)x_n)\| \\
&\qquad + \left\|V_{r_{n+1}}^{(\Theta_2,\varphi_2)}(x_n - r_n(\Psi_2+\Phi_2)x_n) - V_{r_n}^{(\Theta_2,\varphi_2)}(x_n - r_n(\Psi_2+\Phi_2)x_n)\right\| \\
&\quad = \|x_{n+1} - x_n - r_{n+1}[(\Psi_2+\Phi_2)x_{n+1} - (\Psi_2+\Phi_2)x_n] + (r_n - r_{n+1})(\Psi_2+\Phi_2)x_n\| \\
&\qquad + \left\|V_{r_{n+1}}^{(\Theta_2,\varphi_2)}(x_n - r_n(\Psi_2+\Phi_2)x_n) - V_{r_n}^{(\Theta_2,\varphi_2)}(x_n - r_n(\Psi_2+\Phi_2)x_n)\right\| \\
&\quad = \|x_{n+1} - r_{n+1}(\Psi_2+\Phi_2)x_{n+1} - (x_n - r_{n+1}(\Psi_2+\Phi_2)x_n) + (r_n - r_{n+1})(\Psi_2+\Phi_2)x_n\| \\
&\qquad + \left\|V_{r_{n+1}}^{(\Theta_2,\varphi_2)}(x_n - r_n(\Psi_2+\Phi_2)x_n) - V_{r_n}^{(\Theta_2,\varphi_2)}(x_n - r_n(\Psi_2+\Phi_2)x_n)\right\| \\
&\quad \leq \|x_{n+1} - x_n\| + r_{n+1}\left|1 - \frac{r_n}{r_{n+1}}\right|\|(\Psi_2+\Phi_2)x_n\| \\
&\qquad + \left\|V_{r_{n+1}}^{(\Theta_2,\varphi_2)}(x_n - r_n(\Psi_2+\Phi_2)x_n) - V_{r_n}^{(\Theta_2,\varphi_2)}(x_n - r_n(\Psi_2+\Phi_2)x_n)\right\|.
\end{aligned}
\tag{3.16}
$$

In a similar way, we can get

$$\begin{aligned}\|v_{n+1}-v_n\| &= \left\|V_{\mu_{n+1}}^{(\Theta_1,\varphi_1)}\left(u_{n+1}-\mu_{n+1}(\Psi_1+\Phi_1)u_{n+1}\right)-V_{\mu_n}^{(\Theta_1,\varphi_1)}\left(u_n-\mu_n(\Psi_1+\Phi_1)u_n\right)\right\| \\ &\leq \|u_{n+1}-u_n\|+\mu_{n+1}\left|1-\frac{\mu_n}{\mu_{n+1}}\right|\|(\Psi_1+\Phi_1)u_n\| \\ &\quad+\left\|V_{\mu_{n+1}}^{(\Theta_1,\varphi_1)}\left(u_n-\mu_n(\Psi_1+\Phi_1)u_n\right)-V_{\mu_n}^{(\Theta_1,\varphi_1)}\left(u_n-\mu_n(\Psi_1+\Phi_1)u_n\right)\right\|.\end{aligned} \tag{3.17}$$

Substitution (3.16) into (3.17), we obtain

$$\begin{aligned}\|v_{n+1}-v_n\| &\leq \|x_{n+1}-x_n\|+r_{n+1}\left|1-\frac{r_n}{r_{n+1}}\right|\|(\Psi_2+\Phi_2)x_n\|+\mu_{n+1}\left|1-\frac{\mu_n}{\mu_{n+1}}\right|\|(\Psi_1+\Phi_1)u_n\| \\ &\quad+\left\|V_{r_{n+1}}^{(\Theta_2,\varphi_2)}(x_n-r_n(\Psi_2+\Phi_2)x_n)-V_{r_n}^{(\Theta_2,\varphi_2)}(x_n-r_n(\Psi_2+\Phi_2)x_n)\right\| \\ &\quad+\left\|V_{\mu_{n+1}}^{(\Theta_1,\varphi_1)}(u_n-\mu_n(\Psi_1+\Phi_1)u_n)-V_{\mu_n}^{(\Theta_1,\varphi_1)}(u_n-\mu_n(\Psi_1+\Phi_1)u_n)\right\| \\ &\leq \|x_{n+1}-x_n\|+\left(\left|1-\frac{r_n}{r_{n+1}}\right|+\left|1-\frac{\mu_n}{\mu_{n+1}}\right|\right)M_1 \\ &\quad+\left\|V_{r_{n+1}}^{(\Theta_2,\varphi_2)}(x_n-r_n(\Psi_2+\Phi_2)x_n)-V_{r_n}^{(\Theta_2,\varphi_2)}(x_n-r_n(\Psi_2+\Phi_2)x_n)\right\| \\ &\quad+\left\|V_{\mu_{n+1}}^{(\Theta_1,\varphi_1)}(u_n-\mu_n(\Psi_1+\Phi_1)u_n)-V_{\mu_n}^{(\Theta_1,\varphi_1)}(u_n-\mu_n(\Psi_1+\Phi_1)u_n)\right\|,\end{aligned} \tag{3.18}$$

where $M_1=\sup_{n\geq 1}\{\mu_{n+1}\|(\Psi_1+\Phi_1)u_n\|, r_{n+1}\|(\Psi_2+\Phi_2)x_n\|\}$.

On the other hand, notice from Lemma 2.6 that

$$\begin{aligned}&\|y_{n+1}-y_n\| \\ &\quad= \|J_{t_{n+1}}(v_{n+1}-t_{n+1}B_2v_{n+1})-J_{t_n}(v_n-t_nB_2v_n)\| \\ &\quad\leq \|J_{t_{n+1}}(v_{n+1}-t_{n+1}B_2v_{n+1})-J_{t_{n+1}}(v_n-t_nB_2v_n)\|+\|J_{t_{n+1}}(v_n-t_nB_2v_n)-J_{s_n}(v_n-t_nB_2v_n)\| \\ &\quad\leq \|(v_{n+1}-t_{n+1}B_2v_{n+1})-(v_n-t_nB_2v_n)\|+\|J_{t_{n+1}}(v_n-t_nB_2v_n)-J_{s_n}(v_n-t_nB_2v_n)\| \\ &\quad= \|(v_{n+1}-t_{n+1}B_2v_{n+1})-(v_n-t_{n+1}B_2v_n)+(t_n-t_{n+1})B_2v_n\| \\ &\qquad+\|J_{t_{n+1}}(v_n-t_nB_2v_n)-J_{t_n}(v_n-t_nB_2v_n)\| \\ &\quad\leq \|v_{n+1}-v_n\|+t_{n+1}\left|1-\frac{t_n}{t_{n+1}}\right|\|B_2v_n\|+\left|1-\frac{t_n}{t_{n+1}}\right|\|J_{t_{n+1}}(v_n-t_nB_2v_n)-(v_n-t_nB_2v_n)\| \\ &\quad\leq \|v_{n+1}-v_n\|+2\left|1-\frac{t_n}{t_{n+1}}\right|M_2,\end{aligned} \tag{3.19}$$

where $M_2>0$ is an appropriate constant such that $M_2=\sup_{n\geq 1}\{t_{n+1}\|B_2v_n\|, \|J_{t_{n+1}}(v_n-t_nB_2v_n)-(v_n-t_nB_2v_n)\|\}$.

In a similar way, we can get from Lemma 2.6 that

$$\begin{aligned}
&\|z_{n+1} - z_n\| \\
&\quad = \left\| J_{s_{n+1}}(y_{n+1} - s_{n+1}B_1y_{n+1}) - J_{s_n}(y_n - s_nB_1y_n) \right\| \\
&\quad \leq \left\| y_{n+1} - y_n \right\| + s_{n+1}\left|1 - \frac{s_n}{s_{n+1}}\right| \left\|B_1y_n\right\| + \left|1 - \frac{s_n}{s_{n+1}}\right| \left\| J_{s_{n+1}}(y_n - s_nB_1y_n) - (y_n - s_nB_1y_n) \right\| \\
&\quad \leq \left\| y_{n+1} - y_n \right\| + 2\left|1 - \frac{s_n}{s_{n+1}}\right| M_3,
\end{aligned} \tag{3.20}$$

where $M_3 > 0$ is an appropriate constant such that $M_3 = \sup_{n\geq 1}\{s_{n+1}\|B_1y_n\|, \|J_{s_{n+1}}(y_n - s_nB_1y_n) - (y_n - s_nB_1y_n)\|\}$. It follows from (3.18) and (3.19) that

$$\begin{aligned}
\|z_{n+1} - z_n\| \leq \|x_{n+1} - x_n\| &+ \left(\left|1 - \frac{r_n}{r_{n+1}}\right| + \left|1 - \frac{\mu_n}{\mu_{n+1}}\right| + 2\left|1 - \frac{t_n}{t_{n+1}}\right| + 2\left|1 - \frac{s_n}{s_{n+1}}\right|\right)L \\
&+ \left\| V_{r_{n+1}}^{(\Theta_2,\varphi_2)}(x_n - r_n(\Psi_2 + \Phi_2)x_n) - V_{r_n}^{(\Theta_2,\varphi_2)}(x_n - r_n(\Psi_2 + \Phi_2)x_n) \right\| \\
&+ \left\| V_{\mu_{n+1}}^{(\Theta_1,\varphi_1)}(u_n - \mu_n(\Psi_1 + \Phi_1)u_n) - V_{\mu_n}^{(\Theta_1,\varphi_1)}(u_n - \mu_n(\Psi_1 + \Phi_1)u_n) \right\|,
\end{aligned} \tag{3.21}$$

where $L = \max\{M_1, M_2, M_3\}$.

