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Research Article

Tripled Fixed Point Results in Generalized Metric Spaces

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We establish a tripled fixed point result for a mixed monotone mapping satisfying nonlinear contractions in ordered generalized metric spaces. Also, some examples are given to support our result.

1. Introduction and Preliminaries

The study of fixed points of mappings satisfying certain contractive conditions has been at the center of rigorous research activity, see [1–3]. The notion of D-metric space is a generalization of usual metric spaces and it is introduced by Dhage [4–7]. Recently, Mustafa and Sims [8, 9] have shown that most of the results concerning Dhage's D-metric spaces are invalid. In [8, 9], they introduced an improved version of the generalized metric space structure which they called G-metric spaces. For more results on G-metric spaces, one can refer to the papers [10–26].

Now, we give some preliminaries and basic definitions which are used throughout the paper. In 2006, Mustafa and Sims [9] introduced the concept of G-metric spaces as follows.

Definition 1.1 (see [9]). Let X be a nonempty set, $G : X \times X \times X \to \mathbb{R}^+$ be a function satisfying the following properties:

(G1) $G(x, y, z) = 0$ if $x = y = z$,

(G2) $0 < G(x, x, y)$ for all $x, y \in X$ with $x \neq y$, (G3) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $y \neq z$,

(G4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$ (symmetry in all three variables),

(G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$ (rectangle inequality).

Then the function G is called a generalized metric or, more specially, a G-metric on X, and the pair (X, G) is called a G-metric space.

Every G-metric on X will define a metric d_G on X by

$$d_G(x, y) = G(x, y, y) + G(y, x, x), \quad \forall\, x, y \in X. \tag{1.1}$$

Example 1.2. Let (X, d) be a metric space. The function $G : X \times X \times X \to [0, +\infty)$, defined by

$$G(x, y, z) = \max\{d(x, y), d(y, z), d(z, x)\}, \tag{1.2}$$

or

$$G(x, y, z) = d(x, y) + d(y, z) + d(z, x), \tag{1.3}$$

for all $x, y, z \in X$, is a G-metric on X.

Definition 1.3 (see [9]). Let (X, G) be a G-metric space, and let (x_n) be a sequence of points of X; therefore, we say that (x_n) is G-convergent to $x \in X$ if $\lim_{n,m\to+\infty} G(x, x_n, x_m) = 0$, that is, for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x, x_n, x_m) < \varepsilon$, for all $n, m \geq N$. One calls x the limit of the sequence and writes $x_n \to x$ or $\lim_{n\to+\infty} x_n = x$.

Proposition 1.4 (see [9]). *Let (X, G) be a G-metric space. The following are equivalent:*

(1) *(x_n) is G-convergent to x,*

(2) *$G(x_n, x_n, x) \to 0$ as $n \to +\infty$,*

(3) *$G(x_n, x, x) \to 0$ as $n \to +\infty$,*

(4) *$G(x_n, x_m, x) \to 0$ as $n, m \to +\infty$.*

Definition 1.5 (see [9]). Let (X, G) be a G-metric space. A sequence (x_n) is called a G-Cauchy sequence if, for any $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that $G(x_n, x_m, x_l) < \varepsilon$ for all $m, n, l \geq N$, that is, $G(x_n, x_m, x_l) \to 0$ as $n, m, l \to +\infty$.

Proposition 1.6 (see [9]). *Let (X, G) be a G-metric space. Then the following are equivalent:*

(1) *the sequence (x_n) is G-Cauchy,*

(2) *for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x_n, x_m, x_m) < \varepsilon$, for all $m, n \geq N$.*

Definition 1.7 (see [9]). A G-metric space (X, G) is called G-complete if every G-Cauchy sequence is G-convergent in (X, G).

Definition 1.8. Let (X, G) be a G-metric space. A mapping $F : X \times X \times X \to X$ is said to be continuous if for any three G-convergent sequences (x_n), (y_n), and (z_n) converging to x, y, and z, respectively, $(F(x_n, y_n, z_n))$ is G-convergent to $F(x, y, z)$.

Recently, Berinde and Borcut [27] introduced these definitions.

