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*Research Article*

# Numerical Analysis of a Linear-Implicit Average Scheme for Generalized Benjamin-Bona-Mahony-Burgers Equation

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A linear-implicit finite difference scheme is given for the initial-boundary problem of GBBM-Burgers equation, which is convergent and unconditionally stable. The unique solvability of numerical solutions is shown. A priori estimate and second-order convergence of the finite difference approximate solution are discussed using energy method. Numerical results demonstrate that the scheme is efficient and accurate.

## 1. Introduction

The generalized Benjamin-Bona-Mahony-Burgers (GBBM-Burgers) equation is in the form [1]

$$u_t - u_{xxt} - \alpha u_{xx} + \beta u_x + u^p u_x = 0, \tag{1.1}$$

where $\alpha > 0$, $\beta$ are constants, $p \geq 1$ is an integer, and $u(x,t)$ represents the velocity of fluid in the horizontal direction $x$. When $p = 1$, (1.1) is called as the Benjamin-Bona-Mahony-Burgers (BBM-Burgers) equation. In the special case, when $\alpha = 0$, (1.1) is described as the generalized Benjamin-Bona-Mahony equation

$$u_t - u_{xxt} + u_x + u^p u_x = 0. \tag{1.2}$$

The Equation (1.2) which is usually called as the generalized regularized long-wave equation proposed by Peregrine [2] and Benjamin et al. [3], so-called generalized Benjamin-Bona-Mahony equation, has been studied by many authors [4–7]. This equation features a balance between the nonlinear dispersive effect but takes no account of dissipation.

In recent years, a vast amount of work and computation has been devoted to the initial value problem for the GBBM-Burgers equation. In [1], Al-Khaled et al. studied the GBBM-Burgers by Decomposition method. In [8], Hayashi et al. investigated large time asymptotics of solutions to the BBM-Burgers equation. In [9], Jiang and Xu investigated the asymptotic behavior of solutions of the initial-boundary value problem for the GBBM-Burgers equations. In [10], Yin et al. studied the large time behavior of traveling wave solutions to the Cauchy problem of the GBBM-Burgers equations. In [11], Mei studied the large time behavior of global solutions to the Cauchy problem of GBBM-Burgers equations. In [12], Kondo and Webler studied the global existence of solutions for multidimensional GBBM-Burgers equations. Kinami et al. discussed the Cauchy problem of the GBBM-Burgers equations by Fourier transform method and energy method [13]. However, there are few studies on finite difference approximations for (1.1) which we consider in this paper.

In a recent work [14], we have made some preliminary computation by proposing a linearized difference scheme for GRLW equation which is unconditionally stable and reduces the computational work, and the numerical results are encouraging. In this paper, we continue our work and propose a linear-implicit difference scheme for generalized BBM-Burgers equation which is unconditionally stable and second-order convergent.

In this paper, we consider the following initial-boundary value problem of the GBBM-Burgers equation

$$
\begin{gathered}
u_t - u_{xxt} - \alpha u_{xx} + \beta u_x + u^p u_x = 0, \quad x \in [x_L, x_R],\ t \in [0,T], \\
u(x,0) = u_0(x), \quad x \in [x_L, x_R], \\
u(x_L,t) = u(x_R,t) = 0, \quad t \in [0,T].
\end{gathered}
\tag{1.3}
$$

An outline of the paper is as follows. In Section 2, we describe a linear-implicit finite difference scheme for the GBBM-Burgers equation and prove the error estimates of 2 order. In Section 3, we show that the scheme is uniquely solvable. In Section 4, convergence and stability of the scheme are proved. In Section 5, numerical results are provided to test the theoretical results.

## 2. Finite Difference Scheme and Estimate for the Difference Solution

As usual, the following notations will be used:

$$
x_j = x_L + jh, \quad t_n = n\tau, \quad 0 \le j \le J,\ 0 \le n \le N = \left[\frac{T}{\tau}\right], \tag{2.1}
$$

where $h = (x_R - x_L)/J$ and $\tau$ are the uniform spatial and temporal step sizes, respectively,

$$
\left(u_j^n\right)_x = \frac{u_{j+1}^n - u_j^n}{h}, \qquad \left(u_j^n\right)_{\bar{x}} = \frac{u_j^n - u_{j-1}^n}{h},
$$

$$\left(u_j^n\right)_{\hat{x}} = \frac{u_{j+1}^n - u_{j-1}^n}{2h}, \qquad \left(u_j^n\right)_{\hat{t}} = \frac{u_j^{n+1} - u_j^{n-1}}{2\tau},$$
$$\overline{u}_j^n = \frac{u_j^{n+1} + u_j^{n-1}}{2}, \qquad (u^n, v^n) = h\sum_j u_j^n v_j^n,$$
$$\|u^n\|^2 = (u^n, u^n), \qquad \|u^n\|_\infty = \sup_j \left|u_j^n\right|. \tag{2.2}$$

Let $u_j^n$ denote the approximation of $u(x_j, t_n)$, $Z_h^0 = \{u = (u_j) \mid u_0 = u_J = 0,\ 1 \le j \le J\}$. In this paper, we will denote $C$ as a generic constant independent of step sizes $h$ and $\tau$.