Define the sequence $\{l_n\}$ by $l_n := (x_{n+1} - \beta_nx_n)/(1 - \beta_n)$, for all $n \in \mathbb{N}$. Then, $x_{n+1} = \beta_nx_n + (1 - \beta_n)l_n$, for all $n \in \mathbb{N}$, we note that

$$\begin{aligned}
l_{n+1} - l_n &= \frac{x_{n+2} - \beta_{n+1}x_{n+1}}{1 - \beta_{n+1}} - \frac{x_{n+1} - \beta_nx_n}{1 - \beta_n} \\
&= \frac{\epsilon_{n+1}(u + \gamma f(S_{n+1}v_{n+1})) + ((1 - \beta_{n+1})I - \epsilon_{n+1}(I + \mu A))S_{n+1}z_{n+1}}{1 - \beta_{n+1}} \\
&\quad - \frac{\epsilon_n(u + \gamma f(S_nv_n)) + ((1 - \beta_n)I - \epsilon_n(I + \mu A))S_nz_n}{1 - \beta_n} \\
&= \frac{\epsilon_{n+1}}{1 - \beta_{n+1}}(u + \gamma f(S_{n+1}v_{n+1})) - \frac{\epsilon_n}{1 - \beta_n}(u + \gamma f(S_nv_n)) + \frac{\epsilon_n}{1 - \beta_n}(I + \mu A)S_nz_n \\
&\quad - \frac{\epsilon_{n+1}}{1 - \beta_{n+1}}(I + \mu A)S_{n+1}z_{n+1} + S_{n+1}z_{n+1} - S_nz_n.
\end{aligned} \tag{3.22}$$

It follows that

$$\begin{aligned}
\|l_{n+1} - l_n\| \leq &\frac{\epsilon_{n+1}}{1 - \beta_{n+1}}\left\|u + \gamma f(S_{n+1}v_{n+1}) - (I - \mu A)S_{n+1}z_{n+1}\right\| \\
&+ \frac{\epsilon_n}{1 - \beta_n}\left\|(I + \mu A)S_nz_n - u - \gamma f(S_nv_n)\right\| + \|S_{n+1}z_{n+1} - S_nz_n\|.
\end{aligned} \tag{3.23}$$

Now, we estimate $\|S_{n+1}z_n - S_n z_n\|$. By definition of S_n, for all $n \in \mathbb{N}$ and for all $\omega \in \Omega$, we have

$$\begin{aligned}
\|S_{n+1}z_n - S_n z_n\| &= \|U_{n+1,1}z_n - U_{n,1}z_n\| \\
&= \left\|\alpha_1^1 T_1 U_{n+1,2}z_n + \alpha_2^1 U_{n+1,2}z_n + \alpha_3^1 z_n - \alpha_1^1 T_1 U_{n,2}z_n - \alpha_2^1 U_{n,2}z_n - \alpha_3^1 z_n\right\| \\
&= \left(\alpha_1^1 + \alpha_2^1\right)\|T_1 U_{n+1,2}z_n - T_1 U_{n,2}z_n\| \\
&\le \left(1 - \alpha_3^1\right)\|U_{n+1,2}z_n - U_{n,2}z_n\| \\
&\le \left(1 - \alpha_3^1\right)\left(1 - \alpha_3^2\right)\|U_{n+1,3}z_n - U_{n,3}z_n\| \\
&= \prod_{i=1}^{2}\left(1 - \alpha_3^i\right)\|U_{n+1,3}z_n - U_{n,3}z_n\| \\
&\vdots \\
&\le \prod_{i=1}^{n}\left(1 - \alpha_3^i\right)\|U_{n+1,n+1}z_n - U_{n,n+1}z_n\| \\
&\le \|U_{n+1,n+1}z_n - z_n\| \\
&= \left\|\alpha_1^{n+1} T_{n+1}z_n + \left(1 - \alpha_1^{n+1}\right)z_n - z_n\right\| \\
&= \alpha_1^{n+1}\|T_{n+1}z_n - z_n\| \\
&\le \alpha_1^{n+1}(\|T_{n+1}z_n - T_{n+1}\omega\| + \|T_{n+1}\omega - z_n\|) \\
&\le 2\alpha_1^{n+1}\|z_n - \omega\|.
\end{aligned} \tag{3.24}$$

By the condition (C3), we obtain that

$$\lim_{n\to\infty}\|S_{n+1}z_n - S_n z_n\| = 0. \tag{3.25}$$

It follows from (3.21) that

$$\begin{aligned}
\|S_{n+1}z_{n+1} - S_n z_n\| &\le \|S_{n+1}z_{n+1} - S_{n+1}z_n\| + \|S_{n+1}z_n - S_n z_n\| \\
&\le \|z_{n+1} - z_n\| + \|S_{n+1}z_n - S_n z_n\| \\
&\le \|x_{n+1} - x_n\| + \left(\left|1 - \frac{r_n}{r_{n+1}}\right| + \left|1 - \frac{\mu_n}{\mu_{n+1}}\right| + 2\left|1 - \frac{t_n}{t_{n+1}}\right| + 2\left|1 - \frac{s_n}{s_{n+1}}\right|\right)L \\
&\quad + \left\|V_{r_{n+1}}^{(\Theta_2,\varphi_2)}(x_n - r_n(\Psi_2 + \Phi_2)x_n) - V_{r_n}^{(\Theta_2,\varphi_2)}(x_n - r_n(\Psi_2 + \Phi_2)x_n)\right\| \\
&\quad + \left\|V_{\mu_{n+1}}^{(\Theta_1,\varphi_1)}(u_n - \mu_n(\Psi_1 + \Phi_1)u_n) - V_{\mu_n}^{(\Theta_1,\varphi_1)}(u_n - \mu_n(\Psi_1 + \Phi_1)u_n)\right\|.
\end{aligned} \tag{3.26}$$

It follows from (3.23) and (3.26) that

$$\begin{aligned}
&\|l_{n+1} - l_n\| - \|x_{n+1} - x_n\| \\
&\quad \le \frac{\epsilon_{n+1}}{1-\beta_{n+1}} \left\| u + \gamma f(S_{n+1}v_{n+1}) - (I - \mu A)S_{n+1}z_{n+1} \right\| \\
&\qquad + \frac{\epsilon_n}{1-\beta_n} \left\| (I + \mu A)S_n z_n - u - \gamma f(S_n v_n) \right\| + \|S_{n+1}z_{n+1} - S_n z_n\| - \|x_{n+1} - x_n\| \\
&\quad \le \frac{\epsilon_{n+1}}{1-\beta_{n+1}} \left\| u + \gamma f(S_{n+1}v_{n+1}) - (I - \mu A)S_{n+1}z_{n+1} \right\| \\
&\qquad + \frac{\epsilon_n}{1-\beta_n} \left\| (I + \mu A)S_n z_n - u - \gamma f(S_n v_n) \right\| \\
&\qquad + \left(\left|1 - \frac{r_n}{r_{n+1}}\right| + \left|1 - \frac{\mu_n}{\mu_{n+1}}\right| + 2\left|1 - \frac{t_n}{t_{n+1}}\right| + 2\left|1 - \frac{s_n}{s_{n+1}}\right| \right) L \\
&\qquad + \left\| V_{r_{n+1}}^{(\Theta_2,\varphi_2)}(x_n - r_n(\Psi_2 + \Phi_2)x_n) - V_{r_n}^{(\Theta_2,\varphi_2)}(x_n - r_n(\Psi_2 + \Phi_2)x_n) \right\| \\
&\qquad + \left\| V_{\mu_{n+1}}^{(\Theta_1,\varphi_1)}(u_n - \mu_n(\Psi_1 + \Phi_1)u_n) - V_{\mu_n}^{(\Theta_1,\varphi_1)}(u_n - \mu_n(\Psi_1 + \Phi_1)u_n) \right\| \\
&\qquad + \|S_{n+1}z_n - S_n z_n\|.
\end{aligned} \tag{3.27}$$

Note that $0 < \liminf_{n\to\infty} r_n \le \limsup_{n\to\infty} r_n < \min\{\widetilde{\beta}_2, \widetilde{\gamma}_2\}$ and $\lim_{n\to\infty} r_n / r_{n+1} = 1$. Since K_1' is σ_1-strongly monotone and ν_1-Lipschitz continuous, then, from Lemma 2.10, we obtain that

$$\lim_{n\to\infty} \left\| V_{r_{n+1}}^{(\Theta_2,\varphi_2)}(x_n - r_n(\Psi_2 + \Phi_2)x_n) - V_{r_n}^{(\Theta_2,\varphi_2)}(x_n - r_n(\Psi_2 + \Phi_2)x_n) \right\| = 0. \tag{3.28}$$

In a similar way, we can get

$$\lim_{n\to\infty} \left\| V_{\mu_{n+1}}^{(\Theta_1,\varphi_1)}(u_n - \mu_n(\Psi_1 + \Phi_1)u_n) - V_{\mu_n}^{(\Theta_1,\varphi_1)}(u_n - \mu_n(\Psi_1 + \Phi_1)u_n) \right\| = 0. \tag{3.29}$$

Consequently, it follows from (3.25), (3.27), (3.28), (3.29), and the conditions (C4)–(C9) that

$$\limsup_{n\to\infty} (\|l_{n+1} - l_n\| - \|x_{n+1} - x_n\|) \le 0. \tag{3.30}$$

Hence, by Lemma 2.3, we have

$$\lim_{n\to\infty} \|l_n - x_n\| = 0. \tag{3.31}$$

Moreover, we have

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = \lim_{n\to\infty} (1 - \beta_n) \|l_n - x_n\| = 0. \tag{3.32}$$

Since

$$x_{n+1} - x_n = \epsilon_n\big(u + \gamma f(S_n v_n) - (I + \mu A)x_n\big) + \big((1-\beta_n)I - \epsilon_n(I + \mu A)\big)(S_n z_n - x_n) \tag{3.33}$$

it follows that

$$(1 - \beta_n - \epsilon_n(1+\mu)\overline{\gamma})\|x_n - S_n z_n\| \le \|x_{n+1} - x_n\| + \epsilon_n\big\|u + \gamma f(S_n v_n) - (I + \mu A)x_n\big\|. \tag{3.34}$$

It follows from (3.32) and the conditions (C4) and (C5) that

$$\lim_{n\to\infty}\|x_n - S_n z_n\| = 0. \tag{3.35}$$

Next, we show that

$$\begin{aligned}
\lim_{n\to\infty}\|\Psi_1 u_n - \Psi_1\overline{x}\| &= \lim_{n\to\infty}\|\Phi_1 u_n - \Phi_1\overline{x}\| = 0,\\
\lim_{n\to\infty}\|\Psi_2 x_n - \Psi_2\overline{x}\| &= \lim_{n\to\infty}\|\Phi_2 x_n - \Phi_2\overline{x}\| = 0,\\
\lim_{n\to\infty}\big\|B_1 y_n - B_1\overline{x}\big\| &= \lim_{n\to\infty}\|B_2 v_n - B_1\overline{x}\| = 0.
\end{aligned} \tag{3.36}$$

