

## Research Article

# Uniqueness in Inverse Electromagnetic Conductive Scattering by Penetrable and Inhomogeneous Obstacles with a Lipschitz Boundary

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This paper is concerned with the problem of scattering of time-harmonic electromagnetic waves by a penetrable, inhomogeneous, Lipschitz obstacle covered with a thin layer of high conductivity. The well posedness of the direct problem is established by the variational method. The inverse problem is also considered in this paper. Under certain assumptions, a uniqueness result is obtained for determining the shape and location of the obstacle and the corresponding surface parameter  $\lambda(x)$  from the knowledge of the near field data, assuming that the incident fields are electric dipoles located on a large sphere with polarization  $p \in \mathbb{R}^3$ . Our results extend those in the paper by F. Hettlich (1996) to the case of inhomogeneous Lipschitz obstacles.

## 1. Introduction

In this paper we are interested in determining the shape and location of a penetrable, inhomogeneous, isotropic, Lipschitz obstacle surrounded by a piecewise homogeneous, isotropic medium. The obstacle is covered with a thin layer of high conductivity. Such penetrable obstacles lead to conductive boundary conditions; for the precise mathematical description, the reader is referred to [1–3]. In this paper, it is shown that the shape and location of the obstacle and the corresponding surface parameter are uniquely determined from a knowledge of the near field data of the scattered electromagnetic wave at a fixed frequency. To this end, we need a well posedness result for the direct problem.

The well posedness of the Helmholtz equation for a penetrable, inhomogeneous, anisotropic medium has been studied recently in [4]. In [5], the authors provided a proof

for the well posedness of the scattering problem for a dielectric that is partially coated by a highly conductive layer in the TM case in 2007.

In the case of exterior Maxwell problem for the partially coated Lipschitz domains, the authors in [6] have established the well posedness of a unique solution by variational methods in 2004. For the homogeneous isotropic medium problem, by means of an integral equation method, Angell and Kirsch proved the existence and uniqueness of the classical solution for Maxwell's equations with conductive boundary conditions assuming  $\lambda \in C^{0,\alpha}(\partial D)$  in [2]. Variational methods for the homogeneous isotropic medium problem were proposed in [1], under the assumption that the bounded domain  $D \subseteq \mathbb{R}^3$  with boundary  $\partial D$  in the class  $C^2$ , and some additional conditions on  $k, n_D, \mu, \lambda$ . It is also shown that the obstacle is uniquely determined by the far field patterns of all incident waves with a fixed wave number. For the inhomogeneous anisotropic media, the well posedness of the direct problem was proved in [7].

The uniqueness result for the inverse medium scattering problem was first provided by Isakov (see [8, 9]), in which it is shown that the shape of a penetrable, inhomogeneous, isotropic medium is uniquely determined by its far field pattern of all incident plane waves. The idea is to construct singular solutions of the boundary value problem with respect to two different scattering obstacles with identical far field patterns. Our uniqueness proof is based on this idea. The idea of Isakov was modified by Kirsh and Kress [10] using potential theory for the impenetrable obstacle case with Neumann boundary conditions. By the same technique, the authors in [11] proved the case of a penetrable obstacle with constant index of refraction. The use of potential theory will require strong smoothness assumptions on the scattering object. Then D. Mitrea and M. Mitrea [12] improved the previous results to the case of Lipschitz domains. In [13], they extended Isakov's approach to the case of a penetrable obstacle for Helmholtz equations. The uniqueness theorem of Helmholtz equations for partially coated buried obstacle problem was shown in [14, 15], assuming that the scattering fields were known with point sources as incident fields.

Recently, uniqueness for the inverse scattering problem in a layered medium has attracted intensive studies. For the sound-soft or sound-hard obstacle case, based on Schiffer's idea, [16] proved a uniqueness result. But their method can not be extended to other boundary conditions. In recent years, by employing the generalized mixed reciprocity relation, it was proved in [17, 18] that both the obstacle and its physical property can be uniquely determined for different boundary conditions. For the inverse acoustic scattering by an impenetrable obstacle in a two-layered medium case, it is shown in [19] that interface is uniquely determined from the far field pattern. Unfortunately, this method can not be extended to the electromagnetic case, but using ideas in [20], a different method was used in [21] to establish such a uniqueness result for the electromagnetic case.

There are also some uniqueness results for partial differential equations with constant coefficients by integral equation methods. (see [22, 23]). However, integral equation methods are not well tailored for partial differential equations having inhomogeneous coefficients of the highest derivatives. Consequently, in [24], the author brought together the variational approach and the idea from [8, 9] to provide a uniqueness proof of Helmholtz equations with inhomogeneous coefficients for a penetrable, anisotropic obstacle. Their method depends on a regularity theorem for the direct problem and the well posedness of the interior transmission problem related to the direct problem. This idea has been extended to the case of electromagnetic scattering problem for anisotropic media in [25].

The outline of this paper is as follows. In Section 2, besides the formulation of the direct scattering problem in a penetrable, inhomogeneous, Lipschitz domain, we also provide

a proof of the well posedness for the direct problem by using a variational method. The uniqueness result for the inverse problem will be shown in Section 3.

## 2. The Direct Problem

Let  $D \subset \mathbb{R}^3$  be a bounded penetrable, inhomogeneous, isotropic domain with a Lipschitz boundary  $\partial D$  denoted by  $\Gamma$  and covered with a thin layer of high conductivity. Assume that the domain  $D$  is imbedded in a homogeneous background medium. Define  $k_D^2 = k^2 n_D(x)$  and  $k_b^2 = k^2 n_b$  with  $k > 0$  being the wave number, where  $n_D(x)$  and  $n_b$  are the refractive index of the domain  $D$  and the background medium, respectively. Assume that  $n_D \in C(\overline{D})$  with  $\text{Re}[n_D(x)] > 0, \text{Im}[n_D(x)] > 0$  for all  $x \in D$  and  $n_b$  is a complex constant with  $\text{Im}(n_b) \geq 0$ . Assume further that  $\lambda \in L^\infty(\Gamma)$  with  $\text{Re}[\lambda(x)] \geq 0$  is a complex-valued function describing the surface impedance of the coating. The incident field is considered to be an electric dipole located at  $x_0$  on a large sphere  $S_{R_0} = \{x \in \mathbb{R}^3 : |x| = R_0\}$  with polarization  $p \in \mathbb{R}^3$  given by

$$E_e(x, x_0, p, k_b) = \frac{i}{k_b} \text{curl}_x \text{curl}_x p \frac{e^{ik_b|x-x_0|}}{4\pi|x-x_0|}. \quad (2.1)$$

Denote by  $G(x, x_0)$  the free space Green tensor of the background medium and define  $E^i(x) = E^i(x, x_0, p) = pG(x, x_0)$  which satisfies

$$\text{curl curl } E^i(x) - k_b^2 E^i(x) = p\delta(x - x_0) \text{ in } \mathbb{R}^3, \quad (2.2)$$

where  $\delta$  is the Dirac delta function. Note that  $E^i(x)$  can be written as

$$E^i(x) = E_e(x, x_0, p, k_b) + E_b^s(x), \quad (2.3)$$

where  $E_b^s(x)$  is the scattered electric field due to the background medium and the electric dipole  $E_e(x, x_0, p, k_b)$ .