Definition 1.9. Let $(X, \leq)$ be a partially ordered set and $F : X \times X \times X \to X$. The mapping F is said to have the mixed monotone property if, for any $x, y, z \in X$,

$$\begin{aligned} &x_1, x_2 \in X, \quad x_1 \leq x_2 \Longrightarrow F(x_1, y, z) \leq F(x_2, y, z), \\ &y_1, y_2 \in X, \quad y_1 \leq y_2 \Longrightarrow F(x, y_1, z) \geq F(x, y_2, z), \\ &z_1, z_2 \in X, \quad z_1 \leq z_2 \Longrightarrow F(x, y, z_1) \leq F(x, y, z_2). \end{aligned} \tag{1.4}$$

Definition 1.10. Let $F : X \times X \times X \to X$. An element (x, y, z) is called a tripled fixed point of F if

$$F(x, y, z) = x, \qquad F(y, x, y) = y, \qquad F(z, y, x) = z. \tag{1.5}$$

Very recently, Berinde and Borcut [28] proved some tripled coincidence theorems for contractive type mappings in partially ordered metric spaces. Also, Samet and Vetro [29] introduced the notion of fixed point of N-order as natural extension of that of coupled fixed point and established some new coupled fixed point theorems in complete metric spaces, using a new concept of F-invariant set.

Berinde and Borcut [27] proved the following theorem.

Theorem 1.11. *Let $(X, \leq, d)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Suppose $F : X \times X \times X \to X$ such that F has the mixed monotone property and*

$$d(F(x, y, z), F(u, v, w)) \leq jd(x, u) + kd(y, v) + ld(z, w), \tag{1.6}$$

for any $x, y, z \in X$ for which $x \leq u$, $v \leq y$ and $z \leq w$. Suppose either F is continuous or X has the following properties:

(1) *if a nondecreasing sequence $x_n \to x$, then $x_n \leq x$ for all n,*

(2) *if a nonincreasing sequence $y_n \to y$, then $y \leq y_n$ for all n,*

(3) *if a nondecreasing sequence $z_n \to z$, then $z_n \leq z$ for all n.*

If there exist $x_0, y_0, z_0 \in X$ such that $x_0 \leq F(x_0, y_0, z_0)$, $y_0 \geq F(y_0, x_0, z_0)$, and $z_0 \leq F(z_0, y_0, x_0)$, then there exist $x, y, z \in X$ such that

$$F(x, y, z) = x, \qquad F(y, x, y) = y, \qquad F(z, y, x) = z, \tag{1.7}$$

that is, F has a tripled fixed point.

In this paper, we establish a tripled fixed point result for a mapping having a mixed monotone property in G-metric spaces. Also, we give some examples to illustrate our result.

2. Main Results

Let Φ be the set of all non-decreasing functions $\phi : [0, +\infty) \to [0, +\infty)$ such that $\lim_{n \to +\infty} \phi^n(t) = 0$ for all $t > 0$. If $\phi \in \Phi$, then following Matkowski [30], we have

(1) $\phi(t) < t$ for all $t > 0$,

(2) $\phi(0) = 0$.

The aim of this paper is to prove the following theorem.

Theorem 2.1. *Let $(X, \leq)$ be partially ordered set and (X, G) a G-metric space. Let $F : X^3 \to X$ be a continuous mapping having the mixed monotone property on X. Assume there exists $\phi \in \Phi$ such that for $x, y, z, a, b, c, u, v, w \in X$, with $x \geq a \geq u$, $y \leq b \leq v$, and $z \geq c \geq w$, one has*

$$G(F(x,y,z), F(a,b,c), F(u,v,w)) \leq \phi(\max\{G(x,a,u), G(y,b,v), G(z,c,w)\}). \tag{2.1}$$