From (3.8), (3.9), (3.11), and (3.12), we have

$$\begin{aligned}
\|z_n - \overline{x}\|^2 &\le \big\|y_n - \overline{x}\big\|^2 + s_n\Big(s_n - 2\widetilde{\xi}_1\Big)\big\|B_1 y_n - B_1\overline{x}\big\|^2\\
&\le \|v_n - \overline{x}\|^2 + t_n\Big(t_n - 2\widetilde{\xi}_2\Big)\|B_2 v_n - B_2\overline{x}\|^2 + s_n\Big(s_n - 2\widetilde{\xi}_1\Big)\big\|B_1 y_n - B_1\overline{x}\big\|^2\\
&\le \|u_n - \overline{x}\|^2 + 2\mu_n\Big(\mu_n - \widetilde{\beta}_1\Big)\|\Psi_1 u_n - \Psi_1\overline{x}\|^2 + 2\mu_n(\mu_n - \widetilde{\gamma}_1)\|\Phi_1 u_n - \Phi_1\overline{x}\|^2\\
&\quad + t_n\Big(t_n - 2\widetilde{\xi}_2\Big)\|B_2 v_n - B_2\overline{x}\|^2 + s_n\Big(s_n - 2\widetilde{\xi}_1\Big)\big\|B_1 y_n - B_1\overline{x}\big\|^2\\
&\le \|x_n - \overline{x}\|^2 + 2r_n\Big(r_n - \widetilde{\beta}_2\Big)\|\Psi_2 x_n - \Psi_2\overline{x}\|^2 + 2r_n(r_n - \widetilde{\gamma}_2)\|\Phi_2 x_n - \Phi_2\overline{x}\|^2\\
&\quad + 2\mu_n\Big(\mu_n - \widetilde{\beta}_1\Big)\|\Psi_1 u_n - \Psi_1\overline{x}\|^2 + 2\mu_n(\mu_n - \widetilde{\gamma}_1)\|\Phi_1 u_n - \Phi_1\overline{x}\|^2\\
&\quad + t_n\Big(t_n - 2\widetilde{\xi}_2\Big)\|B_2 v_n - B_2\overline{x}\|^2 + s_n\Big(s_n - 2\widetilde{\xi}_1\Big)\big\|B_1 y_n - B_1\overline{x}\big\|^2.
\end{aligned} \tag{3.37}$$

On the other hand, from Lemma 2.4 (ii), we have

$$
\begin{aligned}
&\|x_{n+1}-\overline{x}\|^2 \\
&\quad= \left\|\epsilon_n\left(u+\gamma f(S_n v_n)-(I+\mu A)\overline{x}\right)+\beta_n(x_n-S_nz_n)+\left(I-\epsilon_n(I+\mu A)\right)(S_nz_n-\overline{x})\right\|^2 \\
&\quad\le \left\|\left(I+\epsilon_n(I+\mu A)\right)(S_nz_n-\overline{x})+\beta_n(x_n-S_nz_n)\right\|^2 \\
&\qquad+2\epsilon_n\left\langle u+\gamma f(S_nv_n)-(I+\mu A)\overline{x},x_{n+1}-\overline{x}\right\rangle \\
&\quad\le \left[\left\|\left(I-\epsilon_n(I+\mu A)\right)(S_nz_n-\overline{x})\right\|+\beta_n\|x_n-S_nz_n\|\right]^2 \\
&\qquad+2\epsilon_n\left\langle u+\gamma f(S_nv_n)-(I+\mu A)\overline{x},x_{n+1}-\overline{x}\right\rangle \\
&\quad\le \left[(1-\epsilon_n-\epsilon_n\mu\overline{\gamma})\|z_n-\overline{x}\|+\beta_n\|x_n-S_nz_n\|\right]^2 \\
&\qquad+2\epsilon_n\left\|u+\gamma f(S_nv_n)-(I+\mu A)\overline{x}\right\|\|x_{n+1}-\overline{x}\| \\
&\quad\le (1-\epsilon_n-\epsilon_n\mu\overline{\gamma})^2\|z_n-\overline{x}\|^2+c_n,
\end{aligned}
\tag{3.38}
$$

where

$$
\begin{aligned}
c_n &:= \beta_n^2\|x_n-S_nz_n\|^2+2(1-\epsilon_n-\epsilon_n\mu\overline{\gamma})\beta_n\|z_n-\overline{x}\|\|x_n-S_nz_n\| \\
&\quad+2\epsilon_n\left\|u+\gamma f(S_nv_n)-(I+\mu A)\overline{x}\right\|\|x_{n+1}-\overline{x}\|.
\end{aligned}
\tag{3.39}
$$

It follows from the condition (C4) and (3.35) that

$$
\lim_{n\to\infty} c_n = 0. \tag{3.40}
$$

Substituting (3.37) into (3.38), we have

$$
\begin{aligned}
\|x_{n+1}-\overline{x}\|^2 &\le (1-\epsilon_n-\epsilon_n\mu\overline{\gamma})^2 \\
&\quad\times\Big\{\|x_n-\overline{x}\|^2+2r_n\left(r_n-\widetilde{\beta}_2\right)\|\Psi_2x_n-\Psi_2\overline{x}\|^2+2r_n(r_n-\widetilde{\gamma}_2)\|\Phi_2x_n-\Phi_2\overline{x}\|^2 \\
&\qquad+2\mu_n\left(\mu_n-\widetilde{\beta}_1\right)\|\Psi_1u_n-\Psi_1\overline{x}\|^2+2\mu_n(\mu_n-\widetilde{\gamma}_1)\|\Phi_1u_n-\Phi_1\overline{x}\|^2 \\
&\qquad+t_n\left(t_n-2\widetilde{\xi}_2\right)\|B_2v_n-B_2\overline{x}\|^2+s_n\left(s_n-2\widetilde{\xi}_1\right)\left\|B_1y_n-B_1\overline{x}\right\|^2\Big\}+c_n,
\end{aligned}
\tag{3.41}
$$

which in turn implies that

$$
\begin{aligned}
&(1-\epsilon_n-\epsilon_n\mu\overline{\gamma})^2\Big\{2r_n\Big(\widetilde{\beta}_2-r_n\Big)\|\Psi_2x_n-\Psi_2\overline{x}\|^2+2r_n(\widetilde{\gamma}_2-r_n)\|\Phi_2x_n-\Phi_2\overline{x}\|^2\\
&\qquad+2\mu_n\Big(\widetilde{\beta}_1-\mu_n\Big)\|\Psi_1u_n-\Psi_1\overline{x}\|^2+2\mu_n(\widetilde{\gamma}_1-\mu_n)\|\Phi_1u_n-\Phi_1\overline{x}\|^2\\
&\qquad+t_n\Big(2\widetilde{\xi}_2-t_n\Big)\|B_2v_n-B_2\overline{x}\|^2+s_n\Big(2\widetilde{\xi}_1-s_n\Big)\big\|B_1y_n-B_1\overline{x}\big\|^2\Big\}\\
&\quad\le(1-\epsilon_n-\epsilon_n\mu\overline{\gamma})^2\|x_n-\overline{x}\|^2-\|x_{n+1}-\overline{x}\|^2+c_n\\
&\quad=\Big\{1-2\epsilon_n(1+\mu\overline{\gamma})+\epsilon_n^2(1+\mu\overline{\gamma})^2\Big\}\|x_n-\overline{x}\|^2-\|x_{n+1}-\overline{x}\|^2+c_n\\
&\quad\le\|x_n-\overline{x}\|^2-\|x_{n+1}-\overline{x}\|^2+\epsilon_n^2(1+\mu\overline{\gamma})^2\|x_n-\overline{x}\|^2+c_n\\
&\quad\le(\|x_n-\overline{x}\|+\|x_{n+1}-\overline{x}\|)\|x_{n+1}-x_n\|+\epsilon_n^2(1+\mu\overline{\gamma})^2\|x_n-\overline{x}\|^2+c_n.
\end{aligned}
\tag{3.42}
$$

It follows from the conditions (C4)–(C9), (3.32), and (3.40) that

$$
\begin{aligned}
&\lim_{n\to\infty}\|\Psi_1u_n-\Psi_1\overline{x}\|=\lim_{n\to\infty}\|\Phi_1u_n-\Phi_1\overline{x}\|=0,\\
&\lim_{n\to\infty}\|\Psi_2x_n-\Psi_2\overline{x}\|=\lim_{n\to\infty}\|\Phi_2x_n-\Phi_2\overline{x}\|=0,\\
&\lim_{n\to\infty}\big\|B_1y_n-B_1\overline{x}\big\|=\lim_{n\to\infty}\|B_2v_n-B_1\overline{x}\|=0.
\end{aligned}
\tag{3.43}
$$

Next, we show that $\lim_{n\to\infty}\|Sz_n-z_n\|=0$. Since J_{s_n} is firmly nonexpansive, we have

$$
\begin{aligned}
\|z_n-\overline{x}\|^2&=\big\|J_{s_n}(y_n-s_nB_1y_n)-J_{s_n}(\overline{x}-s_nB_1\overline{x})\big\|^2\\
&\le\big\langle(y_n-s_nB_1y_n)-(\overline{x}-s_nB_1\overline{x}),z_n-\overline{x}\big\rangle\\
&=\frac{1}{2}\Big(\big\|(y_n-s_nB_1y_n)-(\overline{x}-s_nB_1\overline{x})\big\|^2+\|z_n-\overline{x}\|^2\\
&\qquad-\big\|(y_n-s_nB_1y_n)-(\overline{x}-s_nB_1\overline{x})-(z_n-\overline{x})\big\|^2\Big)\\
&\le\frac{1}{2}\Big(\big\|y_n-\overline{x}\big\|^2+\|z_n-\overline{x}\|^2-\big\|y_n-z_n-s_n(B_1y_n-B_1\overline{x})\big\|^2\Big)\\
&=\frac{1}{2}\Big(\big\|y_n-\overline{x}\big\|^2+\|z_n-\overline{x}\|^2-\big\|y_n-z_n\big\|^2\\
&\qquad+2s_n\langle y_n-z_n,B_1y_n-B_1\overline{x}\rangle-s_n^2\big\|B_1y_n-B_1\overline{x}\big\|^2\Big),
\end{aligned}
\tag{3.44}
$$

which in turn implies that

$$
\begin{aligned}
\|z_n-\overline{x}\|^2&\le\big\|y_n-\overline{x}\big\|^2-\big\|y_n-z_n\big\|^2+2s_n\langle y_n-z_n,B_1y_n-B_1\overline{x}\rangle-s_n^2\big\|B_1y_n-B_1\overline{x}\big\|^2\\
&\le\|x_n-\overline{x}\|^2-\big\|y_n-z_n\big\|^2+2s_n\big\|y_n-z_n\big\|\big\|B_1y_n-B_1\overline{x}\big\|.
\end{aligned}
\tag{3.45}
$$