In order to formulate precisely the scattering problem, recall the following Sobolev spaces:

$$H(\text{curl}, D) = \left\{ u \in \left( L^2(D) \right)^3, \text{curl } u \in \left( L^2(D) \right)^3 \right\},$$

$$H(\text{div}, D) = \left\{ u \in \left( L^2(D) \right)^3, \text{div } u \in \left( L^2(D) \right)^3 \right\},$$

$$L_t^2(\partial D) = \left\{ u \in \left( L^2(\partial D) \right)^3, \nu \cdot u = 0 \text{ on } \partial D \right\},$$

$$\begin{aligned}
H_{\text{div}}^{-1/2}(\partial D) &= \left\{ u \in \left( H^{-1/2}(\partial D) \right)^3, \nu \cdot u = 0, \text{div}_{\partial D} u \in H^{-1/2}(\partial D) \right\}, \\
H_{\text{curl}}^{-1/2}(\partial D) &= \left\{ u \in \left( H^{-1/2}(\partial D) \right)^3, \nu \cdot u = 0, \text{curl}_{\partial D} u \in H^{-1/2}(\partial D) \right\}, \\
H_0(\text{curl}, D) &= \left\{ u \in \left( L^2(D) \right)^3, \text{curl} u \in \left( L^2(D) \right)^3, \nu \times u = 0 \text{ on } \partial D \right\},
\end{aligned} \tag{2.4}$$

where  $\nu$  denotes the exterior unit normal to  $\partial D$ . If  $D$  is unbounded, we denote by  $H_{\text{loc}}(\text{curl}, D)$  the space of functions  $u \in H(\text{curl}, K)$  for any compact set  $K \subset\subset D$ . Introduce the space

$$\tilde{X} = \left\{ u \in H_{\text{loc}}(\text{curl}, \mathbb{R}^3), u_T|_{\Gamma} \in L_t^2(\Gamma) \right\}, \tag{2.5}$$

where  $u_T = \nu \times u \times \nu$ . Then the scattering problem can be formulated as follows. Given  $E^i$ , find the field  $V$  and the scattered field  $E^s$  such that

$$\text{curl curl} V - k_D^2 V = 0, \quad \text{in } D, \tag{2.6}$$

$$\text{curl curl} E^s - k_b^2 E^s = 0, \quad \text{in } D_e, \tag{2.7}$$

$$\nu \times E^s - \nu \times V = -\nu \times E^i, \quad \text{on } \Gamma, \tag{2.8}$$

$$\nu \times \text{curl} E^s - \nu \times \text{curl} V = ik\lambda E_T^s + ik\lambda E_T^i - \nu \times \text{curl} E^i, \quad \text{on } \Gamma, \tag{2.9}$$

and the scattered field  $E^s = E - E^i$  is required to satisfy the Silver-Müller radiation condition

$$\lim_{r \rightarrow \infty} r(\text{curl} E^s \times \hat{x} - ikE^s) = 0 \tag{2.10}$$

uniformly in  $\hat{x} = x/|x|$ , where  $r = |x|$ .

We first have the following uniqueness result for the above scattering problem.

**Theorem 2.1.** *The scattering problem (2.6)–(2.9) has at most one solution.*

*Proof.* To prove the theorem, it is enough to consider the case  $E^i = 0$  whence  $E = E^s$ . Taking the dot product of (2.6) with  $V$  over  $D$  and of (2.7) with  $E$  over  $D_R = D_e \cap B_R$  with  $D_e = \mathbb{R}^3 \setminus \bar{D}$ , respectively, and integrating by parts, we obtain by using the conductive conditions (2.8) and (2.9) that

$$\begin{aligned}
& \int_D \left( |\text{curl} V|^2 - k_D^2 |V|^2 \right) dx + \int_{D_R} \left( |\text{curl} E|^2 - k_b^2 |E|^2 \right) dx \\
& - ik \int_{\Gamma} \lambda |E_T|^2 ds + ik \int_{S_R} \nu \times H \cdot \bar{E}_T ds = 0,
\end{aligned} \tag{2.11}$$

where  $H = (1/ik) \operatorname{curl} E$  is the corresponding scattered magnetic field. Taking the complex conjugate of both sides of (2.11) and using the fact that  $\operatorname{Im}(n_D)$ ,  $\operatorname{Im}(n_b)$ , and  $\operatorname{Re}(\lambda)$  are non-negative gives

$$\begin{aligned} \operatorname{Re} \int_{S_R} \nu \times E \cdot \overline{H} ds &= -k^{-1} \int_D \operatorname{Im}(n_D) |V|^2 dx - k^{-1} \int_{D_R} \operatorname{Im}(n_b) |E|^2 dx \\ &\quad - \int_{\Gamma} \operatorname{Re}(\lambda) |E_T|^2 ds \leq 0. \end{aligned} \quad (2.12)$$

An application of the Rellich lemma yields that  $E = 0$  in  $\mathbb{R}^3 \setminus \overline{B}_R$  (see [26, Theorem 6.10]). This, together with the unique continuation principle, implies that  $E = 0$  in  $\mathbb{R}^3 \setminus \overline{D}$ . From the trace theorem, it follows that  $\nu \times E = 0$  on  $\Gamma$ . Thus, taking the imaginary part of (2.11) and using the assumption that  $\operatorname{Im}[n_D(x)] > 0$  for all  $x \in D$ , we have that  $V = 0$  in  $D$ .

Introduce the electric-to-magnetic Calderon operator  $G_e$  (see [27]), which maps the electric field boundary data  $\varphi$  on the surface of a large ball  $B_R = \{x \in \mathbb{R}^3 : |x| < R\}$  to the magnetic boundary data  $\hat{x} \times H$  on  $S_R = \partial B_R$ , where  $(E, H)$  satisfies

$$\operatorname{curl} E - ikH = 0, \quad \text{in } \mathbb{R}^3 \setminus \overline{B}_R, \quad (2.13)$$

$$\operatorname{curl} H + ikE = 0, \quad \text{in } \mathbb{R}^3 \setminus \overline{B}_R, \quad (2.14)$$

$$\hat{x} \times E = \varphi, \quad \text{on } S_R, \quad (2.15)$$

$$\lim_{r \rightarrow \infty} (H \times x - rE) = 0. \quad (2.16)$$

Then the scattering problem (2.6)–(2.10) can be reformulated in the following mixed conductive boundary value problem (MOCKUP) over a bounded domain:

$$\operatorname{curl} \operatorname{curl} V - k_D^2 V = 0, \quad \text{in } D, \quad (2.17)$$

$$\operatorname{curl} \operatorname{curl} E^s - k_b^2 E^s = 0, \quad \text{in } D_R, \quad (2.18)$$

$$\nu \times E^s - \nu \times V = -\nu \times E^i, \quad \text{on } \Gamma, \quad (2.19)$$

$$\nu \times \operatorname{curl} E^s - \nu \times \operatorname{curl} V = ik\lambda E_T^s + ik\lambda E_T^i - \nu \times \operatorname{curl} E^i, \quad \text{on } \Gamma, \quad (2.20)$$

$$\nu \times \frac{1}{ik} \operatorname{curl} E^s = G_e(\nu \times E^s), \quad \text{on } S_R, \quad (2.21)$$

where  $D_R = D_e \cap B_R$ .

In the following, we introduce some properties of the Calderon operator that will be frequently used in the rest of this section. The basis functions for tangential fields on a sphere  $S_R$  are the vector spherical harmonics of order  $n$  given by

$$U_n^m = \frac{1}{\sqrt{n(n+1)}} \nabla_{\tau_1} Y_n^m, \quad V_n^m = \hat{x} \times U_n^m \quad (2.22)$$

for  $n = 1, 2, \dots$  and  $m = -n, \dots, n$ . Here, as usual,  $\nabla_{\tau_1}$  denotes the surface gradient on the surface of the unit sphere  $S_1$ .

For  $\varphi \in H_{\text{div}}^{-1/2}(S_R)$  given by  $\varphi = \sum_{n=1}^{\infty} \sum_{m=-n}^n a_{n,m} U_n^m + b_{n,m} V_n^m$ , the operator  $Ge$  can be defined by

$$Ge\varphi = \sum_{n=1}^{\infty} \sum_{m=-n}^n \left\{ -ikR \frac{b_{n,m}}{\delta_n} U_n^m + \frac{a_{n,m} \delta_n}{ikR} V_n^m \right\}, \quad (2.23)$$

where

$$\delta_n = kR \frac{\left( h_n^{(1)} \right)'(kR)}{h_n^{(1)}(kR)} + 1, \quad (2.24)$$

and  $h_n^{(1)}(kR)$  is the spherical Hankel function.