We propose a three-level linear-implicit difference scheme for the solution of the problem (1.3)

$$\left(u_j^n\right)_{\hat{t}} - \left(u_j^n\right)_{x\overline{x}\hat{t}} - \alpha\left(\overline{u}_j^n\right)_{x\overline{x}} + \beta\left(u_j^n\right)_{\hat{x}} + \frac{1}{p+2}\left[\left(u_j^n\right)^p\left(\overline{u}_j^n\right)_{\hat{x}} + \left(\left(u_j^n\right)^p\overline{u}_j^n\right)_{\hat{x}}\right] = 0,$$
$$1 \le j \le J-1,\ 1 \le n \le N-1, \tag{2.3}$$

$$u_j^0 = u_0(x_j), \quad 1 \le j \le J-1, \tag{2.4}$$

$$u_j^1 - \left(u_j^1\right)_{x\overline{x}} = u_0(x_j) + \frac{d^2u_0}{dx^2}(x_j) - \tau\left[\beta\frac{du_0}{dx}(x_j) - \alpha\frac{d^2u_0}{dx^2}(x_j) + u_0^p(x_j)\frac{du_0}{dx}(x_j)\right], \tag{2.5}$$

$$u_0^n = u_J^n = 0, \quad 1 \le n \le N-1. \tag{2.6}$$

For convenience, the last term of (2.3) is defined by

$$\varphi(u^n,\ \overline{u}^n) = \frac{1}{p+2}\left[\left(u_j^n\right)^p\left(\overline{u}_j^n\right)_{\hat{x}} + \left(\left(u_j^n\right)^p\overline{u}_j^n\right)_{\hat{x}}\right]. \tag{2.7}$$

**Lemma 2.1** (see [15]). *For any two mesh functions $u, v \in Z_h^0$, one has*

$$(u_x, v) = -(u, v_{\overline{x}}), \qquad ((u)_{x\overline{x}}, v) = -(u_x, v_x), \qquad ((u)_{x\overline{x}}, u) = -(u_x,\ u_x) = -\|u_x\|^2. \tag{2.8}$$

**Lemma 2.2.** *For any mesh function $u \in Z_h^0$, one has*

$$(\varphi(u^n,\ \overline{u}^n),\ \overline{u}^n) = 0. \tag{2.9}$$

*Proof.* For $u^n \in Z_h^0$, one has

$$
\begin{aligned}
(\psi(u^n,\ \overline{u}^n),\ \overline{u}^n) &= \frac{1}{8(p+2)}\sum_j\Big[\left(u_j^n\right)^p\left(u_{j+1}^{n+1}-u_{j-1}^{n+1}+u_{j+1}^{n-1}-u_{j-1}^{n-1}\right)\\
&\qquad+\left(u_{j+1}^n\right)^p\left(u_{j+1}^{n+1}+u_{j+1}^{n-1}\right)-\left(u_{j-1}^n\right)^p\left(u_{j-1}^{n+1}+u_{j-1}^{n-1}\right)\Big]\left(u_j^{n+1}+u_j^{n-1}\right)\\
&= \frac{1}{8(p+2)}\sum_j\Big[\left(u_j^n\right)^p\left(u_{j+1}^{n+1}+u_{j+1}^{n-1}\right)+\left(u_{j+1}^n\right)^p\left(u_{j+1}^{n+1}+u_{j+1}^{n-1}\right)\Big]\left(u_j^{n+1}+u_j^{n-1}\right)\\
&\quad-\frac{1}{8(p+2)}\sum_j\Big[\left(u_{j+1}^n\right)^p\left(u_j^{n+1}+u_j^{n-1}\right)+\left(u_j^n\right)^p\left(u_j^{n+1}+u_j^{n-1}\right)\Big]\left(u_{j+1}^{n+1}+u_{j+1}^{n-1}\right)\\
&= 0.
\end{aligned}
\tag{2.10}
$$

□

**Lemma 2.3** (Discrete Sobolev Inequality [16]). *For any discrete function $u_h$ and for any given $\varepsilon > 0$, there exists a constant $K(\varepsilon, n)$, depending only $\varepsilon$ and $n$, such that*

$$\|u^n\|_\infty \le \varepsilon\|u_x^n\| + K(\varepsilon,n)\|u^n\|. \tag{2.11}$$

**Theorem 2.4.** *Assume $u_0 \in H_0^1$, then there is the estimation for the solution of difference scheme* (2.3)–(2.6),

$$\|u^n\| \le C, \qquad \|u_x^n\| \le C, \qquad \|u^n\|_\infty \le C. \tag{2.12}$$