If there exist $x_0, y_0, z_0 \in X$ such that $x_0 \leq F(x_0, y_0, z_0)$, $y_0 \geq F(y_0, x_0, y_0)$, and $z_0 \leq F(z_0, y_0, x_0)$, then F has a tripled fixed point in X, that is, there exist $x, y, z \in X$ such that

$$F(x,y,z) = x, \qquad F(y,x,y) = y, \qquad F(z,y,x) = z. \tag{2.2}$$

Proof. Suppose $x_0, y_0, z_0 \in X$ are such that $x_0 \leq F(x_0, y_0, z_0)$, $y_0 \geq F(y_0, x_0, y_0)$, and $z_0 \leq F(z_0, y_0, x_0)$. Define $x_1 = F(x_0, y_0, z_0)$, $y_1 = F(y_0, x_0, y_0)$, and $z_1 = F(z_0, y_0, x_0)$. Then $x_0 \leq x_1$, $y_0 \geq y_1$, and $z_0 \leq z_1$. Again, define $x_2 = F(x_1, y_1, z_1)$, $y_2 = F(y_1, x_1, y_1)$, and $z_2 = F(z_1, y_1, x_1)$. Since F has the mixed monotone property, we have $x_0 \leq x_1 \leq x_2$, $y_2 \leq y_1 \leq y_0$, and $z_0 \leq z_1 \leq z_2$. Continuing this process, we can construct three sequences (x_n), (y_n), and (z_n) in X such that

$$\begin{aligned}
x_n &= F(x_{n-1}, y_{n-1}, z_{n-1}) \leq x_{n+1} = F(x_n, y_n, z_n), \\
y_{n+1} &= F(y_n, x_n, y_n) \leq y_n = F(y_{n-1}, x_{n-1}, y_{n-1}), \\
z_n &= F(z_{n-1}, y_{n-1}, x_{n-1}) \leq z_{n+1} = F(z_n, y_n, x_n).
\end{aligned} \tag{2.3}$$

If, for some integer n, we have $(x_{n+1}, y_{n+1}, z_{n+1}) = (x_n, y_n, z_n)$, then $F(x_n, y_n, z_n) = x_n$, $F(y_n, x_n, y_n) = y_n$, and $F(z_n, y_n, x_n) = z_n$; that is, (x_n, y_n, z_n) is a tripled fixed point of F. Thus we will assume that $(x_{n+1}, y_{n+1}, z_{n+1}) \neq (x_n, y_n, z_n)$ for all $n \in \mathbb{N}$; that is, we assume that either $x_{n+1} \neq x_n$ or $y_{n+1} \neq y_n$ or $z_{n+1} \neq z_n$. For any $n \in \mathbb{N}^*$, we have from (2.1)

$$\begin{aligned}
&G(x_{n+1}, x_n, x_n) \\
&\quad := G(F(x_n, y_n, z_n), F(x_{n-1}, y_{n-1}, z_{n-1}), F(x_{n-1}, y_{n-1}, z_{n-1})) \\
&\quad \leq \phi(\max\{G(x_n, x_{n-1}, x_{n-1}), G(y_n, y_{n-1}, y_{n-1}), G(z_n, z_{n-1}, z_{n-1})\}), \\
&G(y_{n+1}, y_n, y_n) \\
&\quad := G(F(y_n, x_n, y_n), F(y_{n-1}, x_{n-1}, y_{n-1}), F(y_{n-1}, x_{n-1}, y_{n-1})) \\
&\quad \leq \phi(\max\{G(y_n, y_{n-1}, y_{n-1}), G(x_n, x_{n-1}, x_{n-1})\}) \\
&\quad \leq \phi(\max\{G(y_n, y_{n-1}, y_{n-1}), G(x_n, x_{n-1}, x_{n-1}), G(z_n, z_{n-1}, z_{n-1})\}), \\
&G(z_{n+1}, z_n, z_n) \\
&\quad := G(F(z_n, y_n, x_n), F(z_{n-1}, y_{n-1}, x_{n-1}), F(z_{n-1}, y_{n-1}, x_{n-1})) \\
&\quad \leq \phi(\max\{G(z_n, z_{n-1}, z_{n-1}), G(y_n, y_{n-1}, y_{n-1}), G(x_n, x_{n-1}, x_{n-1})\}).
\end{aligned} \tag{2.4}$$