If  $k = i$  in (2.23), we will obtain another operator  $\widetilde{G}_e$ . Properties of  $Ge$  and  $\widetilde{G}_e$  are collected in the following lemma (for a proof see [27]).  $\square$

**Lemma 2.2.** *The operator  $\widetilde{G}_e$  is negative definite in the sense that*

$$\left\langle \widetilde{G}_e \varphi, \varphi \times \hat{x} \right\rangle_{S_R} < 0 \quad (2.25)$$

for any  $\varphi \in H_{\text{div}}^{-1/2}(S_R)$  with  $\varphi \neq 0$ . Furthermore,

$$\left| \left\langle \widetilde{G}_e \varphi, \varphi \times \hat{x} \right\rangle_{S_R} \right| \geq C \|\varphi\|_{H_{\text{div}}^{-1/2}(S_R)}^2, \quad \forall \varphi \in H_{\text{div}}^{-1/2}(S_R), \quad (2.26)$$

$$Ge + ik\widetilde{G}_e : H_{\text{div}}^{-1/2}(\text{div}, S_R) \longrightarrow H_{\text{div}}^{-1/2}(S_R)$$

is compact, where

$$H_{\text{div}}^{-1/2}(\text{div}, S_R) = \left\{ \varphi = \sum_{n=1}^{\infty} \sum_{m=-n}^n b_{n,m} V_n^m \mid \sum_{n=1}^{\infty} \sum_{m=-n}^n \frac{1}{\sqrt{1+n(n+1)}} |b_{n,m}|^2 < \infty \right\}. \quad (2.27)$$

In the remainder of this paper we will refer to (2.17)–(2.21) as (CBP). Here we will adapt the variational approach used in [6, 27] to prove the existence of a unique solution to our (CBP). Define

$$X = \left\{ u \in H_{\text{loc}}(\text{curl}, B_R), \quad u_T|_{\Gamma} \in L_t^2(\Gamma) \right\}, \quad (2.28)$$

where  $D \subset B_R$ . Then multiplying (2.17) and (2.18) by test function  $\phi \in H_{\text{loc}}(\text{curl}, B_R)$ , using formally integration by parts and using the conductive boundary conditions on  $\Gamma$ , we can derive the following equivalent variational formulation for (CBP). Find  $w \in X$  such that

$$\begin{aligned} & \int_D \left( \text{curl } w \cdot \text{curl } \bar{\phi} - k_D^2 w \cdot \bar{\phi} \right) dx + \int_{D_R} \left( \text{curl } w \cdot \text{curl } \bar{\phi} - k_b^2 w \cdot \bar{\phi} \right) dx - ik \int_{\Gamma} \lambda w_T \cdot \bar{\phi}_T ds \\ & + ik \int_{S_R} G_e(\nu \times w) \cdot \bar{\phi}_T ds = ik \int_{S_R} \left[ G_e(\nu \times E^i) - \nu \times H^i \right] \cdot \bar{\phi}_T ds, \end{aligned} \quad (2.29)$$

where  $H^i = (1/ik_b) \text{curl } E^i$  is the incident magnetic field and

$$w = \begin{cases} V, & x \in D, \\ E = E^i + E^s, & x \in \mathbb{R}^3 \setminus \bar{D}. \end{cases} \quad (2.30)$$

We rewrite (2.29) as the problem of finding  $w \in X$  such that

$$A(w, \phi) = B(\phi), \quad (2.31)$$

where the sesquilinear form  $A : X \times X \rightarrow \mathbb{C}$  is defined by

$$\begin{aligned} A(w, \phi) &= (\text{curl } w, \text{curl } \phi)_D - (k_D^2 w, \phi)_D + (\text{curl } w, \text{curl } \phi)_{D_R} - (k_b^2 w, \phi)_{D_R} \\ &\quad - ik \langle \lambda w_T, \phi_T \rangle_{\Gamma} + ik \langle G_e(\nu \times w), \phi_T \rangle_{S_R}, \\ B(\phi) &= ik \langle G_e(\nu \times E^i) - \nu \times H^i, \phi_T \rangle_{S_R}. \end{aligned} \quad (2.32)$$

Here  $(\cdot, \cdot)_D$  denotes the  $(L^2(D))^3$  scalar product, and  $\langle \cdot, \cdot \rangle_{\partial D}$  denotes the  $(L^2(\partial D))^3$  scalar product. We will use a Helmholtz decomposition to factor out the nullspace of the curl operator and then to prove the existence of a unique solution to (CBP).

Define

$$S = \left\{ p \in H^1(D) \cap H^1(D_R) \mid p = \text{on } \Gamma \right\}, \quad (2.33)$$

then we seek  $p \in S$  such that

$$A(\nabla p, \nabla \xi) = B(\nabla \xi), \quad \forall \xi \in S. \quad (2.34)$$

The variational problem (2.34) can be rewritten as

$$A_1(p, \xi) + A_2(p, \xi) = B(\nabla \xi), \quad \forall \xi \in S, \quad (2.35)$$

where we define

$$\begin{aligned} A_1(p, \xi) &= -\left(k_D^2 \nabla p, \nabla \xi\right)_D - \left(k_b^2 \nabla p, \nabla \xi\right)_{D_R} + k^2 \left\langle \widetilde{G}_e(\nu \times \nabla p), \nabla_{\tau} \xi \right\rangle_{S_R}, \\ A_2(p, \xi) &= ik \left\langle G_e + ik \widetilde{G}_e(\nu \times \nabla p), \nabla_{\tau} \xi \right\rangle_{S_R}. \end{aligned} \quad (2.36)$$

Here we have used  $\nabla_{\tau} \xi$  to write the tangential component of the gradient of  $\xi$  in terms of the tangential gradient on the sphere  $S_R$ . By Lemma 2.2, it follows that  $\widetilde{G}_e$  is negative definite, then we obtain that  $A_1(p, \xi)$  is a coercive sesquilinear form on  $S \times S$ . Further by Lax-Milgram theorem, it is easy to see that  $A_1(p, \xi)$  gives rise to a bijective operator. Since  $\nu \times \nabla p \in H_{\text{div}}^{-1/2}(\text{div}, S_R)$ , still by Lemma 2.2, we know that  $A_2(p, \xi)$  gives rise to a compact operator. In order to apply the Fredholm alternative to the variational problem (2.34), we need to prove the following uniqueness lemma.

**Lemma 2.3.** *The variational problem (2.34) has at most one solution.*

*Proof.* It suffices to consider the following equation:

$$-\left(k_D^2 \nabla p, \nabla \xi\right)_D - \left(k_b^2 \nabla p, \nabla \xi\right)_{D_R} + ik \left\langle G_e(\nu \times \nabla p), \nabla_{\tau} \xi \right\rangle_{S_R} = 0, \quad \forall \xi \in S. \quad (2.37)$$

Choosing  $\xi = p$ , it is easy to see that

$$ik \left\langle G_e(\nu \times \nabla p), \nabla_{\tau} p \right\rangle_{S_R} = \left(k_D^2 \nabla p, \nabla p\right)_D + \left(k_b^2 \nabla p, \nabla p\right)_{D_R}. \quad (2.38)$$