*Proof.* Computing the inner product of (2.3) with $2\overline{u}^n$ (i.e., $u^{n+1}+u^{n-1}$), we obtain

$$
\begin{aligned}
&\frac{1}{2\tau}\left(\left\|u^{n+1}\right\|^2-\left\|u^{n-1}\right\|^2\right)+\frac{1}{2\tau}\left(\left\|u_x^{n+1}\right\|^2-\left\|u_x^{n-1}\right\|^2\right)-\left(\alpha\left(\overline{u}^n\right)_{x\overline{x}},2\overline{u}^n\right)\\
&\quad+\beta h\sum_j\left(u_j^n\right)_{\hat{x}}\left(u_j^{n+1}+u_j^{n-1}\right)+\left(\psi(u^n,\ \overline{u}^n),2\overline{u}^n\right)=0.
\end{aligned}
\tag{2.13}
$$

Now, computing the fourth term of the left-hand side in (2.13), we have

$$h\sum_j\left(u_j^n\right)_{\hat{x}}\left(u_j^{n+1}+u_j^{n-1}\right)=h\left[\sum_j\left(u_j^n\right)_{\hat{x}}u_j^{n+1}-\sum_j\left(u_j^{n-1}\right)_{\hat{x}}u_j^n\right]. \tag{2.14}$$

According to Lemmas 2.1 and 2.2, and using (2.14), we get

$$\begin{aligned}&\frac{1}{2\tau}\left(\left\|u^{n+1}\right\|^2-\left\|u^{n-1}\right\|^2\right)+\frac{1}{2\tau}\left(\left\|u_x^{n+1}\right\|^2-\left\|u_x^{n-1}\right\|^2\right)\\&\quad+\beta h\left[\sum_j\left(u_j^n\right)_{\hat{x}}u_j^{n+1}-\sum_j\left(u_j^{n-1}\right)_{\hat{x}}u_j^n\right]\\&=-2\alpha\left\|\overline{u}_x^n\right\|^2\le 0.\end{aligned}\tag{2.15}$$

We let

$$E^n=\frac{1}{2}\left(\left\|u^{n+1}\right\|^2+\|u^n\|^2\right)+\frac{1}{2}\left(\left\|u_x^{n+1}\right\|^2+\|u_x^n\|^2\right)+\beta h\tau\sum_j\left(u_j^n\right)_{\hat{x}}u_j^{n+1}.\tag{2.16}$$

It follows from (2.15) that

$$\begin{aligned}E^n&=\frac{1}{2}\left(\left\|u^{n+1}\right\|^2+\|u^n\|^2\right)+\frac{1}{2}\left(\left\|u_x^{n+1}\right\|^2+\|u_x^n\|^2\right)+\beta h\tau\sum_j\left(u_j^n\right)_{\hat{x}}u_j^{n+1}\\&\le E^{n-1}\le\cdots\le E^0.\end{aligned}\tag{2.17}$$

Then we have

$$\frac{1}{2}\left(\left\|u^{n+1}\right\|^2+\|u^n\|^2\right)+\frac{1}{2}\left(\left\|u_x^{n+1}\right\|^2+\|u_x^n\|^2\right)\le C+\frac{1}{2}\beta\tau\left(\|u_x^n\|^2+\left\|u^{n+1}\right\|^2\right).\tag{2.18}$$

Using (2.18), we obtain

$$\frac{1}{2}\left[(1-\beta\tau)\left\|u^{n+1}\right\|^2+\|u^n\|^2\right]+\frac{1}{2}\left[\left\|u_x^{n+1}\right\|^2+(1-\beta\tau)\|u_x^n\|^2\right]\le C.\tag{2.19}$$

Equation (2.19) yields

$$\|u^n\|\le C,\qquad \|u_x^n\|\le C.\tag{2.20}$$

Using Lemma 2.3, the proof of Theorem 2.4 is completed. □

*Remark 2.5.* Theorem 2.4 implies that scheme (2.3)–(2.6) is unconditionally stable.

## 3. Solvability

Next, we will discuss the solvability of the scheme (2.3) based on the technique of Omrani et al. [17].