Substituting (3.45) into (3.38), we have

$$\|x_{n+1}-\overline{x}\|^2 \le (1-\epsilon_n-\epsilon_n\mu\overline{\gamma})^2\left\{\|x_n-\overline{x}\|^2-\left\|y_n-z_n\right\|^2+2s_n\left\|y_n-z_n\right\|\left\|B_1y_n-B_1\overline{x}\right\|\right\}+c_n, \tag{3.46}$$

which in turn implies that

$$\begin{aligned}(1-\epsilon_n-\epsilon_n\mu\overline{\gamma})^2\left\|y_n-z_n\right\|^2 &\le (1-\epsilon_n-\epsilon_n\mu\overline{\gamma})^2\|x_n-\overline{x}\|^2-\|x_{n+1}-\overline{x}\|^2\\ &\quad+2(1-\epsilon_n-\epsilon_n\mu\overline{\gamma})^2 s_n\left\|y_n-z_n\right\|\left\|B_1y_n-B_1\overline{x}\right\|+c_n\\ &\le \|x_n-\overline{x}\|^2-\|x_{n+1}-\overline{x}\|^2+\epsilon_n^2(1+\mu\overline{\gamma})^2\|x_n-\overline{x}\|^2\\ &\quad+2s_n(1-\epsilon_n-\epsilon_n\mu\overline{\gamma})^2\left\|y_n-z_n\right\|\left\|B_1y_n-B_1\overline{x}\right\|+c_n\\ &\le (\|x_n-\overline{x}\|+\|x_{n+1}-\overline{x}\|)\|x_{n+1}-x_n\|\\ &\quad+\epsilon_n^2(1+\mu\overline{\gamma})^2\|x_n-\overline{x}\|^2\\ &\quad+2s_n(1-\epsilon_n-\epsilon_n\mu\overline{\gamma})^2\left\|y_n-z_n\right\|\left\|B_1y_n-B_1\overline{x}\right\|+c_n.\end{aligned} \tag{3.47}$$

Since $\epsilon_n \to 0$, $\|B_1y_n-B_1\overline{x}\| \to 0$ and from (3.32), we obtain that

$$\lim_{n\to\infty}\left\|y_n-z_n\right\|=0. \tag{3.48}$$

In a similar way, we can get

$$\left\|y_n-\overline{x}\right\|^2 \le \|x_n-\overline{x}\|^2-\left\|v_n-y_n\right\|^2+2t_n\left\|v_n-y_n\right\|\|B_2v_n-B_2\overline{x}\|. \tag{3.49}$$

It follows from (3.38) and (3.49) that

$$\begin{aligned}\|x_{n+1}-\overline{x}\|^2 &\le (1-\epsilon_n-\epsilon_n\mu\overline{\gamma})^2\|z_n-\overline{x}\|^2+c_n\\ &\le (1-\epsilon_n-\epsilon_n\mu\overline{\gamma})^2\left\|y_n-\overline{x}\right\|^2+c_n\\ &\le (1-\epsilon_n-\epsilon_n\mu\overline{\gamma})^2\left\{\|x_n-\overline{x}\|^2-\left\|v_n-y_n\right\|^2+2t_n\left\|v_n-y_n\right\|\|B_2v_n-B_2\overline{x}\|\right\}+c_n,\end{aligned} \tag{3.50}$$

which in turn implies that

$$
\begin{aligned}
(1-\epsilon_n-\epsilon_n\mu\overline{\gamma})^2\|v_n-y_n\|^2 &\le (1-\epsilon_n-\epsilon_n\mu\overline{\gamma})^2\|x_n-\overline{x}\|^2-\|x_{n+1}-x_n\|^2 \\
&\quad +2(1-t_n\epsilon_n-\epsilon_n\mu\overline{\gamma})^2\|v_n-y_n\|\|B_2v_n-B_2\overline{x}\|+c_n \\
&\le \|x_n-\overline{x}\|^2-\|x_{n+1}-\overline{x}\|^2+\epsilon_n^2(1+\mu\overline{\gamma})^2\|x_n-\overline{x}\|^2 \\
&\quad +2t_n(1-t_n\epsilon_n-\epsilon_n\mu\overline{\gamma})^2\|v_n-y_n\|\|B_2v_n-B_2\overline{x}\|+c_n \\
&\le (\|x_n-\overline{x}\|+\|x_{n+1}-\overline{x}\|)\|x_{n+1}-x_n\|+\epsilon_n^2(1+\mu\overline{\gamma})^2\|x_n-\overline{x}\|^2 \\
&\quad +2t_n(1-\epsilon_n-\epsilon_n\mu\overline{\gamma})^2\|v_n-y_n\|\|B_2v_n-B_2\overline{x}\|+c_n.
\end{aligned}
\tag{3.51}
$$

Since $\epsilon_n \to 0$, $\|B_2v_n-B_2\overline{x}\| \to 0$ and from (3.32), we obtain that

$$
\lim_{n\to\infty}\|v_n-y_n\|=0. \tag{3.52}
$$

In addition, from the firmly nonexpansivity of $V_{\mu_n}^{(\Theta_1,\varphi_1)}$, we have

$$
\begin{aligned}
\|v_n-\overline{x}\|^2 &= \left\|V_{\mu_n}^{(\Theta_1,\varphi_1)}(u_n-\mu_n(\Psi_1+\Phi_1)u_n)-V_{\mu_n}^{(\Theta_1,\varphi_1)}(\overline{x}-\mu_n(\Psi_1+\Phi_1)\overline{x})\right\|^2 \\
&\le \langle (u_n-\mu_n(\Psi_1+\Phi_1)u_n)-(\overline{x}-\mu_n(\Psi_1+\Phi_1)\overline{x}), v_n-\overline{x}\rangle \\
&= \frac{1}{2}\Big(\|(u_n-\mu_n(\Psi_1+\Phi_1)u_n)-(\overline{x}-\mu_n(\Psi_1+\Phi_1)\overline{x})\|^2+\|v_n-\overline{x}\|^2 \\
&\qquad -\|(u_n-\mu_n(\Psi_1+\Phi_1)u_n)-(\overline{x}-\mu_n(\Psi_1+\Phi_1)\overline{x})-(v_n-\overline{x})\|^2\Big) \\
&\le \frac{1}{2}\Big(\|u_n-\overline{x}\|^2+\|v_n-\overline{x}\|^2-\|(u_n-v_n)-\mu_n((\Psi_1+\Phi_1)u_n-(\Psi_1+\Phi_1)\overline{x})\|^2\Big) \\
&= \frac{1}{2}\Big(\|u_n-\overline{x}\|^2+\|v_n-\overline{x}\|^2-\|u_n-v_n\|^2+2\mu_n\langle u_n-v_n,(\Psi_1+\Phi_1)u_n-(\Psi_1+\Phi_1)\overline{x}\rangle \\
&\qquad -\mu_n^2\|(\Psi_1+\Phi_1)u_n-(\Psi_1+\Phi_1)\overline{x}\|^2\Big),
\end{aligned}
\tag{3.53}
$$

which in turn implies that

$$
\begin{aligned}
\|v_n-\overline{x}\|^2 &\le \|u_n-\overline{x}\|^2-\|u_n-v_n\|^2+2\mu_n\langle u_n-v_n,(\Psi_1+\Phi_1)u_n-(\Psi_1+\Phi_1)\overline{x}\rangle \\
&\quad -\mu_n^2\|(\Psi_1+\Phi_1)u_n-(\Psi_1+\Phi_1)\overline{x}\|^2 \\
&\le \|x_n-\overline{x}\|^2-\|u_n-v_n\|^2+2\mu_n\|u_n-v_n\|\|(\Psi_1+\Phi_1)u_n-(\Psi_1+\Phi_1)\overline{x}\|.
\end{aligned}
\tag{3.54}
$$

It follows from (3.38) and (3.54) that

$$\begin{aligned}
&\|x_{n+1}-\overline{x}\|^2 \\
&\quad \leq \left(1-\epsilon_n-\epsilon_n\mu\overline{\gamma}\right)^2\|z_n-\overline{x}\|^2+c_n \\
&\quad \leq \left(1-\epsilon_n-\epsilon_n\mu\overline{\gamma}\right)^2\|v_n-\overline{x}\|^2+c_n \\
&\quad \leq \left(1-\epsilon_n-\epsilon_n\mu\overline{\gamma}\right)^2\left\{\|x_n-\overline{x}\|^2-\|u_n-v_n\|^2+2\mu_n\|u_n-v_n\|\|(\Psi_1+\Phi_1)u_n-(\Psi_1+\Phi_1)\overline{x}\|\right\}+c_n,
\end{aligned} \tag{3.55}$$

which in turn implies that

$$\begin{aligned}
&\left(1-\epsilon_n-\epsilon_n\mu\overline{\gamma}\right)^2\|u_n-v_n\|^2 \\
&\quad \leq \left(1-\epsilon_n-\epsilon_n\mu\overline{\gamma}\right)^2\|x_n-\overline{x}\|^2-\|x_{n+1}-\overline{x}\|^2 \\
&\qquad +2\mu_n\left(1-\epsilon_n-\epsilon_n\mu\overline{\gamma}\right)^2\|u_n-v_n\|\|(\Psi_1+\Phi_1)u_n-(\Psi_1+\Phi_1)\overline{x}\|+c_n \\
&\quad \leq \|x_n-\overline{x}\|^2-\|x_{n+1}-\overline{x}\|^2+\epsilon_n^2\left(1+\mu\overline{\gamma}\right)^2\|x_n-\overline{x}\|^2 \\
&\qquad +2\mu_n\left(1-\epsilon_n-\epsilon_n\mu\overline{\gamma}\right)^2\|u_n-v_n\|\|(\Psi_1+\Phi_1)u_n-(\Psi_1+\Phi_1)\overline{x}\|+c_n \\
&\quad \leq (\|x_n-\overline{x}\|+\|x_{n+1}-\overline{x}\|)\|x_{n+1}-x_n\|+\epsilon_n^2\left(1+\mu\overline{\gamma}\right)^2\|x_n-\overline{x}\|^2 \\
&\qquad +2\mu_n\left(1-\epsilon_n-\epsilon_n\mu\overline{\gamma}\right)^2\|u_n-v_n\|\|(\Psi_1+\Phi_1)u_n-(\Psi_1+\Phi_1)\overline{x}\|+c_n.
\end{aligned} \tag{3.56}$$