By the definition of the operator  $G_e$ , if  $E^s \in H_{\text{loc}}(\text{curl}, \mathbb{R}^3 \setminus \overline{B})$  is the weak solution of the problem

$$\begin{aligned} \text{curl curl } E^s - k^2 E^s &= 0, \quad \text{in } \mathbb{R}^3 \setminus \overline{B}, \\ \nu \times E^s &= \nu \times \nabla p, \quad \text{on } S_R, \\ \lim_{r \rightarrow \infty} (\text{curl } E^s \times x - ikr E^s) &= 0, \end{aligned} \quad (2.39)$$

then we have

$$G_e(\nu \times \nabla p) = \nu \times H^s, \quad \text{on } S_R, \quad (2.40)$$

where

$$H^s = \frac{1}{ik} \text{curl } E^s. \quad (2.41)$$



Furthermore, we can compute that

$$\begin{aligned} \int_{S_R} H^s \cdot \overline{\nu \times E^s} ds &= -\langle \nu \times H^s, \nabla_\tau p \rangle_{S_R} = -\langle Ge(\nu \times \nabla p), \nabla_\tau p \rangle_{S_R} \\ &= -\frac{1}{ik} (k_D^2 \nabla p, \nabla p)_D - \frac{1}{ik} (k_b^2 \nabla p, \nabla p)_{D_R}, \end{aligned} \quad (2.42)$$

which together with the fact  $\text{Im } n_D \geq 0$ ,  $\text{Im } n_b \geq 0$  implies

$$\begin{aligned} \text{Re} \int_{S_R} \nu \times E^s \cdot \overline{H^s} ds &= \text{Re} \int_{S_R} H^s \cdot \overline{\nu \times E^s} ds \\ &= -\frac{1}{k} (\text{Im } k_D^2 \nabla p, \nabla p)_D - \frac{1}{k} (\text{Im } k_b^2 \nabla p, \nabla p)_{D_R} \leq 0. \end{aligned} \quad (2.43)$$

Therefore the Rellich lemma ensures us that  $E^s = 0$  in  $\mathbb{R}^3 \setminus \overline{B}$ . From (2.39), we see that  $\nabla_\tau p = 0$  on  $S_R$  and then  $(k_D^2 \nabla p, \nabla p)_D + (k_b^2 \nabla p, \nabla p)_{D_R} = 0$  which, together with the fact that  $p|_\Gamma = 0$ , implies  $p = 0$ . This completes the proof of Lemma 2.3.  $\square$

Lemma 2.3 together with the Fredholm alternative implies that there exists a unique solution  $p_0 \in S$  of the variational problem (2.34).

**Lemma 2.4.** *The space*

$$X_0 = \{u \in X \mid A(u, \nabla \xi) = 0, \forall \xi \in S\} \quad (2.44)$$

is compactly imbedded in  $(L^2(B))^3$ , where  $B$  is a ball with  $D \subset B \subset B_R$ .

*Proof.* Consider a bounded set of functions  $\{u_j\}_{j=1}^\infty \subset X_0$ . Each function  $u_j \in X_0$  can be extended to all of  $\mathbb{R}^3$  by solving the exterior Maxwell equation

$$\begin{aligned} \nabla \times (\nabla \times v_j) - k_b^2 v_j &= 0, \quad \text{in } \mathbb{R}^3 \setminus \overline{B}, \\ \nu \times v_j &= \nu \times u_j, \quad \text{on } S_R, \end{aligned} \quad (2.45)$$

$$\lim_{|x| \rightarrow \infty} |x| ((\nabla \times v_j) \times \nu - ik v_j) = 0.$$

Define

$$u_j^e = \begin{cases} u_j, & \text{if } x \in B, \\ v_j, & \text{if } x \in \mathbb{R}^3 \setminus \overline{B}. \end{cases} \quad (2.46)$$

Since the tangential components of  $u_j^e$  are continuous across  $S_R$ , it follows that  $u_j^e \in H_{\text{loc}}(\text{curl}, \mathbb{R}^3)$ . By using the properties of the Calderon operator  $G_e$  and the conditions in  $X_0$ , we see that the following equations hold true

$$\begin{aligned} \text{div}(k_D^2 u) &= 0, \quad \text{in } D, \\ \text{div}(k_b^2 u) &= 0, \quad \text{in } D_R, \\ k_b^2 \nu \cdot u &= -ik \nabla_\tau \cdot G_e(\nu \times u), \quad \text{on } S_R. \end{aligned} \tag{2.47}$$

Then, by the definition of  $G_e$  that  $G_e(\nu \times u_j) = (1/ik)\nu \times \text{curl } v_j$  and the relationship  $\nabla_\tau \cdot (\nu \times v_j) = -\nu \cdot \text{curl } v_j$  on  $S_R$ , we immediately have

$$\begin{aligned} k_b^2 \nu \cdot u_j &= -ik \nabla_\tau \cdot G_e(\nu \times u_j) = -\nabla_\tau \cdot (\nu \times \text{curl } v_j) = \nu \cdot \text{curl curl } v_j \\ &= k_b^2 \nu \cdot v_j, \quad \text{on } S_R. \end{aligned} \tag{2.48}$$

Thus,  $u_j^e$  has a well-defined divergence and  $\nabla \cdot (\widehat{k}^2 u_j^e) = 0$  in  $\mathbb{R}^3 \setminus \Gamma$ , where

$$\widehat{k}^2 = \begin{cases} k_D^2, & x \in D, \\ k_b^2, & x \in \mathbb{R}^3 \setminus \overline{D}. \end{cases} \tag{2.49}$$

Now we choose a cut-off function  $\chi \in C_0^\infty(\mathbb{R}^3)$  such that  $\chi = 1$  in  $\overline{B}$  and  $\chi$  is supported in a ball  $B_{R_1} \supset \overline{B}$ . Then one can use the general compactness theorem (Theorem 4.7 in [27]) to the sequence  $\{\chi u_j^e\}$  and extract a subsequence converging strongly in  $(L^2(B))^3$ . This proves the lemma.  $\square$

From the above definitions of  $S$  and  $X_0$ , we have the following Helmholtz decomposition lemma.

**Lemma 2.5.** *The spaces  $\nabla S$  and  $X_0$  are closed subspaces of  $X$ . The space  $X$  is the direct sum of the spaces  $\nabla S$  and  $X_0$ , that is,*

$$X = X_0 \oplus \nabla S. \tag{2.50}$$

The proof of this Helmholtz decomposition Lemma is entirely classical (see [27, 28]).

We now look for a solution of the variational problem (2.31) in the form  $w = w_0 + \nabla p_0$ , where  $w_0 \in X_0$  and  $p_0 \in S$  is the unique solution of (2.34). We observe that  $A(w_0, \nabla \xi) = 0$  for all  $\xi \in S$  by the definition of  $X_0$ . Hence the problem of determining  $w \in X$  is equivalent to the problem of determining  $w_0 \in X_0$  such that

$$A(w_0, \phi_0) = B(\phi_0) - A(\nabla p_0, \phi_0), \quad \forall \phi_0 \in X_0. \tag{2.51}$$

From Chapter 10.3.2 in [27] we know that for  $\varphi \in H_{\text{div}}^{-1/2}(S_R)$

$$G_e \varphi = G_{1e} \varphi + G_{2e} \varphi, \quad (2.52)$$

where the operator  $G_{1e}$  is a compact operator from  $X_0$  into  $H_{\text{div}}^{-1/2}(S_R)$  and the operator  $G_{2e}$  satisfies  $ik \langle G_{2e}(\nu \times \varphi), \varphi_T \rangle_{S_R} \geq 0$ . We now split the sesquilinear form  $A(\cdot, \cdot)$  into  $A = a + b$  with

$$\begin{aligned} a(w_0, \phi_0) &= (\text{curl } w_0, \text{curl } \phi_0)_D + (w_0, \phi_0)_D + (\text{curl } w_0, \text{curl } \phi_0)_{D_R} + (w_0, \phi_0)_{D_R} \\ &\quad - ik \langle \lambda(w_0)_T, (\phi_0)_T \rangle_T + ik \langle G_{2e}(\nu \times w_0), (\phi_0)_T \rangle_{S_R}, \end{aligned} \quad (2.53)$$

$$b(w_0, \phi_0) = -\left((k_D^2 + 1)w_0, \phi_0\right)_D - \left((k_b^2 + 1)w_0, \phi_0\right)_{D_R} + ik \langle G_{1e}(\nu \times w_0), (\phi_0)_T \rangle_{S_R}. \quad (2.54)$$

The sesquilinear form  $a(\cdot, \cdot)$  is obviously bounded and a direct computation verifies that

$$|a(w_0, w_0)| \geq \alpha \|w_0\|_{X_0}^2, \quad \forall w_0 \in X_0, \quad (2.55)$$

with some constant  $\alpha > 0$ .