**Theorem 3.1.** *The finite difference scheme* (2.3) *is uniquely solvable.*

*Proof.* It is obvious that $u^0$ and $u^1$ are uniquely determined by (2.4)-(2.5). Now suppose $u^0, u^1, \dots, u^n$ $(1 \le n \le N-1)$ be solved uniquely. Considering the equation of (2.3) for $u^{n+1}$, we have

$$\frac{1}{2\tau}u_j^{n+1} - \frac{1}{2\tau}\left(u_j^{n+1}\right)_{x\bar{x}} - \frac{\alpha}{2}\left(u_j^{n+1}\right)_{x\bar{x}} + \frac{1}{2(p+2)}\left[\left(u_j^n\right)^p\left(u_j^{n+1}\right)_{\hat{x}} + \left(\left(u_j^n\right)^p u_j^{n+1}\right)_{\hat{x}}\right] = 0. \quad (3.1)$$

Computing the inner product of (3.1) with $u^{n+1}$, we have

$$\frac{1}{2\tau}\left\|u^{n+1}\right\|^2 + \frac{1}{2\tau}\left\|u_x^{n+1}\right\|^2 + \frac{\alpha}{2}\left\|u_x^{n+1}\right\|^2 + \left(\phi\left(u^n, u^{n+1}\right), u^{n+1}\right) = 0, \quad (3.2)$$

where $\phi(u^n, u^{n+1}) = (1/2(p+2))[(u_j^n)^p(u_j^{n+1})_{\hat{x}} + ((u_j^n)^p u_j^{n+1})_{\hat{x}}]$.

In view of difference properties and the boundary conditions (2.6), we obtain

$$\begin{aligned}\left(\phi\left(u^n, u^{n+1}\right), u^{n+1}\right) &= \frac{1}{2(p+2)}h\sum_{j=1}^{J-1}\left[\left(u_j^n\right)^p\left(u_j^{n+1}\right)_{\hat{x}} + \left(\left(u_j^n\right)^p u_j^{n+1}\right)_{\hat{x}}\right]u_j^{n+1} \\ &= \frac{1}{4(p+2)}h\sum_{j=1}^{J-1}\left[\left(u_j^n\right)^p u_{j+1}^{n+1}u_j^{n+1} + \left(u_{j+1}^n\right)^p u_{j+1}^{n+1}u_j^{n+1}\right] \\ &\quad - \frac{1}{4(p+2)}h\sum_{j=1}^{J-1}\left[\left(u_j^n\right)^p u_{j-1}^{n+1}u_j^{n+1} + \left(u_{j-1}^n\right)^p u_{j-1}^{n+1}u_j^{n+1}\right] \\ &= 0.\end{aligned} \quad (3.3)$$

It follows from (3.2) and (3.3) that

$$\left\|u^{n+1}\right\|^2 + \left\|u_x^{n+1}\right\|^2 + \alpha\tau\left\|u_x^{n+1}\right\|^2 = 0. \quad (3.4)$$

Noting that $\alpha > 0$ and following from (3.4), we have

$$\left\|u^{n+1}\right\|^2 + \left\|u_x^{n+1}\right\|^2 = 0. \quad (3.5)$$

That is (3.1) has only a trivial solution. Therefore, the scheme (2.3) determines $u_j^{n+1}$ uniquely. This completes the proof. □

*Remark 3.2.* All results above in this paper are correct for IBV problem of the BBM-Burgers equation with finite or infinite boundary.

## 4. Convergence and Stability of the Difference Scheme

First, we consider the truncation error of the difference scheme (2.3)–(2.6).

Suppose $v_j^n = u(x_j, t_n)$. Making use of Taylor expansion, we find

$$Er_j^n = \left(v_j^n\right)_{\hat{t}} - \left(v_j^n\right)_{x\overline{x}\hat{t}} - \alpha\left(\overline{v}_j^n\right)_{x\overline{x}} + \beta\left(v_j^n\right)_{\hat{x}} + \frac{1}{p+2}\left[\left(v_j^n\right)^p\left(\overline{v}_j^n\right)_{\hat{x}} + \left(\left(v_j^n\right)^p\overline{v}_j^n\right)_{\hat{x}}\right],$$
$$u_j^0 = u_0(x_j), \tag{4.1}$$
$$u_j^1 - \left(u_j^1\right)_{x\overline{x}} = u_0(x_j) + \frac{d^2u_0}{dx^2}(x_j) - \tau\left[\beta\frac{du_0}{dx}(x_j) - \alpha\frac{d^2u_0}{dx^2}(x_j) + u_0^p(x_j)\frac{du_0}{dx}(x_j)\right] + r_i,$$

where $Er_j^n$ and $r_i$ are the truncation errors of the difference scheme (2.3)–(2.6). It can be easily obtained that (see [18, 19])

$$\left|Er_j^n\right| = O\left(h^2 + \tau^2\right), \tag{4.2}$$

$$\left|r_j^n\right| = O\left(h^2 + \tau^2\right). \tag{4.3}$$

**Lemma 4.1.** *Assume $u(x,t)$ is smooth enough, then the local truncation error of the finite difference scheme* (2.3)–(2.6) *is*

$$\left|Er_j^n\right| = O\left(h^2 + \tau^2\right). \tag{4.4}$$

**Lemma 4.2** (see [16]). *Suppose that the discrete function $w_h$ satisfies recurrence formula*

$$w_n - w_{n-1} \le A\tau w_n + B\tau w_{n-1} + C_n\tau, \tag{4.5}$$

*where $A, B, C_n$ $(n = 1, \cdots N)$ are nonnegative constants. Then*

$$\|w_n\|_\infty \le \left(w_0 + \tau\sum_{k=1}^{N} C_k\right)e^{2(A+B)\tau}, \tag{4.6}$$