From (2.4), it follows that

$$\begin{aligned}
&\max\{G(x_{n+1},x_n,x_n),G(y_n,y_n,y_{n+1}),G(z_{n+1},z_n,z_n)\}\\
&\quad\le \phi(\max\{G(x_n,x_{n-1},x_{n-1}),G(y_n,y_{n-1},y_{n-1}),G(z_n,z_{n-1},z_{n-1})\}).
\end{aligned}\tag{2.5}$$

By repeating (2.5) n-times and using the fact that ϕ is non-decreasing, we get that

$$\begin{aligned}
&\max\{G(x_{n+1},x_n,x_n),G(y_{n+1},y_n,y_n),G(z_{n+1},z_n,z_n)\}\\
&\quad\le \phi(\max\{G(x_n,x_{n-1},x_{n-1}),G(y_n,y_{n-1},y_{n-1}),G(z_n,z_{n-1},z_{n-1})\})\\
&\quad\le \phi^2(\max\{G(x_{n-1},x_{n-2},x_{n-2}),G(y_{n-1},y_{n-2},y_{n-2}),G(z_{n-1},z_{n-2},z_{n-2})\})\\
&\quad\ \vdots\\
&\quad\le \phi^n(\max\{G(x_1,x_0,x_0),G(y_1,y_0,y_0),G(z_1,z_0,z_0)\}).
\end{aligned}\tag{2.6}$$

Now, we shill show that (x_n) is a G-Cauchy sequence in X. Let $\epsilon > 0$. Since

$$\lim_{n\to+\infty}\phi^n(\max\{G(x_1,x_0,x_0),G(y_1,y_0,y_0),G(z_1,z_0,z_0)\}) = 0,\tag{2.7}$$

and $\epsilon > \phi(\epsilon)$, there exists $n_0 \in \mathbb{N}$ such that

$$\phi^n(\max\{G(x_1,x_0,x_0),G(y_1,y_0,y_0),G(z_1,z_0,z_0)\}) < \epsilon - \phi(\epsilon) \quad \forall n \ge n_0.\tag{2.8}$$

By (2.6), this implies that

$$\max\{G(x_{n+1},x_n,x_n),G(y_{n+1},y_n,y_n),G(z_{n+1},z_n,z_n)\} < \epsilon - \phi(\epsilon) \quad \forall n \ge n_0.\tag{2.9}$$

For $m, n \in \mathbb{N}$, we prove by induction on m that

$$\max\{G(x_n,x_n,x_m),G(y_n,y_n,y_m),G(z_n,z_n,z_m)\} < \epsilon \quad \forall m \ge n \ge n_0.\tag{2.10}$$

Since $\epsilon - \phi(\epsilon) \le \epsilon$, then by using (2.9) and the property (G4), we conclude that (2.10) holds when $m = n + 1$. Now suppose that (2.10) holds for $m = k$. For $m = k + 1$, we have

$$\begin{aligned}
&G(x_n,x_n,x_{k+1})\\
&\quad\le G(x_n,x_n,x_{n+1}) + G(x_{n+1},x_{n+1},x_{k+1})\\
&\quad< \epsilon - \phi(\epsilon) + G(F(x_n,y_n,z_n),F(x_n,y_n,z_n),F(x_k,y_k,z_k))\\
&\quad\le \epsilon - \phi(\epsilon) + \phi(\max\{G(x_n,x_n,x_k),G(y_n,y_n,y_k),G(z_n,z_n,z_k)\})\\
&\quad\le \epsilon - \phi(\epsilon) + \phi(\epsilon) = \epsilon.
\end{aligned}\tag{2.11}$$

Similarly, we show that

$$\begin{aligned}
G(y_n,y_n,y_{k+1}) &< \epsilon,\\
G(z_n,z_n,z_{k+1}) &< \epsilon.
\end{aligned}\tag{2.12}$$

Hence, we have

$$\max\{G(x_n, x_n, x_{k+1}), G(y_n, y_n, y_{k+1}), G(z_n, z_n, z_{k+1})\} < \epsilon. \tag{2.13}$$