Hence by Lax-Milgram theorem,  $a(\cdot, \cdot)$  gives rise to a bijective operator and by the compact embedding of  $X_0$  in  $(L^2(B))^3$  and the fact that  $G_{1e}$  is a compact operator from  $X_0$  into  $H_{\text{div}}^{-1/2}(S_R)$ , the second term  $b(\cdot, \cdot)$  gives rise to a compact operator. Then a standard argument implies that the Fredholm alternative can be applied. Finally, the uniqueness theorem yields the existence result. We summarize the above analysis in the following theorem.

**Theorem 2.6.** *For any incident field  $E^i$ , there exists a unique solution  $w \in X$  of (CBP) which depends continuously on the incident field  $E^i$ .*

### 3. Uniqueness for the Inverse Problem

In this section we will show that the scattering obstacle  $D$  and the corresponding parameter  $\lambda$  are uniquely determined from the knowledge of the scattered fields  $E_{1,x_0}^s(x, x_0)|_{x \in \partial\Omega}$  for all  $x_0 \in S_{R_0}$ , where  $S_{R_0}$  is the surface of a large ball  $B_{R_0}$  with  $\overline{D} \subset \overline{\Omega} \subset B_{R_0}$ . By some properties of the scattered fields, we can derive a relationship between them, then constructing special singular solutions which satisfy the relationship. Finally, we can obtain the uniqueness result by using the singularities of the singular solutions that we constructed.

**Lemma 3.1.** *Assume that  $k_b^2$  is not an eigenvalue of Maxwell equation for the domain  $\Omega_0$ . Then we have*

- (i) *the restriction to  $\partial\Omega_0$  of  $\{\nu \times p(x_0)G(x, x_0) : x_0 \in S_{R_0}\}$  is complete in  $H_{\text{div}}^{-1/2}(\partial\Omega_0)$ ;*
- (ii) *the restriction to  $\partial\Omega_0$  of  $\{\nu \times \text{curl}_x p(x_0)G(x, x_0) - ik\lambda p(x_0)G(x, x_0)_T : x_0 \in S_{R_0}\}$  is complete in  $L_T^2(\partial\Omega_0)$ .*

*Proof.* For simplicity, we only prove statement (ii). Case (i) can be proved similarly.

Let  $g \in L^2_\Gamma(\partial\Omega_0)$  be such that

$$\int_{\partial\Omega_0} [\nu(x) \times \text{curl}_x p(x_0)G(x, x_0) - ik\lambda p(x_0)G(x, x_0)_T] \cdot \bar{g}(x) ds(x) = 0. \quad (3.1)$$

Then it follows that

$$\int_{\partial\Omega_0} [\text{curl}_{x_0}(\bar{g}(x) \times \nu(x))G(x, x_0) - ik\lambda \bar{g}(x)G(x, x_0)] ds(x) \cdot p(x_0) = 0. \quad (3.2)$$

Define

$$F_1^*(x_0) = \int_{\partial\Omega_0} \text{curl}_{x_0}(\bar{g}(x) \times \nu(x))G(x, x_0) ds(x), \quad (3.3)$$

$$F_2^*(x_0) = - \int_{\partial\Omega_0} ik\lambda \bar{g}(x)G(x, x_0) ds(x), \quad (3.4)$$

$$F^*(x_0) = F_1^*(x_0) + F_2^*(x_0). \quad (3.5)$$

By (3.2), it is easy to see that for arbitrary polarization  $p(x_0)$  in the tangential plane to  $S_{R_0}$  at  $x_0$ , we have

$$F^*(x_0) \cdot p(x_0) = 0, \quad \forall x_0 \in S_{R_0}. \quad (3.6)$$

From the definition of (3.3), we immediately have

$$\text{curl}_{x_0} \text{curl}_{x_0} F_1^*(x_0) - k_b^2 F_1^*(x_0) = 0, \quad \forall x_0 \in \mathbb{R}^3 \setminus \partial\Omega_0. \quad (3.7)$$

Due to the symmetry of the background Green function,  $F_2^*(x_0)$  as a function of  $x_0$  solves  $\text{curl}_{x_0} \text{curl}_{x_0} F_2^*(x_0) - k_b^2 F_2^*(x_0) = 0, \forall x_0 \in \mathbb{R}^3 \setminus \partial\Omega_0$ . Hence,  $F^*(x_0)$  satisfies the Maxwell's equation in  $\mathbb{R}^3 \setminus \partial\Omega_0$ . By (3.6) and the fact that  $p(x_0)$  is an arbitrary polarization in the tangential plane to  $S_{R_0}$  at  $x_0$ , we immediately have that  $\nu \times F^*(x_0)|_{S_{R_0}} = 0$ .

The uniqueness of the exterior problem

$$\begin{aligned} \text{curl}_{x_0} \text{curl}_{x_0} F^*(x_0) - k_b^2 F^*(x_0) &= 0, \quad \text{in } \mathbb{R}^3 \setminus \bar{B}_{R_0}, \\ \nu \times F^*(x_0) &= 0, \quad \text{on } S_{R_0}, \end{aligned} \quad (3.8)$$

$$\lim_{|x_0| \rightarrow \infty} |x_0| (\text{curl}_{x_0} F^*(x_0) \times \nu(x_0) - ikF^*(x_0)) = 0$$

implies that  $F^*(x_0) = 0$  in  $\mathbb{R}^3 \setminus \bar{B}_{R_0}$ . Thus, the unique continuation principle ensures us that  $F^*(x_0) = 0$  in  $\Omega_0^e = \mathbb{R}^3 \setminus \bar{\Omega}_0$ . By trace theorem, it follows that  $\nu \times F^*(x_0) = 0$  and  $\nu \times \text{curl}_{x_0} F^*(x_0) =$

0 on  $\partial\Omega_0$ . By the definition of  $F^*(x_0)$  and the jump relations of the vector potential across  $\partial\Omega_0$ , it can be checked that  $F^*(x_0)$  satisfies the following equations:

$$\begin{aligned} \operatorname{curl}_{x_0} \operatorname{curl}_{x_0} F^*(x_0) - k_b^2 F^*(x_0) &= 0, \quad \text{in } \Omega_0, \\ \nu \times \operatorname{curl}_{x_0} F^*(x_0) - ik\lambda\nu \times F^*(x_0) &= 0, \quad \text{on } \partial\Omega_0. \end{aligned} \quad (3.9)$$

Therefore, the uniqueness theorem of the interior problem for Maxwell's equations implies that  $F^*(x_0) = 0$  in  $\Omega_0$ . Finally, from the jump relations of the vector potential across  $\partial\Omega_0$ , we have

$$0 = [\nu \times F^*(x_0) \times \nu]_{\partial\Omega_0} = -\bar{g}(x_0), \quad (3.10)$$

which completes the proof.  $\square$

We now consider two obstacles  $D_1$  and  $D_2$  with the refractive index  $n_{D_j}$  and the surface impedance  $\lambda_j$ ,  $j = 1, 2$ . Let  $U$  denote the unbounded part of  $\mathbb{R}^3 \setminus (\bar{D}_1 \cup \bar{D}_2)$  and  $D_0 = \mathbb{R}^3 \setminus \bar{U}$  its open complement. From the proof of Theorem 2.6, it follows that the total field  $w_j$ ,  $j = 1, 2$  satisfies

$$(\operatorname{curl} w_j, \operatorname{curl} \phi)_{B_R} - (k_{j*}^2 w_j, \phi)_{B_R} - ik \langle \lambda_j w_{jT}, \phi_T \rangle_{\Gamma_j} = -\langle \nu \times \operatorname{curl} w_j, \phi \rangle_{S_R}, \quad (3.11)$$

for any large ball  $B_R$  with  $D \subset B_R$  and all test function  $\phi \in \tilde{X}$ , where

$$k_{j*}^2 = \begin{cases} k_{D_j}^2, & x \in D_j; \\ k_b^2, & x \in \mathbb{R}^3 \setminus \bar{D}_j. \end{cases} \quad (3.12)$$

It is convenient to introduce the following space:

$$X_j = \left\{ E \in H(\operatorname{curl}, B_R), E_T|_{\Gamma_j} \in L_t^2(\Gamma_j) \right\}, \quad j = 1, 2, \quad (3.13)$$

where  $D_j \subset B_R$ . The relationship derived in the following lemma plays a central role in the proof of the main result in this section.