*where $\tau$ is small, such that $(A+B)\tau \le ((N-1)/2N)(N > 1)$.*

**Theorem 4.3.** *Assume $u_0 \in H_0^1[x_L, x_R]$ and $u \in C^{(4,3)}$, then the solution of the difference scheme* (2.3)–(2.6) *converges to the solution of the problem* (1.3) *with order $O(h^2 + \tau^2)$ by the $\|\cdot\|_\infty$ norm.*

*Proof.* Let $e_j^n = v_j^n - u_j^n$. Subtracting (2.3)-(2.5) from (4.1)–(4.3), respectively, we have

$$\begin{aligned} Er_j^n &= \left(e_j^n\right)_{\hat{t}} - \left(e_j^n\right)_{x\overline{x}\hat{t}} - \alpha\left(\overline{e}_j^n\right)_{x\overline{x}} \\ &\quad + \beta\left(e_j^n\right)_{\hat{x}} + \frac{1}{p+2}\left[\left(v_j^n\right)^p\left(\overline{v}_j^n\right)_{\hat{x}} + \left(\left(v_j^n\right)^p\overline{v}_j^n\right)_{\hat{x}}\right] \\ &\quad - \frac{1}{p+2}\left[\left(u_j^n\right)^p\left(\overline{u}_j^n\right)_{\hat{x}} + \left(\left(u_j^n\right)^p\overline{u}_j^n\right)_{\hat{x}}\right], \\ e_j^0 &= 0, \\ e_j^1 &= r_j. \end{aligned} \tag{4.7}$$

For a simple notation, the last two terms of (4.7) are defined by

$$\begin{aligned} I &= \frac{1}{p+2}\left(v_j^n\right)^p\left(\overline{v}_j^n\right)_{\hat{x}} - \frac{1}{p+2}\left(u_j^n\right)^p\left(\overline{u}_j^n\right)_{\hat{x}}, \\ II &= \frac{1}{p+2}\left(\left(v_j^n\right)^p\overline{v}_j^n\right)_{\hat{x}} - \frac{1}{p+2}\left(\left(u_j^n\right)^p\overline{u}_j^n\right)_{\hat{x}}. \end{aligned} \tag{4.8}$$

Computing the inner product of (4.7) with $e^{n+1} + e^{n-1}$ (i.e., $2\overline{e}^n$), we get

$$\begin{aligned} \left(Er_j^n, 2\overline{e}^n\right) &= \frac{1}{2\tau}\left(\left\|e^{n+1}\right\|^2 - \left\|e^{n-1}\right\|^2\right) + \frac{1}{2\tau}\left(\left\|e_x^{n+1}\right\|^2 - \left\|e_x^{n-1}\right\|^2\right) - \left(\alpha\left(\overline{e}^n\right)_{x\overline{x}}, 2\overline{e}^n\right) \\ &\quad + \beta h\sum_j\left(e_j^n\right)_{\hat{x}}\left(e_j^{n+1} + e_j^{n-1}\right) + \left(I + II, 2\overline{e}^n\right). \end{aligned} \tag{4.9}$$

Similarly to the proof of Theorem 2.4, we obtain

$$\begin{aligned} \left(\alpha\left(\overline{e}^n\right)_{x\overline{x}}, 2\overline{e}^n\right) &= -2\alpha\left\|\overline{e}_x^n\right\|^2, \\ \beta h\sum_j\left(e_j^n\right)_{\hat{x}}\left(e_j^{n+1} + e_j^{n-1}\right) &= \beta\left[h\sum_j\left(e_j^n\right)_{\hat{x}}e_j^{n+1} - h\sum_j\left(e_{j-1}^n\right)_{\hat{x}}e_j^n\right]. \end{aligned} \tag{4.10}$$

According to Theorem 2.4, we obtain

$$\begin{aligned} \left(I, 2\overline{e}^n\right) &= \frac{1}{p+2}h\sum_j\left[\left(v_j^n\right)^p\left(e_j^{n+1} + e_j^{n-1}\right)_{\hat{x}} + \left(\left(v_j^n\right)^p - \left(u_j^n\right)^p\right)\left(u_j^{n+1} + u_j^{n-1}\right)_{\hat{x}}\right]\left(e_j^{n+1} + e_j^{n-1}\right) \\ &\le Ch\sum_j\left[\left|\left(e_j^{n+1} + e_j^{n-1}\right)_{\hat{x}}\right| + \left|\left(u_j^{n+1} + u_j^{n-1}\right)_{\hat{x}}\right|\right]\left(e_j^{n+1} + e_j^{n-1}\right) \\ &\le C\left(\left\|e_x^{n+1}\right\|^2 + \left\|e_x^{n-1}\right\|^2 + \left\|e^{n+1}\right\|^2 + \left\|e^{n-1}\right\|^2\right), \end{aligned}$$