Thus (2.10) holds for all $m \geq n \geq n_0$. Hence (x_n), (y_n), and (z_n) are G-Cauchy sequences in X. Since X is a G-complete metric space, there exist $x, y, z \in X$ such that (x_n), (y_n), and (z_n) converge to x, y, and z, respectively. Finally, we show that (x, y, z) is a tripled fixed point of F. Since F is continuous and $(x_n, y_n, z_n) \to (x, y, z)$, we have $x_{n+1} = F(x_n, y_n, z_n) \to F(x, y, z)$. By the uniqueness of limit, we get that $x = F(x, y, z)$. Similarly, we show that $y = F(y, x, y)$ and $z = F(z, y, x)$. So (x, y, z) is a tripled fixed point of F. □

Corollary 2.2. *Let $(X, \leq)$ be partially ordered set and (X, G) a G-metric space. Let $F : X^3 \to X$ be a continuous mapping having the mixed monotone property on X. Suppose that there exists $k \in [0, 1)$ such that for $x, y, z, a, b, c, u, v, w \in X$, with $x \geq a \geq u$, $y \leq b \leq v$, and $z \geq c \geq w$ one has*

$$G(F(x, y, z), F(a, b, c), F(u, v, w)) \leq k \max\{G(x, a, u), G(y, b, v), G(z, c, w)\}. \tag{2.14}$$

If there exist $x_0, y_0, z_0 \in X$ such that $x_0 \leq F(x_0, y_0, z_0)$, $y_0 \geq F(y_0, x_0, y_0)$, and $z_0 \leq F(z_0, y_0, x_0)$, then F has a tripled fixed point in X, that is, there exist $x, y, z \in X$ such that

$$F(x, y, z) = x, \qquad F(y, x, y) = y, \qquad F(z, y, x) = z. \tag{2.15}$$

Proof. It follows from Theorem 2.1 by taking $\phi(t) = kt$. □

Corollary 2.3. *Let $(X, \leq)$ be partially ordered set and (X, G) be a G-metric space.*

Let $F : X^3 \to X$ be a continuous mapping having the mixed monotone property on X. Suppose that there exists $k \in [0, 1)$ such that for $x, y, z, a, b, c, u, v, w \in X$, with $x \geq a \geq u$, $y \leq b \leq v$, and $z \geq c \geq w$ one has

$$G(F(x, y, z), F(a, b, c), F(u, v, w)) \leq \frac{k}{3}(G(x, a, u) + G(y, b, v) + G(z, c, w)). \tag{2.16}$$

If there exist $x_0, y_0, z_0 \in X$ such that $x_0 \leq F(x_0, y_0, z_0)$, $y_0 \geq F(y_0, x_0, y_0)$, and $z_0 \leq F(z_0, y_0, x_0)$, then F has a tripled fixed point in X, that is, there exist $x, y, z \in X$ such that

$$F(x, y, z) = x, \qquad F(y, x, y) = y, \qquad F(z, y, x) = z. \tag{2.17}$$

Proof. Note that

$$G(x, a, u) + G(y, b, v) + G(z, c, w) \leq 3 \max\{G(x, a, u), G(y, b, v), G(z, c, w)\}. \tag{2.18}$$

Then, the proof follows from Corollary 2.2. □

By adding an additional hypothesis, the continuity of F in Theorem 2.1 can be dropped.

Theorem 2.4. *Let $(X, \leq)$ be a partially ordered set and (X, d) a complete metric space. Let $F : X \times X \times X \to X$ be a mapping having the mixed monotone property. Assume that there exists $\phi \in \Phi$ such that*

$$G(F(x,y,z), F(a,b,c), F(u,v,w)) \leq \phi(\max\{G(x,a,u), G(y,b,v), G(z,c,w)\}) \tag{2.19}$$

for all $x, y, z, a, b, c, u, v, w \in X$ with $x \geq a \geq u$, $y \leq b \leq v$, and $z \geq c \geq w$. Assume also that X has the following properties:

(i) *if a nondecreasing sequence $x_n \to x$, then $x_n \leq x$ for all $n \in \mathbb{N}$,*

(ii) *if a nonincreasing sequence $y_n \to y$, then $y_n \geq y$ for all $n \in \mathbb{N}$.*

If there exist $x_0, y_0, z_0 \in X$ such that $x_0 \leq F(x_0, y_0, z_0)$, $y_0 \geq F(y_0, x_0, y_0)$, and $z_0 \leq F(z_0, y_0, x_0)$, then F has a tripled fixed point.