**Lemma 3.2.** *Assume that  $k_b^2$  is not an eigenvalue of Maxwell equation in  $\Omega$ . Let  $B_R \subset \mathbb{R}^3$  be a ball with  $\bar{D}_0 \subset \bar{\Omega} \subset B_R$ . Let  $E_{1,x_0}^s(x, x_0)$  and  $E_{2,x_0}^s(x, x_0)$  be the scattered fields with respect to  $D_1$  and  $D_2$ , respectively, produced by the same incident field  $pG(\cdot, x_0)$ . Assume that  $\nu \times E_{1,x_0}^s|_{\partial\Omega} = \nu \times E_{2,x_0}^s|_{\partial\Omega}$  for all  $x_0 \in S_{R_0}$  with the radius  $R_0 > R$  for a fixed wave number  $k$ . Then we have*

$$\begin{aligned} k^2 \left( (n_{D_1} - n_b) V_1, \bar{V}_2 \right)_{D_1} + ik \left\langle \lambda_1 (V_1)_T, (\bar{V}_2)_T \right\rangle_{\Gamma_1} \\ = k^2 \left( (n_{D_2} - n_b) V_1, \bar{V}_2 \right)_{D_2} + ik \left\langle \lambda_2 (V_1)_T, (\bar{V}_2)_T \right\rangle_{\Gamma_2}. \end{aligned} \quad (3.14)$$

Here  $V_j \in X_1 \cap X_2$  satisfies the following variational problem:

$$(a_j \operatorname{curl} V_j, \operatorname{curl} V)_{B_R} - (\kappa_j V_j, V)_{B_R} - ik \langle \tilde{\lambda}_j (V_j)_T, V_T \rangle_{\Gamma_j} = 0, \quad (3.15)$$

for all  $V \in H_0(\operatorname{curl}, B_R)$ , where the coefficients  $a_j, \kappa_j \in L^\infty(B_R)$  and  $\tilde{\lambda}_j \in L^\infty(\Gamma_j)$  satisfy that  $a_j|_{D_0} = 1$ ,  $\kappa_j|_{D_0} = (\kappa_{j*}^2)|_{D_0}$  and  $\tilde{\lambda}_j|_{\Gamma_j \cap D_0} = \lambda_j|_{\Gamma_j}$ ,  $j = 1, 2$ .

*Proof.* (i) We first prove that for any fixed  $z \in U$ , the scattered fields  $E_{1,z}^s(x, z) = E_{2,z}^s(x, z)$ , where  $(V_{j,z}, E_{j,z}^s)$ ,  $j = 1, 2$  is the solution of the following problem:

$$\begin{aligned} \operatorname{curl} \operatorname{curl} V - k_{D_j}^2 V &= 0, \quad \text{in } D_j, \\ \operatorname{curl} \operatorname{curl} E^s - k_b^2 E^s &= 0, \quad \text{in } \mathbb{R}^3 \setminus \bar{D}_j, \\ \nu \times (V - E^s) &= f, \quad \text{on } \Gamma_j, \\ \nu \times \operatorname{curl}(V - E^s) &= -ik\lambda_j E_T^s + r + h, \quad \text{on } \Gamma_j, \end{aligned} \quad (3.16)$$

with the incident field  $E^i = pG(\cdot, z)$  and  $f = \nu \times E^i$ ,  $h = \nu \times \operatorname{curl} E^i$ ,  $r = -ik\lambda_j E_T^i$ . By Lemma 3.1 and the fact that  $k_b^2$  is not an eigenvalue of Maxwell equation in  $\Omega$ , it follows that there exists a sequence  $a_n \in \mathbb{R}$  and  $x_0^{(n)} \in S_{R_0}$  such that

$$\left\| \nu \times pG(\cdot, z) - \sum_n a_n \nu \times pG(\cdot, x_0^{(n)}) \right\|_{H_{\operatorname{div}}^{-1/2}(\partial\Omega)} < \varepsilon, \quad \forall \varepsilon > 0. \quad (3.17)$$

Let  $E(x) = pG(\cdot, z) - \sum_n a_n pG(\cdot, x_0^{(n)})$ , then it satisfies the Maxwell equation  $\operatorname{curl} \operatorname{curl} E - k_b^2 E = 0$  in  $\Omega \setminus \{z\}$ . Let  $f = (\nu \times E)|_{\partial\Omega}$ , then the well posedness of the problem

$$\begin{aligned} \operatorname{curl} \operatorname{curl} E - k_b^2 E &= 0, \quad \text{in } \Omega \setminus \{z\}, \\ \nu \times E &= f, \quad \text{on } \partial\Omega \end{aligned} \quad (3.18)$$

and (3.17) imply that

$$\|E\|_{H(\operatorname{curl}, \Omega)} \leq C_1 \|\nu \times E\|_{H_{\operatorname{div}}^{-1/2}(\partial\Omega)} < C_1 \varepsilon, \quad \forall \varepsilon > 0. \quad (3.19)$$

This, together with the fact that  $\operatorname{div} E = 0$ , implies (see [28])

$$\|E\|_{H^1(\Omega)} \leq C_2 \left( \|E\|_{H(\operatorname{curl}, \Omega)} + \|E\|_{H(\operatorname{div}, \Omega)} + \|\nu \times E\|_{H_{\operatorname{div}}^{-1/2}(\partial\Omega)} \right) \leq C_2 \varepsilon, \quad \forall \varepsilon > 0. \quad (3.20)$$

Then by (3.19), (3.20), and the trace theorem, it can be proved that

$$\begin{aligned} & \|\nu \times E\|_{H_{\text{div}}^{-1/2}(\Gamma_j)} + \|\nu \times \text{curl } E\|_{L_t^2(\Gamma_j)} + \|\nu \times \text{curl } E - ik\lambda_j E_T\|_{L_t^2(\Gamma_j)} \\ & \leq C_3 \left( \|E\|_{H(\text{curl}, D_j)} + \|E_T\|_{L_t^2(\Gamma_j)} \right) \leq C_4 \left( \|E\|_{H(\text{curl}, D_j)} + \|E\|_{H^1(D_j)} \right) \\ & \leq C_5 \left( \|E\|_{H(\text{curl}, \Omega)} + \|E\|_{H^1(\Omega)} \right) \leq C_5 \varepsilon, \quad j = 1, 2, \quad \forall \varepsilon > 0. \end{aligned} \quad (3.21)$$

Denote by  $\tilde{E}_1^s$  and  $\tilde{E}_2^s$  the scattered fields with respect to  $D_1$  and  $D_2$  produced by the same incident field  $\sum_n a_n pG(\cdot, x_0^{(n)})$ . By the assumption  $\nu \times E_{1, x_0}^s|_{\partial\Omega} = \nu \times E_{2, x_0}^s|_{\partial\Omega}$  for all  $x_0 \in S_{R_0}$ , it is easy to see that  $\nu \times \tilde{E}_1^s|_{\partial\Omega} = \nu \times \tilde{E}_2^s|_{\partial\Omega}$ . Then by the uniqueness theorem of the exterior scattering problem, it follows that  $\tilde{E}_1^s = \tilde{E}_2^s$  in  $\mathbb{R}^3 \setminus \bar{\Omega}$ , which together with the unique continuation principle ensures that  $\tilde{E}_1^s = \tilde{E}_2^s$  in  $U$ . Now, by (3.21) and the well posedness of the direct problem (2.18), it can be checked that for any compact set  $K \subset\subset U$ , we have

$$\begin{aligned} & \left\| E_{1,z}^s(x, z) - \tilde{E}_1^s(x) \right\|_{H(\text{curl}, K)} < \varepsilon, \quad \forall \varepsilon > 0, \\ & \left\| E_{1,z}^s(x, z) - \tilde{E}_2^s(x) \right\|_{H(\text{curl}, K)} < \varepsilon, \quad \forall \varepsilon > 0, \end{aligned} \quad (3.22)$$

for any fixed  $z \in K$ .