$$
\begin{aligned}
(II,2\overline{e}^n) &= \frac{1}{p+2}\sum_j\left\{\left[\left(v_j^n\right)^p\left(e_j^{n+1}+e_j^{n-1}\right)\right]_{\hat{x}}+\left[((v_j^n)^p-(u_j^n)^p)(u_j^{n+1}+u_j^{n-1})\right]_{\hat{x}}\right\}\left(e_j^{n+1}+e_j^{n-1}\right)\\
&= -\frac{1}{3}h\sum_j\left[\left(v_j^n\right)^p\left(e_j^{n+1}+e_j^{n-1}\right)+\left(\left(v_j^n\right)^p-\left(u_j^n\right)^p\right)\left(u_j^{n+1}+u_j^{n-1}\right)\right]\left(e_j^{n+1}+e_j^{n-1}\right)_{\hat{x}}\\
&\le Ch\sum_j\left[\left|\left(e_j^{n+1}+e_j^{n-1}\right)\right|+\left|e_j^n\right|\right]\left(e_j^{n+1}+e_j^{n-1}\right)_{\hat{x}}\\
&\le C\left(\left\|e_x^{n+1}\right\|^2+\left\|e_x^{n-1}\right\|^2+\left\|e^{n+1}\right\|^2+\|e^n\|^2+\left\|e^{n-1}\right\|^2\right).
\end{aligned}
\tag{4.11}
$$

In addition, there exists obviously that

$$
\left|\left(Er_j^n,e^{n+1}+e^{n-1}\right)\right| \le \|Er^n\|^2+\frac{1}{2}\left(\left\|e^{n+1}\right\|^2+\left\|e^{n-1}\right\|^2\right). \tag{4.12}
$$

Substituting (4.10)–(4.12) into (4.9), we have

$$
\begin{aligned}
&\frac{1}{2\tau}\left(\left\|e^{n+1}\right\|^2-\left\|e^{n-1}\right\|^2\right)+\frac{1}{2\tau}\left(\left\|e_x^{n+1}\right\|^2-\left\|e_x^{n-1}\right\|^2\right)\\
&\quad\le \|Er^n\|^2+\frac{1}{2}\left(\left\|e^{n+1}\right\|^2+\left\|e^{n-1}\right\|^2\right)+\beta\|e_x^n\|^2+\frac{1}{2}\beta\left(\left\|e^{n+1}\right\|^2+\|e^n\|^2\right)\\
&\qquad+C\left(\left\|e_x^{n+1}\right\|^2+\|e_x^n\|^2+\left\|e_x^{n-1}\right\|^2+\left\|e^{n+1}\right\|^2+\|e^n\|^2+\left\|e^{n-1}\right\|^2\right).
\end{aligned}
\tag{4.13}
$$

Let

$$
B^n=\frac{1}{2}\left(\left\|e^{n+1}\right\|^2+\|e^n\|^2\right)+\frac{1}{2}\left(\left\|e_x^{n+1}\right\|^2+\|e_x^n\|^2\right). \tag{4.14}
$$

Then (4.13) can be rewritten as

$$
B^n-B^{n-1}\le\tau\|Er^n\|^2+C\tau\left(B^n+B^{n-1}\right). \tag{4.15}
$$

By Lemma 4.2, it can immediately be obtained that

$$
B^N\le\left(B^0+T\sup_{1\le n\le N}\|Er^n\|^2\right)e^{CT}. \tag{4.16}
$$

To complete the proof, it is enough to find $B^0$ estimate. From (4.7), we obtain

$$
\left\|e^0\right\|=0. \tag{4.17}
$$

Using (4.3) and (4.8), we get

$$\left\| e^1 \right\| \leq O\left(h^2 + \tau^2\right). \tag{4.18}$$

It follows from (4.17) and (4.18) that

$$B^0 \leq \left[O\left(\tau^2 + h^2\right)\right]^2. \tag{4.19}$$

Thus

$$\|e^n\| \leq O\left(\tau^2 + h^2\right), \qquad \|e_x^n\| \leq O\left(\tau^2 + h^2\right). \tag{4.20}$$

According to Lemma 2.3, there exists that

$$\|e^n\|_\infty \leq O\left(\tau^2 + h^2\right). \tag{4.21}$$

□

Similarly, the following theorem can be proved.

**Theorem 4.4.** *Under the conditions of Theorem 4.3, the solution of finite difference scheme* (2.3)–(2.6) *is stable by the* $\|\cdot\|_\infty$ *norm.*

## 5. Numerical Experiments

In this section, we will compute several numerical experiments to verify the correction of our theoretical analysis in the above sections.