Since $\epsilon_n \to 0$, $\|\Psi_1 u_n-\Psi_1\overline{x}\| \to 0$, $\|\Phi_1 u_n-\Phi_1\overline{x}\| \to 0$ and from (3.32), we obtain that

$$\lim_{n\to\infty}\|u_n-v_n\|=0. \tag{3.57}$$

In a similar way, we can get

$$\|u_n-\overline{x}\|^2 \leq \|x_n-\overline{x}\|^2-\|x_n-u_n\|^2+2r_n\|x_n-u_n\|\|(\Psi_2+\Phi_2)x_n-(\Psi_2+\Phi_2)\overline{x}\|. \tag{3.58}$$

It follows from (3.38) and (3.58) that

$$\begin{aligned}
&\|x_{n+1}-\overline{x}\|^2 \\
&\quad \leq \left(1-\epsilon_n-\epsilon_n\mu\overline{\gamma}\right)^2\|z_n-\overline{x}\|^2+c_n \\
&\quad \leq \left(1-\epsilon_n-\epsilon_n\mu\overline{\gamma}\right)^2\|u_n-\overline{x}\|^2+c_n \\
&\quad \leq \left(1-\epsilon_n-\epsilon_n\mu\overline{\gamma}\right)^2\left\{\|x_n-\overline{x}\|^2-\|x_n-u_n\|^2+2r_n\|x_n-u_n\|\|(\Psi_2+\Phi_2)x_n-(\Psi_2+\Phi_2)\overline{x}\|\right\}+c_n,
\end{aligned} \tag{3.59}$$

which in turn implies that

$$
\begin{aligned}
&(1-\epsilon_n-\epsilon_n\mu\overline{\gamma})^2\|x_n-u_n\|^2\\
&\quad\le(1-\epsilon_n-\epsilon_n\mu\overline{\gamma})^2\|x_n-\overline{x}\|^2-\|x_{n+1}-\overline{x}\|^2\\
&\qquad+2r_n(1-\epsilon_n-\epsilon_n\mu\overline{\gamma})^2\|x_n-u_n\|\|(\Psi_2+\Phi_2)x_n-(\Psi_2+\Phi_2)\overline{x}\|+c_n\\
&\quad\le\|x_n-\overline{x}\|^2-\|x_{n+1}-\overline{x}\|^2+\epsilon_n^2(1+\mu\overline{\gamma})^2\|x_n-\overline{x}\|^2\\
&\qquad+2r_n(1-\epsilon_n-\epsilon_n\mu\overline{\gamma})^2\|x_n-u_n\|\|(\Psi_2+\Phi_2)x_n-(\Psi_2+\Phi_2)\overline{x}\|+c_n\\
&\quad\le(\|x_n-\overline{x}\|+\|x_{n+1}-\overline{x}\|)\|x_{n+1}-x_n\|+\epsilon_n^2(1+\mu\overline{\gamma})^2\|x_n-\overline{x}\|^2\\
&\qquad+2r_n(1-\epsilon_n-\epsilon_n\mu\overline{\gamma})^2\|x_n-u_n\|\|(\Psi_2+\Phi_2)x_n-(\Psi_2+\Phi_2)\overline{x}\|+c_n.
\end{aligned}
\tag{3.60}
$$

Since $\epsilon_n \to 0$, $\|\Psi_2x_n-\Psi_2\overline{x}\| \to 0$, $\|\Phi_2x_n-\Phi_2\overline{x}\| \to 0$ and from (3.32), we obtain that

$$\lim_{n\to\infty}\|x_n-u_n\|=0. \tag{3.61}$$

Notice that

$$\|S_nz_n-z_n\|\le\|S_nz_n-x_n\|+\|x_n-u_n\|+\|u_n-v_n\|+\left\|v_n-y_n\right\|+\left\|y_n-z_n\right\|. \tag{3.62}$$

It follows from (3.35), (3.48), (3.52), (3.57), and (3.61) that

$$\lim_{n\to\infty}\|S_nz_n-z_n\|=0. \tag{3.63}$$

Moreover, we note that

$$
\begin{aligned}
\|Sz_n-z_n\|&\le\|Sz_n-S_nz_n\|+\|S_nz_n-z_n\|\\
&\le\sup_{x\in\widetilde{K}}\|Sx-S_nx\|+\|S_nz_n-z_n\|,
\end{aligned}
\tag{3.64}
$$

where $\widetilde{K}$ is any bounded subset of C. From Lemma 2.2 and (3.63), we obtain that

$$\lim_{n\to\infty}\|Sz_n-z_n\|=0. \tag{3.65}$$

Next, we show that

$$\limsup_{n\to\infty}\langle u+\gamma f(x^*)-(I+\mu A)x^*,x_n-x^*\rangle\le 0, \tag{3.66}$$

where x^* is a solution of optimization problem (3.4). To show this, we choose a subsequence $\{z_{n_j}\}$ of $\{z_n\}$ such that

$$\begin{aligned}\limsup_{n\to\infty}&\langle u+\gamma f(x^*)-(I+\mu A)x^*, z_n-x^*\rangle\\ &=\lim_{j\to\infty}\left\langle u+\gamma f(x^*)-(I+\mu A)x^*, z_{n_j}-x^*\right\rangle.\end{aligned}\tag{3.67}$$

Since $\{z_n\}$ is bounded, there exists a subsequence $\{z_{n_j}\}$ of $\{z_n\}$ such that $z_{n_j}\rightharpoonup\omega$ as $j\to\infty$. On the other hand, we note that

$$\|x_n-z_n\|\le\|x_n-u_n\|+\|u_n-v_n\|+\|v_n-y_n\|+\|y_n-z_n\|.\tag{3.68}$$

It follows from (3.48), (3.52), (3.57), and (3.61) that

$$\lim_{n\to\infty}\|x_n-z_n\|=0.\tag{3.69}$$

Now, we show that $\omega\in\Omega:=\bigcap_{i=1}^{\infty}F(T_i)\cap\mathrm{GMEP}(\Theta_1,\varphi_1,\Psi_1+\Phi_1)\cap\mathrm{GMEP}(\Theta_2,\varphi_2,\Psi_2+\Phi_2)\cap(B_1+W_1)^{-1}(0)\cap(B_2+W_2)^{-1}(0)$.

(i) First, we show that $\omega\in\bigcap_{i=1}^{\infty}F(T_i)$. By Lemma 2.2, we have $F(S)=\bigcap_{i=1}^{\infty}F(T_i)$. Suppose the contrary, $\omega\neq S\omega$, from (3.65) and Opial's condition (see [63]), we have

$$\begin{aligned}\liminf_{n\to\infty}\left\|z_{n_j}-\omega\right\|&<\liminf_{n\to\infty}\left\|z_{n_j}-S\omega\right\|\\ &\le\liminf_{n\to\infty}\left\{\left\|z_{n_j}-Sz_{n_j}\right\|+\left\|Sz_{n_j}-S\omega\right\|\right\}\\ &\le\liminf_{n\to\infty}\left\|z_{n_j}-\omega\right\|,\end{aligned}\tag{3.70}$$

which is a contradiction. So, we obtain $\omega\in F(S)=\bigcap_{i=1}^{\infty}F(T_i)$.

(ii) Now, we show that $\omega\in\mathrm{GMEP}(\Theta_1,\varphi_1,\Psi_1+\Phi_1)$. From $v_n=V_{\mu_n}^{(\Theta_1,\varphi_1)}(u_n-\mu_n(\Psi_1+\Phi_1)u_n)$, we know that

$$\begin{aligned}\Theta_1(v_n,y)&+\varphi_1(y)-\varphi_1(v_n)+\langle(\Psi_1+\Phi_1)u_n,y-v_n\rangle\\ &+\frac{1}{\mu_n}\langle K_1'(v_n)-K_1'(u_n),y-v_n\rangle\ge 0,\quad\forall y\in C.\end{aligned}\tag{3.71}$$

From (H2), we have

$$\begin{aligned}\varphi_1(y)&-\varphi_1(v_n)+\langle(\Psi_1+\Phi_1)u_n,y-v_n\rangle\\ &+\frac{1}{\mu_n}\langle K_1'(v_n)-K_1'(u_n),y-v_n\rangle\ge\Theta_1(y,v_n),\quad\forall y\in C.\end{aligned}\tag{3.72}$$

Now, replacing n by n_j in (3.72), we have

$$\varphi_1(y) - \varphi_1\left(v_{n_j}\right) + \left\langle (\Psi_1 + \Phi_1)u_{n_j}, y - v_{n_j} \right\rangle + \frac{1}{\mu_{n_j}} \left\langle K_1'\left(v_{n_j}\right) - K_1'\left(u_{n_j}\right), y - v_{n_j} \right\rangle \geq \Theta_1\left(y, v_{n_j}\right), \quad \forall y \in C. \tag{3.73}$$

Put $v_t = ty + (1-t)\omega$ for all $t \in (0,1]$ and $y \in C$. Then, from (3.73), we have

$$\begin{aligned}
&\left\langle v_t - v_{n_j}, (\Psi_1 + \Phi_1)v_t \right\rangle \\
&\quad \geq \left\langle v_t - v_{n_j}, (\Psi_1 + \Phi_1)v_t \right\rangle - \varphi_1(v_t) + \varphi_1\left(v_{n_j}\right) - \left\langle v_t - v_{n_j}, (\Psi_1 + \Phi_1)u_{n_j} \right\rangle \\
&\qquad - \left\langle \frac{K_1'\left(v_{n_j}\right) - K_1'\left(u_{n_j}\right)}{\mu_{n_j}}, v_t - v_{n_j} \right\rangle + \Theta_1\left(v_t, v_{n_j}\right) \\
&\quad = \left\langle v_t - v_{n_j}, (\Psi_1 + \Phi_1)v_t - (\Psi_1 + \Phi_1)v_{n_j} \right\rangle + \left\langle v_t - v_{n_j}, (\Psi_1 + \Phi_1)v_{n_j} - (\Psi_1 + \Phi_1)u_{n_j} \right\rangle \\
&\qquad - \varphi_1(v_t) + \varphi_1\left(v_{n_j}\right) - \left\langle \frac{K_1'\left(v_{n_j}\right) - K_1'\left(u_{n_j}\right)}{\mu_{n_j}}, v_t - v_{n_j} \right\rangle + \Theta_1\left(v_t, v_{n_j}\right).
\end{aligned} \tag{3.74}$$