Therefore, the fact  $\tilde{E}_1^s = \tilde{E}_2^s$  in  $U$  ensures us that

$$\left\| E_{1,z}^s(x, z) - E_{2,z}^s(x, z) \right\|_{H(\text{curl}, K)} < \varepsilon, \quad \forall \varepsilon > 0. \quad (3.23)$$

The arbitrariness of  $\varepsilon$  implies that  $E_{1,z}^s(x, z) = E_{2,z}^s(x, z)$  for any fixed  $z \in K$ .

(ii) Next we will show that the identity (3.14) holds. Set  $w^* = w_2 - w_1$ , then it follows from (3.11) that

$$\begin{aligned} & (\text{curl } w^*, \text{curl } V)_{B_R} - (k_{2*}^2 w^*, V)_{B_R} - ik \langle \lambda_2 w_T^*, V_T \rangle_{\Gamma_2} \\ & = - \left( (k_{1*}^2 - k_{2*}^2) w_1, V \right)_{B_R} - ik \langle \lambda_1 (w_1)_T, V_T \rangle_{\Gamma_1} + ik \langle \lambda_2 (w_1)_T, V_T \rangle_{\Gamma_2} \end{aligned} \quad (3.24)$$

for all  $V \in X^* = \{V \in X_1 \cap X_2, V_T|_{S_R} = 0\}$ . Choose two domains  $\Omega_1, \Omega_2 \subset B_R$  with  $\bar{D}_0 \subset \Omega_1 \subset \Omega_2$  and define a smooth function  $\psi \in C^\infty(\mathbb{R}^3)$  with  $\psi = 1$  in  $\Omega_1$  and  $\psi = 0$  in  $\mathbb{R}^3 \setminus \Omega_2$ . Let  $a_2 = \psi$ ,  $\kappa_2 = \psi k_{2*}^2$ ,  $\tilde{\lambda}_2 = \lambda_2$ , it is easy to see that  $a_2, \kappa_2, \tilde{\lambda}_2$  satisfy the assumptions of the lemma. We further assume  $V_2$  satisfies (3.15) with respect to  $a_2 = \psi$ ,  $\kappa_2 = \psi k_{2*}^2$ ,  $\tilde{\lambda}_2 = \lambda_2$ , so that substituting  $V = \overline{\psi V_2}$  into the left hand of (3.24) and noting that  $w^* = 0$  in  $U$  yield that

$$\begin{aligned} & \left( \text{curl } w^*, \text{curl}(\overline{\psi V_2}) \right)_{B_R} - (k_{2*}^2 w^*, \overline{\psi V_2})_{B_R} - ik \langle \lambda_2 w_T^*, \overline{\psi(V_2)_T} \rangle_{\Gamma_2} \\ & = -(a_2 \text{curl } V_2, \text{curl } \overline{w^*})_{B_R} - (\kappa_2 V_2, \overline{w^*})_{B_R} - ik \langle \tilde{\lambda}_2 V_{2T}, \overline{w_T^*} \rangle_{\Gamma_2} = 0. \end{aligned} \quad (3.25)$$

Hence substituting  $V = \overline{\psi V_2}$  into (3.24), it follows from the right hand of (3.24) that

$$-\left(\left(k_{1*}^2 - k_{2*}^2\right)w_1, \overline{V_2}\right)_{B_R} - ik\langle\lambda_1(w_1)_T, (\overline{V_2})_T\rangle_{\Gamma_1} + ik\langle\lambda_2(w_1)_T, (\overline{V_2})_T\rangle_{\Gamma_2} = 0. \quad (3.26)$$

We define  $f \in X_1$  by

$$(f, \phi)_{X_1} = -\left(\left(k_{1*}^2 - k_{2*}^2\right)V_2, \phi\right)_{B_R} - ik\langle\lambda_1(V_2)_T, \phi_T\rangle_{\Gamma_1} + ik\langle\lambda_2(V_2)_T, \phi_T\rangle_{\Gamma_2}, \quad (3.27)$$

for all  $\phi \in X_1$ . By Theorem 2.6, it follows that there exists a unique solution  $w_0 \in X_1$  of the problem

$$(\operatorname{curl} w_0, \operatorname{curl} \phi)_{B_R} - \left(k_{1*}^2 w_0, \phi\right)_{B_R} - ik\langle\lambda_1(w_0)_T, \phi_T\rangle_{\Gamma_1} + ik\langle Ge(\nu \times w_0), \phi_T\rangle_{S_R} = (f, \phi)_{X_1}, \quad (3.28)$$

for all  $\phi \in X_1$ . Choose two domains  $\Omega_e, \Omega_i \subseteq B_R$  with  $\overline{\Omega} \subseteq \Omega_e, \overline{D_0} \subset \Omega_i \subset \Omega$  and define smooth functions  $\psi_e, \psi_i \in C^\infty(\mathbb{R}^3)$  with  $\psi_e = 1, x \in \Omega, \psi_e = 0, x \in \mathbb{R}^3 \setminus \Omega_e$  and  $\psi_i = 1, x \in \Omega_i, \psi_i = 0, x \in \mathbb{R}^3 \setminus \Omega$ . Take  $j = 1, \phi = \overline{\psi_e w_0}$  in (3.11), it is seen that

$$(\operatorname{curl} w_1, \operatorname{curl}(\overline{\psi_e w_0}))_{B_R} - \left(k_{1*}^2 w_1, \overline{\psi_e w_0}\right)_{B_R} - ik\langle\lambda_1(w_1)_T, (\overline{w_0})_T\rangle_{\Gamma_1} = 0. \quad (3.29)$$

Equation (3.28) with  $\phi$  replaced by  $\overline{\psi_i w_1}$  yields

$$(\operatorname{curl} w_0, \operatorname{curl}(\overline{\psi_i w_1}))_{B_R} - \left(k_{1*}^2 w_0, \overline{\psi_i w_1}\right)_{B_R} - ik\langle\lambda_1(w_0)_T, (\overline{w_1})_T\rangle_{\Gamma_1} = (f, \overline{\psi_i w_1})_{X_1}. \quad (3.30)$$

By (3.26) and (3.27), it can be shown that

$$\begin{aligned} (f, \overline{\psi_i w_1})_{X_1} &= -\left(\left(k_{1*}^2 - k_{2*}^2\right)V_2, \overline{w_1}\right)_{B_R} - ik\langle\lambda_1(V_2)_T, (\overline{w_1})_T\rangle_{\Gamma_1} \\ &\quad + ik\langle\lambda_2(V_2)_T, (\overline{w_1})_T\rangle_{\Gamma_2} = 0. \end{aligned} \quad (3.31)$$