Proof. Following proof of Theorem 2.1 step by step, we construct three G-Cauchy sequences (x_n), (y_n), and (z_n) in X with

$$\begin{gathered} x_1 \leq x_2 \leq \cdots \leq x_n \leq \cdots, \\ y_1 \geq y_2 \geq \cdots \geq y_n \geq \cdots, \\ z_1 \leq z_2 \leq \cdots \leq z_n \leq \cdots \end{gathered} \tag{2.20}$$

such that $x_n \to x \in X$, $y_n \to y \in X$, and $z_n \to z \in X$. By the hypotheses on X, we have $x_n \leq x$, $y_n \geq y$, and $z_n \leq z$ for all $n \in \mathbb{N}$. If for some $n \geq 0$, $x_n = x$, $y_n = y$, and $z_n = z$, then

$$x = x_n \leq x_{n+1} \leq x = x_n, \qquad y = y_n \geq y_{n+1} \leq y = y_n, \qquad z = z_n \leq z_{n+1} \leq z = z_n, \tag{2.21}$$

which implies that $x_n = x_{n+1} = F(x_n, y_n, z_n)$, $y_n = y_{n+1} = F(y_n, x_n, y_n)$, and $z_n = z_{n+1} = F(z_n, y_n, x_n)$; that is, (x_n, y_n, z_n) is a tripled fixed point of F. Now, assume that, for all $n \geq 0$, $(x_n, y_n, z_n) \neq (x, y, z)$. Thus, for each $n \geq 0$,

$$\max\{G(x,x,x_n), G(y,y,y_n), G(z,z,z_n)\} > 0. \tag{2.22}$$

From (2.19), we have

$$\begin{aligned} &G(F(x,y,z), F(x,y,z), x_{n+1}) := G(F(x,y,z), F(x,y,z), F(x_n,y_n,z_n)) \\ &\qquad \leq \phi(\max\{G(x,x,x_n), G(y,y,y_n), G(z,z,z_n)\}), \\ &G(y_{n+1}, F(y,x,y), F(y,x,y)) := G(F(y_n,x_n,y_n), F(y,x,y), F(y,x,y)) \\ &\qquad \leq \phi(\max\{G(y_n,y,y), G(x_n,x,x)\}) \\ &G(F(z,y,x), F(z,y,x), z_{n+1}) := G(F(z,y,x), F(z,y,x), F(z_n,y_n,x_n)) \\ &\qquad \leq \phi(\max\{G(x,x,x_n), G(y,y,y_n), G(z,z,z_n)\}). \end{aligned} \tag{2.23}$$

Letting $n \to +\infty$ in (2.23) and using (2.22) in the fact that $\phi(t) < t$ for all $t > 0$, it follows that $x = F(x,y,z)$, $y = F(y,x,y)$, and $z = F(z,y,x)$. Hence (x,y,z) is a tripled fixed point of F. □

Now we give some examples illustrating our results.

Example 2.5. Take $X = [0, +\infty)$ endowed with the complete G-metric:

$$G(x,y,z) = \max\{|x-y|, |x-z|, |y-z|\}, \tag{2.24}$$

for all $x, y, z \in X$. Set $k = 1/2$ and $F : X^3 \to X$ defined by $F(x,y,z) = (1/6)x$. The mapping F has the mixed monotone property. We have

$$G(F(x,y,z), F(a,b,c), F(u,v,w)) = \frac{1}{6}G(x,a,u) \leq \frac{k}{3}\max\{G(x,a,u), G(y,b,v), G(z,c,w)\} \tag{2.25}$$

for all $x \geq a \geq u$, $y \leq b \leq v$, and $z \geq c \geq w$, that is, (2.14) holds. Take $x_0 = y_0 = z_0 = 0$, then all the hypotheses of Corollary 2.2 are verified, and $(0,0,0)$ is the unique tripled fixed point of F.