Since $\|v_{n_j} - u_{n_j}\| \to 0$, we have $\|(\Psi_1+\Phi_1)v_{n_j} - (\Psi_1+\Phi_1)u_{n_j}\| \to 0$ and from Lipschitz continuity of K_1', we have $\|K_1'(v_{n_j}) - K_1'(u_{n_j})\| \to 0$. Further, from the monotonicity of $\Psi_1 + \Phi_1$, we have

$$\left\langle v_t - v_{n_j}, (\Psi_1 + \Phi_1)v_t - (\Psi_1 + \Phi_1)v_{n_j} \right\rangle \geq 0. \tag{3.75}$$

Since $x_{n_j} \rightharpoonup \omega$, $\|x_{n_j} - u_{n_j}\| \to 0$ and $\|v_{n_j} - u_{n_j}\| \to 0$, we have $v_{n_j} \rightharpoonup \omega$. Then, from (H4), (H5), the weakly lower semicontinuous of φ_1, and $(K_1'(v_{n_j}) - K'(u_{n_j}))/\mu_{n_j} \to 0$, we obtain that

$$\langle v_t - \omega, (\Psi_1 + \Phi_1)v_t \rangle \geq -\varphi_1(v_t) + \varphi_1(\omega) + \Theta_1(v_t, \omega), \quad \text{as } j \longrightarrow \infty. \tag{3.76}$$

From (H1), (H4), and (3.76), we also have

$$\begin{aligned}
0 &= \Theta_1(v_t, v_t) + \varphi_1(v_t) - \varphi_1(v_t) \\
&\leq t\Theta_1(v_t, y) + (1-t)\Theta_1(v_t, \omega) + t\varphi_1(y) + (1-t)\varphi_1(\omega) - \varphi_1(v_t) \\
&= t\left[\Theta_1(v_t, y) + \varphi_1(y) - \varphi_1(v_t)\right] + (1-t)\left[\Theta_1(v_t, \omega) + \varphi_1(\omega) - \varphi_1(v_t)\right] \\
&\leq t\left[\Theta_1(v_t, y) + \varphi_1(y) - \varphi_1(v_t)\right] + (1-t)\langle v_t - \omega, (\Psi_1 + \Phi_1)v_t \rangle \\
&= t\left[\Theta_1(v_t, y) + \varphi_1(y) - \varphi_1(v_t)\right] + (1-t)t\langle y - \omega, (\Psi_1 + \Phi_1)v_t \rangle,
\end{aligned} \tag{3.77}$$

and hence

$$0 \leq \Theta_1(v_t, y) + \varphi_1(y) - \varphi_1(v_t) + (1-t)\langle y - \omega, (\Psi_1 + \Phi_1)v_t \rangle. \tag{3.78}$$

Letting $t \to 0$, we have, for all $y \in C$,

$$0 \leq \Theta_1(\omega, y) + \varphi_1(y) - \varphi_1(\omega) + \langle (\Psi_1 + \Phi_1)\omega, y - \omega \rangle. \tag{3.79}$$

This implies that $\omega \in \mathrm{GMEP}(\Theta_1, \varphi_1, \Psi_1 + \Phi_1)$. In a similar way, we can get $\omega \in \mathrm{GMEP}(\Theta_2, \varphi_2, \Psi_2 + \Phi_2)$.

(iii) Now, we show that $\omega \in (B_1 + W_1)^{-1}(0)$. In fact, notice that

$$y_n - s_n B_1 y_n \in z_n + s_n W_1 z_n. \tag{3.80}$$

Let $\rho \in W_1 \eta$. Since W_1 is monotone, we have

$$\left\langle \frac{y_n - z_n}{s_n} - B_1 y_n - \rho, z_n - \eta \right\rangle \geq 0. \tag{3.81}$$

It follows from the condition (C8) and (3.48) that

$$\langle -B_1\omega - \rho, \omega - \eta \rangle \geq 0. \tag{3.82}$$

This implies that $-B_1\omega \in W_1\omega$, that is $\omega \in (B_1 + W_1)^{-1}(0)$. In a similar way, we can get $\omega \in (B_2 + W_2)^{-1}(0)$. Therefore $\omega \in \Omega := \bigcap_{i=1}^{\infty} F(T_i) \cap \mathrm{GMEP}(\Theta_1, \varphi_1, \Psi_1 + \Phi_1) \cap \mathrm{GMEP}(\Theta_2, \varphi_2, \Psi_2 + \Phi_2) \cap (B_1 + W_1)^{-1}(0) \cap (B_2 + W_2)^{-1}(0)$. Now, from Lemma 2.11, (3.67), and (3.69), we obtain that

$$\begin{aligned}
&\limsup_{n \to \infty} \langle u + \gamma f(x^*) - (I + \mu A)x^*, x_n - x^* \rangle \\
&\quad = \limsup_{n \to \infty} \langle u + \gamma f(x^*) - (I + \mu A)x^*, z_n - x^* \rangle \\
&\quad = \lim_{j \to \infty} \left\langle u + \gamma f(x^*) - (I + \mu A)x^*, z_{n_j} - x^* \right\rangle \\
&\quad = \langle u + \gamma f(x^*) - (I + \mu A)x^*, \omega - x^* \rangle \\
&\quad \leq 0.
\end{aligned} \tag{3.83}$$

Finally, we show that $x_n \to x^*$ as $n \to \infty$. Again, from Lemma 2.4 (ii), we compute

$$\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
&\quad = \left\|\epsilon_n\left(u + \gamma f(S_n v_n) - (I + \mu A)x^*\right) + \beta_n(x_n - x^*) + \left((1-\beta_n)I - \epsilon_n(I + \mu A)\right)(S_n z_n - x^*)\right\|^2 \\
&\quad \le \left\|\left((1-\beta_n)I - \epsilon_n(I + \mu A)\right)(S_n z_n - x^*) + \beta_n(x_n - x^*)\right\|^2 \\
&\qquad + 2\epsilon_n\left\langle u + \gamma f(S_n v_n) - (I + \mu A)x^*, x_{n+1} - x^*\right\rangle \\
&\quad \le \left[\left\|\left((1-\beta_n)I - \epsilon_n(I + \mu A)\right)(S_n z_n - x^*)\right\| + \beta_n\|x_n - x^*\|\right]^2 \\
&\qquad + 2\epsilon_n\gamma\left\langle f(S_n v_n) - f(x^*), x_{n+1} - x^*\right\rangle + 2\epsilon_n\left\langle u + \gamma f(x^*) - (I + \mu A)x^*, x_{n+1} - x^*\right\rangle \\
&\quad \le \left[\left(1 - \beta_n - \epsilon_n(1+\mu)\overline{\gamma}\right)\|z_n - x^*\| + \beta_n\|x_n - x^*\|\right]^2 + 2\epsilon_n\gamma\alpha\|S_n v_n - x^*\|\|x_{n+1} - x^*\| \\
&\qquad + 2\epsilon_n\left\langle u + \gamma f(x^*) - (I + \mu A)x^*, x_{n+1} - x^*\right\rangle \\
&\quad \le \left[\left(1 - \beta_n - \epsilon_n(1+\mu)\overline{\gamma}\right)\|x_n - x^*\| + \beta_n\|x_n - x^*\|\right]^2 + 2\epsilon_n\gamma\alpha\|x_n - x^*\|\|x_{n+1} - x^*\| \\
&\qquad + 2\epsilon_n\left\langle u + \gamma f(x^*) - (I + \mu A)x^*, x_{n+1} - x^*\right\rangle \\
&\quad \le \left(1 - \epsilon_n(1+\mu)\overline{\gamma}\right)^2\|x_n - x^*\|^2 + \epsilon_n\gamma\alpha\left(\|x_n - x^*\|^2 + \|x_{n+1} - x^*\|^2\right) \\
&\qquad + 2\epsilon_n\left\langle u + \gamma f(x^*) - (I + \mu A)x^*, x_{n+1} - x^*\right\rangle \\
&\quad = \left(1 - 2\epsilon_n(1+\mu)\overline{\gamma} + \epsilon_n^2\left[(1+\mu)\overline{\gamma}\right]^2 + \epsilon_n\gamma\alpha\right)\|x_n - x^*\|^2 + \epsilon_n\gamma\alpha\|x_{n+1} - x^*\|^2 \\
&\qquad + 2\epsilon_n\left\langle u + \gamma f(x^*) - (I + \mu A)x^*, x_{n+1} - x^*\right\rangle.
\end{aligned} \tag{3.84}$$