*Example 5.1* (see [20]). Consider the following initial-boundary problem of BBM-Burgers equation:

$$u_t - u_{xxt} - \alpha u_{xx} + u_x + uu_x = 0, \quad x \in [0,1], \ t \in [0,10], \tag{5.1}$$

$$u(x,0) = u_0(x), \quad x \in [0,1], \tag{5.2}$$

$$u(0,t) = u(1,t) = 0, \quad t \in [0,10]. \tag{5.3}$$

We denote the scheme proposed in [20] as Scheme I and the scheme (2.3) in present paper as Scheme II. In computations, we choose the initial condition $u_0(x) = \exp(-x^2)$ [20]. The maximal errors of both schemes are listed in Table 1. We get that a second-order linear scheme is as accurate as Scheme I which is a nonlinear one.

*Example 5.2* (see [13]). Consider the GBBM-Burgers equation

$$u_t - u_{xxt} - \alpha u_{xx} + \beta u_x + u^p u_x = 0, \quad x \in [0,1], \ t \in [0,T], \tag{5.4}$$

**Table 1:** The maximal errors of numerical solutions at $t = 10$ with $\tau = 0.1$ for $\alpha = 0.5$ when $p = 1$.

| | $h = 1/4$ | $h = 1/8$ | $h = 1/16$ | $h = 1/32$ |
|---|---|---|---|---|
| Scheme I | $2.486233e - 4$ | $6.519728e - 5$ | $1.618990e - 5$ | $4.929413e - 6$ |
| Scheme II | $2.438693e - 4$ | $6.418263e - 5$ | $1.594145e - 5$ | $3.867502e - 6$ |

**Table 2:** The maximal errors of numerical solutions at $t = 10$ with $\tau = 0.1$ for $\alpha = 0.5$ when $p = 4$.

| | $h = 1/4$ | $h = 1/8$ | $h = 1/16$ | $h = 1/32$ |
|---|---|---|---|---|
| Scheme II | $5.293584e - 4$ | $1.416254e - 4$ | $3.480022e - 5$ | $8.423768e - 6$ |
| Scheme III | $5.069513e - 3$ | $3.444478e - 3$ | $1.916013e - 3$ | $9.262223e - 4$ |

**Table 3:** The errors of numerical solutions at $t = 10$ with $\tau = 0.1$ when $p = 2$.

| $h$ | $\|\|v^n - u^n\|\|$ | $\|\|v^n - u^n\|\|_\infty$ | $\|\|v^{n/4} - u^{n/4}\|\|/\|\|v^n - u^n\|\|$ | $\|\|v^{n/4} - u^{n/4}\|\|_\infty/\|\|v^n - u^n\|\|_\infty$ |
|---|---|---|---|---|
| 0.25 | $6.377969e - 4$ | $9.352639e - 4$ | — | — |
| 0.125 | $1.582597e - 4$ | $2.314686e - 4$ | 4.030065 | 4.040566 |
| 0.0625 | $3.920742e - 5$ | $5.893641e - 5$ | 4.036473 | 3.927429 |
| 0.03125 | $9.501117e - 6$ | $1.428261e - 5$ | 4.126612 | 4.126445 |

**Table 4:** The errors of numerical solutions at $t = 10$ with $\tau = 0.1$ when $p = 4$.

| $h$ | $\|\|v^n - u^n\|\|$ | $\|\|v^n - u^n\|\|_\infty$ | $\|\|v^{n/4} - u^{n/4}\|\|/\|\|v^n - u^n\|\|$ | $\|\|v^{n/4} - u^{n/4}\|\|_\infty/\|\|v^n - u^n\|\|_\infty$ |
|---|---|---|---|---|
| 0.25 | $6.316492e - 4$ | $9.262624e - 4$ | — | — |
| 0.125 | $1.568213e - 4$ | $2.294480e - 4$ | 4.027828 | 4.036916 |
| 0.0625 | $3.885715e - 5$ | $5.828155e - 5$ | 4.035841 | 3.936889 |
| 0.03125 | $9.416614e - 6$ | $1.412454e - 5$ | 4.126446 | 4.126262 |

with an initial condition

$$u(x,0) = u_0(x), \quad x \in [0,1], \tag{5.5}$$

and boundary conditions

$$u(0,t) = u(1,t) = 0, \quad t \in [0,T]. \tag{5.6}$$

In computations, we choose the initial condition $u_0(x) = 1/(1 + x^4)$ [13]. Without loss of generality, We take $p = 2, 4, 8$ and $\alpha = 0.5$, $\beta = 1$. Since we do not know the exact solution of (5.4)–(5.6), an error estimate method in [21] is used. A comparison between the numerical solutions on a coarse mesh and those on a refine mesh is made. In order to obtain the error estimates, we consider the solution on mesh $h = 1/160$ as reference solution and obtain error estimates on mesh $h = 1/4$, $1/8$, $1/16$, and $1/32$, respectively. We denote the scheme proposed in [13] as Scheme III and make a comparison with the scheme (2.3) in present paper as Scheme II when $p = 4$ in Table 2. The corresponding errors in the sense of $L_\infty$-norm and $L_2$-norm are listed in Tables 3, 4, and 5, respectively. These three tables verify the second-order convergence and good stability of the numerical solutions.