Example 2.6. As in Example 2.5, take $X = [0, +\infty)$ and

$$G(x,y,z) = \max\{|x-y|, |x-z|, |y-z|\}, \tag{2.26}$$

for all $x, y, z \in X$. Set $k = 1/2$ and $F : X^3 \to X$ defined by $F(x,y,z) = (1/36)(6x - 6y + 6z + 5)$. The mapping F has the mixed monotone property. For all $x \geq a \geq u$, $y \leq b \leq v$, and $z \geq c \geq w$, we have

$$\begin{aligned} G(F(x,y,z), F(a,b,c), F(u,v,w)) &= \frac{1}{6}(|x-u| + |y-v| + |z-w|) \\ &= \frac{1}{6}(G(x,a,u) + G(y,b,v) + G(z,c,w)) \\ &= \frac{k}{3}(G(x,a,u) + G(y,b,v) + G(z,c,w)), \end{aligned} \tag{2.27}$$

that is, (2.16) holds. Take $x_0 = y_0 = z_0 = 1/6$, then all the hypotheses of Corollary 2.3 hold, and $(1/6, 1/6, 1/6)$ is the unique tripled fixed point of F.

Remark 2.7. In our main results (Theorems 2.1 and 2.4), the considered contractions are of nonlinear type. Then, inequality (2.1) does not reduce to any metric inequality with the metric d_G (this metric is given by (1.1)). Hence our theorems do not reduce to fixed point problems in the corresponding metric space (X, d_G).

References

[1] L. Ćirić, M. Abbas, B. Damjanović, and R. Saadati, "Common fuzzy fixed point theorems in ordered metric spaces," *Mathematical and Computer Modelling*, vol. 53, no. 9-10, pp. 1737–1741, 2011.

[2] Y. J. Cho, R. Saadati, and S. Wang, "Common fixed point theorems on generalized distance in ordered cone metric spaces," *Computers & Mathematics with Applications*, vol. 61, no. 4, pp. 1254–1260, 2011.

[3] R. Saadati, S. M. Vaezpour, and Lj. B. Ćirića, "Generalized distance and some common fixed point theorems," *Journal of Computational Analysis and Applications*, vol. 12, no. 1-A, pp. 157–162, 2010.

[4] B. C. Dhage, "Generalized metric space and mapping with fixed point," *Bulletin of Calcutta Mathematical Society*, vol. 84, pp. 329–336, 1992.

[5] B. C. Dhage, "On generalized metric spaces and topological structure. II," *Pure and Applied Mathematika Sciences*, vol. 40, no. 1-2, pp. 37–41, 1994.

[6] B. C. Dhage, "A common fixed point principle in *D*-metric spaces," *Bulletin of the Calcutta Mathematical Society*, vol. 91, no. 6, pp. 475–480, 1999.

[7] B. C. Dhage, "Generalized metric spaces and topological structure. I," *Annalele Stintifice ale Universitatii Al.I. Cuza*, vol. 46, no. 1, pp. 3–24, 2000.

[8] Z. Mustafa and B. Sims, "Some remarks concerning *D*-metric spaces," in *Proceedings of the International Conference on Fixed Point Theory and Applications*, pp. 189–198, Yokohama, Japan, 2004.

[9] Z. Mustafa and B. Sims, "A new approach to generalized metric spaces," *Journal of Nonlinear and Convex Analysis*, vol. 7, no. 2, pp. 289–297, 2006.

[10] M. Abbas, A. R. Khan, and T. Nazir, "Coupled common fixed point results in two generalized metric spaces," *Applied Mathematics and Computation*, vol. 217, no. 13, pp. 6328–6336, 2011.

[11] M. Abbas and B. E. Rhoades, "Common fixed point results for noncommuting mappings without continuity in generalized metric spaces," *Applied Mathematics and Computation*, vol. 215, no. 1, pp. 262–269, 2009.

[12] H. Aydi, B. Damjanović, B. Samet, and W. Shatanawi, "Coupled fixed point theorems for nonlinear contractions in partially ordered *G*-metric spaces," *Mathematical and Computer Modelling*, vol. 54, no. 9-10, pp. 2443–2450, 2011.