It follows that

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\le \frac{1 - 2\epsilon_n(1+\mu)\overline{\gamma} + \epsilon_n^2\left[(1+\mu)\overline{\gamma}\right]^2 + \epsilon_n\gamma\alpha}{1 - \epsilon_n\gamma\alpha}\|x_n - x^*\|^2 \\
&\quad + \frac{2\epsilon_n}{1 - \epsilon_n\gamma\alpha}\left\langle u + \gamma f(x^*) - (I + \mu A)x^*, x_{n+1} - x^*\right\rangle \\
&= \frac{1 - 2\epsilon_n(1+\mu)\overline{\gamma} + \epsilon_n\gamma\alpha}{1 - \epsilon_n\gamma\alpha}\|x_n - x^*\|^2 + \frac{\epsilon_n^2\left[(1+\mu)\overline{\gamma}\right]^2}{1 - \epsilon_n\gamma\alpha}\|x_n - x^*\|^2 \\
&\quad + \frac{2\epsilon_n}{1 - \epsilon_n\gamma\alpha}\left\langle u + \gamma f(x^*) - (I + \mu A)x^*, x_{n+1} - x^*\right\rangle \\
&\le \left[1 - \frac{2\left((1+\mu)\overline{\gamma} - \gamma\alpha\right)\epsilon_n}{1 - \epsilon_n\gamma\alpha}\right]\|x_n - x^*\|^2 + \frac{\epsilon_n^2\left[(1+\mu)\overline{\gamma}\right]^2}{1 - \epsilon_n\gamma\alpha}M_4 \\
&\quad + \frac{2\epsilon_n}{1 - \epsilon_n\gamma\alpha}\left\langle u + \gamma f(x^*) - (I + \mu A)x^*, x_{n+1} - x^*\right\rangle,
\end{aligned} \tag{3.85}$$

where $M_4 = \sup_{n\geq 1}\{\|x_n - x^*\|^2\}$. Put $\tau_n := 2((1+\mu)\overline{\gamma} - \gamma\alpha)\epsilon_n/(1-\epsilon_n\gamma\alpha)$ and

$$\delta_n := \frac{\epsilon_n^2\left[(1+\mu)\overline{\gamma}\right]^2}{1-\epsilon_n\gamma\alpha}M + \frac{2\epsilon_n}{1-\epsilon_n\gamma\alpha}\langle u + \gamma f(x^*) - (I+\mu A)x^*, x_{n+1} - x^*\rangle. \tag{3.86}$$

Then, (3.85) reduces to formula

$$\|x_{n+1} - x^*\|^2 \leq (1-\tau_n)\|x_n - x^*\|^2 + \delta_n. \tag{3.87}$$

It follows from the condition (C4) and (3.83) that $\sum_{n=1}^{\infty}\tau_n = \infty$ and

$$\begin{aligned}\limsup_{n\to\infty}\frac{\delta_n}{\tau_n} &= \limsup_{n\to\infty}\frac{1}{2((1+\mu)\overline{\gamma} - \gamma\alpha)} \\ &\quad\times\left[\epsilon_n\left[(1+\mu)\overline{\gamma}\right]^2 M_4 + 2\langle u + \gamma f(x^*) - (I+\mu A)x^*, x_{n+1} - x^*\rangle\right] \leq 0.\end{aligned} \tag{3.88}$$

Hence, by Lemma 2.12, we obtain that $x_n \to x^*$ as $n \to \infty$. This completes the proof. □

Remark 3.2. The control condition $\lim_{n\to\infty}|\mu_{n+1} - \mu_n| = 0$ and other on (C6)–(C9) are replaced by the *strictly weaker conditions*: $\lim_{n\to\infty}\mu_n/\mu_{n+1} = 1$ as shown in the next example.

Example 3.3. (a) If $\lim_{n\to\infty}|\mu_{n+1} - \mu_n| = 0$, then $\lim_{n\to\infty}\mu_n/\mu_{n+1} = 1$.

(b) The converse of (a) is not true.

Proof. (a) Since $0 < \mu_n < \min\{\widetilde{\beta}_1, \widetilde{\gamma}_1\}$ and $\liminf_{n\to\infty}\mu_n > 0$, there exists a constant $\widetilde{\mu}$ such that $\mu_n \geq \widetilde{\mu} > 0$ for all $n \in \mathbb{N}$. We observe that

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{\mu_n}{\mu_{n+1}} - 1\right| &= \lim_{n\to\infty}\left|\frac{\mu_n - \mu_{n+1}}{\mu_{n+1}}\right| \\ &\leq \lim_{n\to\infty}\frac{|\mu_{n+1} - \mu_n|}{\widetilde{\mu}} \\ &= 0.\end{aligned} \tag{3.89}$$

Hence, we obtain that $\lim_{n\to\infty}\mu_n/\mu_{n+1} = 1$.

(b) Let $\mu_n = ((n+1)/n)^n$, then $\mu_{n+1} = ((n+2)/(n+1))^{n+1}$. We see that

$$\begin{aligned}\lim_{n\to\infty}\frac{\mu_n}{\mu_{n+1}} &= \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n\cdot\frac{(1+1/n)^{n+1}}{(1+2/n)^{n+1}} \\ &= e\cdot\frac{e}{e^2} = 1,\end{aligned} \tag{3.90}$$

but

$$\lim_{n\to\infty}\left|\mu_{n+1}-\mu_n\right| = \lim_{n\to\infty}\left|\frac{(1+1/n)^{n+1}}{(1+2/n)^{n+1}} - \left(1+\frac{1}{n}\right)^n\right| = e-\frac{1}{e} \neq 0. \tag{3.91}$$

Then, converse of (a) is not true. Hence, (b) is proved. □

Remark 3.4. Theorem 3.1 improves and extends [45, Theorem 3.3] in the following respects.

(1) The problem of finding the common element $\bigcap_{i=1}^{\infty} F(T_i) \cap \mathrm{GMEP}(\Theta_1, \varphi_1, \Psi_1 + \Phi_1) \cap \mathrm{GMEP}(\Theta_2, \varphi_2, \Psi_2 + \Phi_2) \cap (B_1 + W_1)^{-1}(0) \cap (B_2 + W_2)^{-1}(0)$ is more general and more complex than the one of finding the common element $F(T) \cap \mathrm{GMEP}(\Theta_1, \varphi, \Psi_1) \cap \mathrm{GMEP}(\Theta_2, \varphi, \Psi_2) \cap \mathrm{VI}(C, B)$ in [44, Theorem 3.3].

(2) In [44, Algorithm 3.13], the function $K_i : C \to \mathbb{R}$ is chosen as $K_i(x) = \|x\|^2/2$ for $i = 1, 2$.

(3) The conditions (C3) and (C4) of Chantarangsi et al. [44, Theorem 3.3] are replaced by the strictly weaker conditions.

Remark 3.5. Theorem 3.1 improves and extends [45, Theorem 2.1] in the following respects.

(1) Theorem 3.1 extended [64, Theorem 2.1] to finding common element of the set of solutions of a system of generalized mixed equilibrium problem, and the set of infinite family of nonexpansive mapping involves strongly positive linear bounded operator and the optimization problem.

(2) The conditions (C3) and (C4) of Yu and Liang [45, Theorem 2.1] are replaced by the strictly weaker conditions.

4. Some Applications

Let H be a real Hilbert space and $g : H \to (-\infty, +\infty]$ a proper convex lower semicontinuous function. Then, the *subdifferential* ∂g of g is defined as follows:

$$\partial g(x) = \{y \in H : g(z) \geq g(x) + \langle z - x, y\rangle,\ \forall z \in H\}, \quad \forall x \in H. \tag{4.1}$$

From Rockafellar [65], we know that ∂g is maximal monotone. Let C be a closed and convex subset of H, and let δ_C be the indicator function of C, that is,

$$\delta_C(x) = \begin{cases} 0, & x \in C, \\ +\infty, & x \notin C. \end{cases} \tag{4.2}$$

Since δ_C is a proper lower semicontinuous convex function on H, the subdifferential $\partial\delta_C$ of δ_C is a maximal monotone operator.

Lemma 4.1 (see [57]). *Let C be a nonempty closed convex subset of a real Hilbert space H, let P_C be the metric projection from H onto C, and let $\partial\delta_C$ be the subdifferential of δ_C, where δ_C is as defined in* (4.2) *and $J_r = (I + r\partial\delta_C)^{-1}$. Then,*

$$y = J_r x \Longleftrightarrow y = P_C x, \quad \forall x \in H,\ y \in C. \tag{4.3}$$

Now, we consider the existence of solution of the variational inequality (1.6).

Theorem 4.2. *Let C be a nonempty closed convex subset of a Hilbert space H. Let $\varphi_1, \varphi_2 : C \to \mathbb{R}$ be two lower semicontinuous and convex functionals, and let $\Theta_1, \Theta_2 : C \times C \to \mathbb{R}$ be two bifunctions satisfying conditions (H1)–(H5). Let $\Psi_1, \Psi_2, \Phi_1, \Phi_2 : C \to H$ be $\widetilde{\beta}_1$-inverse-strongly monotone mapping, $\widetilde{\beta}_2$-inverse-strongly monotone mapping, $\widetilde{\gamma}_1$-inverse-strongly monotone mapping, and $\widetilde{\gamma}_2$-inverse-strongly monotone mapping, respectively, and let $B_1, B_2 : C \to H$ be $\widetilde{\xi}_1$-inverse-strongly monotone mapping and $\widetilde{\xi}_2$-inverse-strongly monotone mapping, respectively. Let $\{T_i\}_{i=1}^{\infty} : C \to C$ be an infinite family of nonexpansive mappings, and let $\alpha_i = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, where $I = [0,1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j + \alpha_2^j \le b < 1$, and $\alpha_1^j, \alpha_2^j, \alpha_3^j \in (0,1)$ for all $j = 1, 2, \ldots$. For all $n \in \mathbb{N}$, let S_n and S be S-mappings generated by $T_n, T_{n-1}, \ldots, T_1$ and $\alpha_n, \alpha_{n-1}, \ldots, \alpha_1$ and $T_n, T_{n-1}, \ldots$ and $\alpha_n, \alpha_{n-1}, \ldots$, respectively. Assume that*

$$\begin{aligned}\Omega := \bigcap_{i=1}^{\infty} F(T_i) &\cap \mathrm{GMEP}(\Theta_1, \varphi_1, \Psi_1 + \Phi_1) \\ &\cap \mathrm{GMEP}(\Theta_2, \varphi_2, \Psi_2 + \Phi_2) \cap \mathrm{VI}(C, B_1) \cap \mathrm{VI}(C, B_2) \ne \emptyset.\end{aligned} \tag{4.4}$$