**Table 5:** The errors of numerical solutions at $t = 10$ with $\tau = 0.1$ when $p = 8$.

| $h$ | $\|\|v^n - u^n\|\|$ | $\|\|v^n - u^n\|\|_\infty$ | $\|\|v^{n/4} - u^{n/4}\|\|/\|\|v^n - u^n\|\|$ | $\|\|v^{n/4} - u^{n/4}\|\|_\infty/\|\|v^n - u^n\|\|_\infty$ |
|---|---|---|---|---|
| 0.25 | $1.150448e-4$ | $1.822979e-4$ | — | — |
| 0.125 | $2.981547e-5$ | $4.674950e-5$ | 3.858561 | 3.899462 |
| 0.0625 | $7.426232e-6$ | $1.167644e-5$ | 4.014885 | 4.003745 |
| 0.03125 | $1.801424e-6$ | $2.879611e-6$ | 4.122423 | 4.054867 |

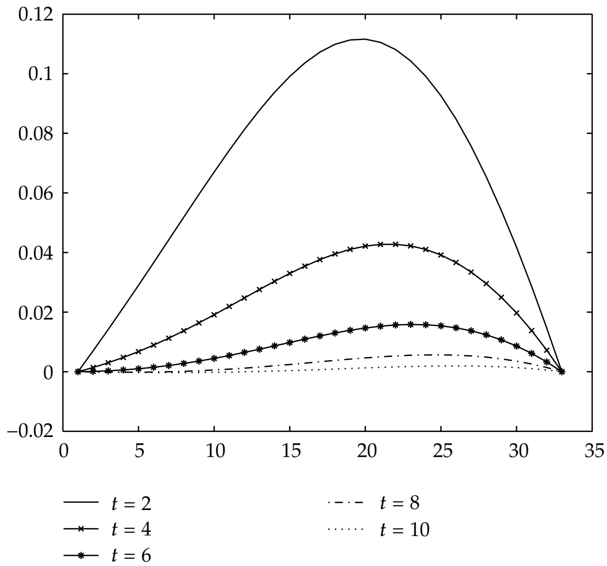


**Figure 1:** Numerical solution of $u(x,t)$ with $h = 0.03125, \tau = 0.1$ when $p = 2$.

Figures 1 and 2 plot the numerical solutions computed by the linearly implicit scheme (2.3) with $\tau = 0.1$, $h = 0.03125$, and $\alpha = 0.5$ when $p = 2, 8$ at $t = 2, 4, 6, 8$, and 10, respectively. From Figures 1 and 2, it is easy to observe that the height of the numerical approximation to $u$ is more and more low with time elapsing due to the effect of dissipative term $\alpha u_{xx}$. Both of them simulates that the continuous energy $E(t)$ of the problem (1.3) in Theorem 2.4 decreases in time. Numerical experiments show our scheme is accurate and efficient.

## 6. Conclusions

In this paper, we have presented a three-level linear-implicit finite difference scheme for the GBBM-Burgers equation, which has a wide range of applications in physics. The convergence and stability as well as second-order error estimate of the finite difference approximate solutions were discussed in detail. Numerical experiments show our scheme is accurate and efficient.

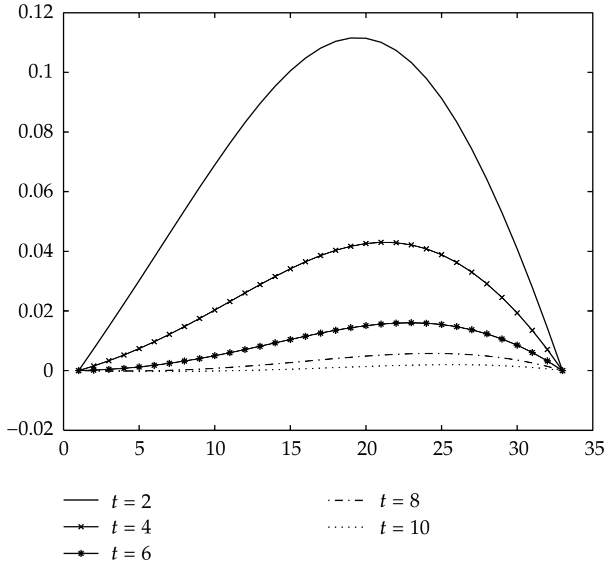


**Figure 2:** Numerical solution of $u(x,t)$ with $h = 0.03125, \tau = 0.1$ when $p = 8$.


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