[13] H. Aydi, W. Shatanawi, and C. Vetro, "On generalized weakly *G*-contraction mapping in *G*-metric spaces," *Computers & Mathematics with Applications* , vol. 62, pp. 4222–4229, 2011.

[14] H. Aydi, "A fixed point result involving a generalized weakly contractive condition in *G*-metric spaces," *Bulletin of Mathematical Analysis and Applications*, vol. 3, no. 4, pp. 180–188, 2011.

[15] H. Aydi, W. Shatanawi, and M. Postolache, "Coupled fixed point results for (ψ, ϕ)-weakly contractive mappings in ordered *G*-metric spaces," *Computers & Mathematics with Applications*, vol. 63, pp. 298–309, 2012.

[16] H. Aydi, "A common fixed point of integral type contraction in generalized metric spaces," *Journal of Advanced Mathematical Studies*, vol. 5, no. 1, pp. 111–117, 2012.

[17] B. S. Choudhury and P. Maity, "Coupled fixed point results in generalized metric spaces," *Mathematical and Computer Modelling*, vol. 54, no. 1-2, pp. 73–79, 2011.

[18] Z. Mustafa, *A new structure for generalized metric spaces with applications to fixed point theory*, Ph.D. thesis, The University of Newcastle, Callaghan, Australia, 2005.

[19] Z. Mustafa, H. Obiedat, and F. Awawdeh, "Some fixed point theorem for mapping on complete *G*-metric spaces," *Fixed Point Theory and Applications*, vol. 2008, Article ID 189870, 12 pages, 2008.

[20] Z. Mustafa and B. Sims, "Fixed point theorems for contractive mappings in complete *G*-metric spaces," *Fixed Point Theory and Applications*, vol. 2009, Article ID 917175, 10 pages, 2009.

[21] Z. Mustafa, W. Shatanawi, and M. Bataineh, "Existence of fixed point results in *G*-metric spaces," *International Journal of Mathematics and Mathematical Sciences*, vol. 2009, Article ID 283028, 10 pages, 2009.

[22] R. Saadati, S. M. Vaezpour, P. Vetro, and B. E. Rhoades, "Fixed point theorems in generalized partially ordered *G*-metric spaces," *Mathematical and Computer Modelling*, vol. 52, no. 5-6, pp. 797–801, 2010.

[23] W. Shatanawi, "Fixed point theory for contractive mappings satisfying Φ-maps in *G*-metric spaces," *Fixed Point Theory and Applications*, vol. 2010, Article ID 181650, 9 pages, 2010.

[24] W. Shatanawi, "Some fixed point theorems in ordered *G*-metric spaces and applications," *Abstract and Applied Analysis*, vol. 2011, Article ID 126205, 11 pages, 2011.

[25] W. Shatanawi, "Coupled fixed point theorems in generalized metric spaces," *Hacettepe Journal of Mathematics and Statistics*, vol. 40, no. 3, pp. 441–447, 2011.

[26] N. Tahat, H. Aydi, E. Karapinar, and W. Shatanawi, "Common fixed points for single-valued and multi-valued maps satisfying a generalized contraction in $_G$-metric spaces," *Fixed Point Theory and Applications*, vol. 2012, 48 pages, 2012.

[27] V. Berinde and M. Borcut, "Tripled fixed point theorems for contractive type mappings in partially ordered metric spaces," *Nonlinear Analysis A*, vol. 74, no. 15, pp. 4889–4897, 2011.

[28] V. Berinde and M. Borcut, "Tripled coincidence theorems for contractive type mappings in partially ordered metric spaces," *Applied Mathematics & Computation*, vol. 218, no. 10, pp. 5929–5936, 2012.

[29] B. Samet and C. Vetro, "Coupled fixed point, f-invariant set and fixed point of N-order," *Annals of Functional Analysis*, vol. 1, no. 2, pp. 46–56, 2010.

[30] J. Matkowski, "Fixed point theorems for mappings with a contractive iterate at a point," *Proceedings of the American Mathematical Society*, vol. 62, no. 2, pp. 344–348, 1977.