Let $f : C \to H$ be a contraction mapping with a coefficient $\alpha \in (0,1)$, and let $A : C \to H$ be a strongly positive linear bounded operator with a coefficient $\overline{\gamma} \in (0,1)$. Let $\mu > 0$ and $\gamma > 0$ be two constants such that $0 < \gamma < (1+\mu)\overline{\gamma}/\alpha$. Let $\{x_n\}$ be a sequence defined by $u, x_1 \in H$ and

$$\begin{aligned} u_n &= V_{r_n}^{(\Theta_2,\varphi_2)}(x_n - r_n(\Psi_2 + \Phi_2)x_n), \\ v_n &= V_{\mu_n}^{(\Theta_1,\varphi_1)}(u_n - \mu_n(\Psi_1 + \Phi_1)u_n), \\ y_n &= P_C(v_n - t_n B_2 v_n), \\ x_{n+1} &= \epsilon_n(u + \gamma f(S_n v_n)) + \beta_n x_n + ((1-\beta_n)I - \epsilon_n(I + \mu A))S_n P_C(y_n - s_n B_1 y_n), \quad \forall n \ge 1, \end{aligned} \tag{4.5}$$

where $\{s_n\} \subset (0, 2\widetilde{\xi}_1)$, $\{t_n\} \subset (0, 2\widetilde{\xi}_2)$, $\{\mu_n\} \subset (0, \min\{\widetilde{\beta}_1, \widetilde{\gamma}_1\})$, $\{r_n\} \subset (0, \min\{\widetilde{\beta}_2, \widetilde{\gamma}_2\})$, and $\{\epsilon_n\}$ and $\{\beta_n\} \subset (0,1)$. Assume the following conditions are satisfied:

(C1) *for all $i = 1, 2, K_i : C \to \mathbb{R}$ is strongly convex with constant $\sigma_i > 0$ and its derivative K_i' is Lipschitz continuous with constant $\nu_i > 0$ such that the function $x \mapsto \langle y - x, K_i'(x) \rangle$ is weakly upper semicontinuous for each $y \in C$,*

(C2) *for all $i = 1, 2$ and for each $x \in C$, there exist a bounded subset $D_x \subset C$ and $z_x \in C$ such that, for any $y \notin D_x$,*

$$\Theta_i(y, z_x) + \varphi_i(z_x) - \varphi_i(y) + \frac{1}{r_i}\langle K_i'(y) - K_i'(x), z_x - y \rangle < 0, \tag{4.6}$$

(C3) $\lim_{n\to\infty}\alpha_n^1 = 0$,

(C4) $\lim_{n\to\infty}\epsilon_n = 0$ *and* $\sum_{n=1}^{\infty}\epsilon_n = \infty$,

(C5) $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$,

(C6) $0 < \liminf_{n\to\infty}\mu_n \le \limsup_{n\to\infty}\mu_n < \min\{\widetilde{\beta}_1, \widetilde{\gamma}_1\}$ *and* $\lim_{n\to\infty}\mu_n/\mu_{n+1} = 1$,

(C7) $0 < \liminf_{n\to\infty}r_n \le \limsup_{n\to\infty}r_n < \min\{\widetilde{\beta}_2, \widetilde{\gamma}_2\}$ *and* $\lim_{n\to\infty}r_n/r_{n+1} = 1$,

(C8) $0 < \liminf_{n\to\infty}s_n \le \limsup_{n\to\infty}s_n < 2\widetilde{\xi}_1$ *and* $\lim_{n\to\infty}s_n/s_{n+1} = 1$,

(C9) $0 < \liminf_{n\to\infty}t_n \le \limsup_{n\to\infty}t_n < 2\widetilde{\xi}_2$ *and* $\lim_{n\to\infty}t_n/t_{n+1} = 1$.

Then, the sequence $\{x_n\}$ *defined by* (4.5) *converges strongly to* $x^* \in \Omega$, *provided* $V_{r,n}^{(\Theta,\varphi)}$ *is firmly nonexpansive, where* x^* *solves the following optimization problem:*

$$\min_{x\in\Omega} \frac{\mu}{2}\langle Ax, x\rangle + \frac{1}{2}\|x-u\|^2 - h(x). \tag{4.7}$$

Proof. Now, we show that $\mathrm{VI}(C, B_1) = (B_1 + \partial\delta_C)^{-1}(0)$ and $\mathrm{VI}(C, B_2) = (B_2 + \partial\delta_C)^{-1}(0)$, respectively. Set $W_1 = W_2 = \partial\delta_C$ in Theorem 3.1. Notice that

$$\begin{aligned} x^* \in (B_1 + \partial\delta_C)^{-1}(0) &\Longleftrightarrow 0 \in B_1x^* + \partial\delta_C x^* \\ &\Longleftrightarrow -B_1x^* \in \partial\delta_C x^* \\ &\Longleftrightarrow \langle B_1x^*, y - x^*\rangle \ge 0 \\ &\Longleftrightarrow x^* \in \mathrm{VI}(C, B_1). \end{aligned} \tag{4.8}$$

In a similar way, we can get

$$x^* \in (B_2 + \partial\delta_C)^{-1}(0) \Longleftrightarrow x^* \in \mathrm{VI}(C, B_2). \tag{4.9}$$

From Lemma 4.1, we can conclude the desired conclusion immediately. This completes the proof. □

Let $\widehat{T} : C \to C$ be k-strict pseudocontraction mapping. Setting $B = I - \widehat{T}$, we see that, for all $x, y \in C$,

$$\|(I-B)x - (I-B)y\|^2 \le \|x-y\|^2 + k\|Bx - By\|^2. \tag{4.10}$$

On the other hand, we note that

$$\|(I-B)x - (I-B)y\|^2 \le \|x-y\|^2 - 2\langle Bx - By, x - y\rangle + \|Bx - By\|^2. \tag{4.11}$$

For all $x, y \in C$, we obtain that

$$\langle Bx - By, x - y \rangle \geq \frac{1-k}{2} \|Bx - By\|^2. \tag{4.12}$$

Then, B is $((1-k)/2)$-inverse-strongly monotone mapping.

Theorem 4.3. *Let C be a nonempty closed convex subset of a Hilbert space H. Let $\varphi_1, \varphi_2 : C \to \mathbb{R}$ be two lower semicontinuous and convex functionals, and let $\Theta_1, \Theta_2 : C \times C \to \mathbb{R}$ be two bifunctions satisfying conditions (H1)–(H5). Let $\Psi_1, \Psi_2, \Phi_1, \Phi_2 : C \to H$ be $\widetilde{\beta}_1$-inverse-strongly monotone mapping, $\widetilde{\beta}_2$-inverse-strongly monotone mapping, $\widetilde{\gamma}_1$-inverse-strongly monotone mapping, and $\widetilde{\gamma}_2$-inverse-strongly monotone mapping, respectively, and let $B_1, B_2 : C \to H$ be $\widetilde{\xi}_1$-inverse-strongly monotone mapping and $\widetilde{\xi}_2$-inverse-strongly monotone mapping, respectively. Let $\{T_i\}_{i=1}^{\infty} : C \to C$ be an infinite family of nonexpansive mappings, and let $\alpha_i = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, where $I = [0,1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j + \alpha_2^j \leq b < 1$, and $\alpha_1^j, \alpha_2^j, \alpha_3^j \in (0,1)$ for all $j = 1, 2, \ldots$. For all $n \in \mathbb{N}$, let S_n and S be S-mappings generated by $T_n, T_{n-1}, \ldots, T_1$ and $\alpha_n, \alpha_{n-1}, \ldots, \alpha_1$ and $T_n, T_{n-1}, \ldots$ and $\alpha_n, \alpha_{n-1}, \ldots$, respectively. Let $\widehat{T}_i : C \to C$ be a k_i-pseudocontraction mapping for all $i = 1, 2$. Assume that*

$$\Omega := \bigcap_{i=1}^{\infty} F(T_i) \cap \mathrm{GMEP}(\Theta_1, \varphi_1, \Psi_1 + \Phi_1) \cap \mathrm{GMEP}(\Theta_2, \varphi_2, \Psi_2 + \Phi_2) \cap F\left(\widehat{T}_1\right) \cap F\left(\widehat{T}_2\right) \neq \emptyset. \tag{4.13}$$

Let $f : C \to H$ be a contraction mapping with a coefficient $\alpha \in (0,1)$, and let $A : C \to H$ be a strongly positive linear bounded operator with a coefficient $\overline{\gamma} \in (0,1)$. Let $\mu > 0$ and $\gamma > 0$ be two constants such that $0 < \gamma < (1+\mu)\overline{\gamma}/\alpha$. Let $\{x_n\}$ be a sequence defined by $u, x_1 \in H$ and

$$\begin{aligned}
u_n &= V_{r_n}^{(\Theta_2, \varphi_2)}(x_n - r_n(\Psi_2 + \Phi_2)x_n), \\
v_n &= V_{\mu_n}^{(\Theta_1, \varphi_1)}(u_n - \mu_n(\Psi_1 + \Phi_1)u_n), \\
y_n &= (1 - t_n)v_n + t_n \widehat{T}_2 v_n, \\
x_{n+1} &= \epsilon_n(u + \gamma f(S_n v_n)) + \beta_n x_n + ((1-\beta_n)I - \epsilon_n(I + \mu A)) \\
&\quad \times S_n\left[(1 - s_n)y_n + s_n \widehat{T}_1 y_n\right], \quad \forall n \geq 1,
\end{aligned} \tag{4.14}$$

where $\{s_n\} \subset (0, 2\widetilde{\xi}_1)$, $\{t_n\} \subset (0, 2\widetilde{\xi}_2)$, $\{\mu_n\} \subset (0, \min\{\widetilde{\beta}_1, \widetilde{\gamma}_1\})$, $\{r_n\} \subset (0, \min\{\widetilde{\beta}_2, \widetilde{\gamma}_2\})$, and $\{\epsilon_n\}$ and $\{\beta_n\} \subset (0,1)$. Assume the conditions (C1)– (C9) in Theorem 4.2. Then, the sequence $\{x_n\}$ defined by (4.14) converges strongly to $x^* \in \Omega := \bigcap_{i=1}^{\infty} F(T_i) \cap \mathrm{GMEP}(\Theta_1, \varphi_1, \Psi_1 + \Phi_1) \cap \mathrm{GMEP}(\Theta_2, \varphi_2, \Psi_2 + \Phi_2) \cap F(\widehat{T}_1) \cap F(\widehat{T}_2)$, provided $V_{r,n}^{(\Theta,\varphi)}$ is firmly nonexpansive, where x^* solves the following optimization problem:

$$\min_{x \in \Omega} \frac{\mu}{2}\langle Ax, x \rangle + \frac{1}{2}\|x - u\|^2 - h(x). \tag{4.15}$$

Proof. Taking $B_i = I - \widehat{T}_i : C \to H$, we see that $B_i : C \to H$ is λ_i-strict pseudocontraction mapping with $\lambda_i = (1 - k_i)/2$ and $F(\widehat{T}_i) = \text{VI}(C, B_i)$ for $i = 1, 2$. From Theorem 4.2, we can conclude the desired conclusion easily. This completes the proof. □

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