Taking the difference of (3.29) and (3.30), we have that

$$\begin{aligned} 0 &= \int_{\Omega \setminus \Omega_i} \operatorname{curl} w_0 \cdot \operatorname{curl}(\psi_i w_1 - w_1) - k_b^2 w_0 \cdot (\psi_i w_1 - w_1) dx \\ &\quad - \int_{\Omega_e \setminus \Omega} \operatorname{curl}(\psi_e w_0) \cdot \operatorname{curl} w_1 - k_b^2(\psi_e w_0) \cdot w_1 dx. \end{aligned} \quad (3.32)$$

By (3.28), we can deduce that  $w_0$  is a radiating solution of the corresponding Maxwell's equations in  $B_R$ , then it can be extended to all of  $\mathbb{R}^3$  denoted by  $w_0^e$  by solving the exterior



Maxwell's equation in  $\mathbb{R}^3 \setminus \overline{B}_R$  with  $\nu \times w_0^e = \nu \times w_0$  on  $S_R$ , which also satisfies the Silver-Müller radiation condition at infinity. By applying the vector Green formula to (3.32), it can be proved that

$$0 = \int_{\partial\Omega} [w_1 \cdot (\nu \times \text{curl } w_0^e) - w_0^e \cdot (\nu \times \text{curl } w_1)] ds. \quad (3.33)$$

In view of the fact  $w_1 = E_1 = E_1^s + E^i$  and  $E^i = pG(\cdot, x_0)$  in  $\mathbb{R}^3 \setminus \overline{D}_1$ , we immediately have

$$\begin{aligned} 0 &= \int_{\partial\Omega} E_1^s \cdot (\nu \times \text{curl } w_0^e) - w_0^e \cdot (\nu \times \text{curl } E_1^s) ds \\ &+ \int_{\partial\Omega} [(\nu \times \text{curl } w_0^e) \cdot pG(\cdot, x_0) - w_0^e \cdot (\nu \times \text{curl}(pG(\cdot, x_0)))] ds. \end{aligned} \quad (3.34)$$

Application of the vector Green formula again and noting that both  $E_1^s$  and the extended function  $w_0^e$  satisfy the Silver-Müller radiation condition, it follows that

$$\begin{aligned} &\int_{\partial\Omega} E_1^s \cdot (\nu \times \text{curl } w_0^e) - w_0^e \cdot (\nu \times \text{curl } E_1^s) ds \\ &= \int_{S_R} E_1^s \cdot (\nu \times \text{curl } w_0^e) - w_0^e \cdot (\nu \times \text{curl } E_1^s) ds = 0. \end{aligned} \quad (3.35)$$

Hence the Stratton-Chu formula combines with (3.34) implies that

$$0 = p(x_0) \cdot w_0^e(x_0), \quad \forall x_0 \in S_{R_0}, \quad \forall p(x_0) \in \mathbb{R}^3. \quad (3.36)$$

Since  $p(x_0)$  is an arbitrary polarization in the tangential plane to  $S_{R_0}$  at  $x_0$ , we obtain that  $\nu \times w_0^e(x_0)|_{S_{R_0}} = 0$ . By the fact that  $w_0^e$  is a radiating solution of Maxwell's equation in  $\mathbb{R}^3 \setminus \overline{B}_{R_0}$ , it follows that  $w_0^e = 0$  in  $\mathbb{R}^3 \setminus \overline{B}_{R_0}$ . Hence the unique continuation principle implies that  $w_0^e = 0$  in  $\mathbb{R}^3 \setminus \overline{U}$ . Therefore,  $w_0^e$  can be used as a test function for  $V_1$ , which satisfies (3.15) with  $a_1 = \psi_i$ ,  $\kappa_1 = \psi_i k_{1*}^2$ ,  $\tilde{\lambda}_1 = \lambda_1$ . So that from the left hand of (3.30), we deduce that

$$\begin{aligned} &\left( \text{curl } w_0^e, \text{curl}(\overline{\psi_i V_1}) \right)_{B_R} - \left( k_{1*}^2 w_0^e, \overline{\psi_i V_1} \right)_{B_R} - ik \left\langle \lambda_1 (w_0^e)_T, (\overline{V_1})_T \right\rangle_{\Gamma_1} \\ &= (a_1 \text{curl } V_1, \text{curl } \overline{w_0^e})_{B_R} - (\kappa_1 V_1, \overline{w_0^e})_{B_R} - ik \left\langle \tilde{\lambda}_1 (V_1)_T, (\overline{w_0^e})_T \right\rangle_{\Gamma_1} = 0. \end{aligned} \quad (3.37)$$

Thus, it follows from the right hand of (3.30) that  $(f, \overline{\psi_i V_1})_{X_1} = 0$ . Furthermore, from (3.27) with  $\phi$  replaced by  $\overline{\psi_i V_1}$ , it can be shown that

$$-\left( (k_{1*}^2 - k_{2*}^2) V_2, V_1 \right)_{B_R} - ik \left\langle \lambda_1 (V_1)_T, (V_2)_T \right\rangle_{\Gamma_1} + ik \left\langle \lambda_2 (V_2)_T, (V_1)_T \right\rangle_{\Gamma_2} = 0. \quad (3.38)$$

From the definitions of  $k_{j*}^2$ , we observe that

$$k_{1*}^2 - k_{2*}^2 = \begin{cases} k^2 n_{D_1} - k^2 n_b, & x \in D_1, \\ k^2 n_{D_2} - k^2 n_b, & x \in D_2, \\ 0, & x \in B_R \setminus (\overline{D_1} \cup \overline{D_2}), \end{cases} \quad (3.39)$$

which combines (3.38), the definition of the scalar product  $(\cdot, \cdot)_D$ , and the fact that  $((k_{1*}^2 - k_{2*}^2)V_2, V_1)_{B_R} = k^2((n_{D_1} - n_b)V_2, V_1)_{D_1} - k^2((n_{D_2} - n_b)V_2, V_1)_{D_2}$  implies that (3.14) holds. This ends the proof of this lemma.  $\square$

The main result of this section is contained in the following theorem.

**Theorem 3.3.** *Let  $E_1^s$  and  $E_2^s$  be the scattered fields with respect to  $D_1$  and  $D_2$ , respectively, and  $\lambda_1, \lambda_2$  the corresponding impedances. Suppose that the assumptions in Lemma 3.2 hold true and  $\Gamma_j \cap (D_i \setminus D_j)$  is not empty for  $i, j = 1, 2$ ,  $i \neq j$ . If one of the following assumptions holds, then we have  $D_1 = D_2$ . Consider*

- (i)  $\operatorname{Re} \lambda_j \geq \delta > 0$ ;
- (ii)  $\operatorname{Im} \lambda_j \geq \delta > 0$  or  $\operatorname{Im} \lambda_j \leq -\delta < 0$ .

*Proof.* Let us assume that  $\overline{D_1}$  is not included in  $\overline{D_2}$ . Since  $D_2^e = \mathbb{R}^3 \setminus \overline{D_2}$  is connected, we can find a point  $z \in \Gamma_1 \setminus D_2$  and a sufficiently small  $\varepsilon > 0$  with the following properties:

- (i)  $B_{2\varepsilon}(z) \cap \overline{D_2} = \emptyset$ ;
- (ii) the points  $z_n = z + (\varepsilon/n)v(z)$  lie in  $B_{2\varepsilon}(z)$  for all  $n \in \mathbb{N}$ , where  $v(z)$  is the unit normal to  $\Gamma_1$  at  $z$ .

Denote  $D = (\overline{D_1} \setminus \overline{D_2})^\circ$ , the inner part of the domain  $\overline{D_1} \setminus \overline{D_2}$ . We consider the unique solution of the following problem:

$$\begin{aligned} \operatorname{curl} \operatorname{curl} V - k_D^2 V &= 0, \quad \text{in } D, \\ \operatorname{curl} \operatorname{curl} E^s - k_b^2 E^s &= 0, \quad \text{in } \mathbb{R}^3 \setminus \overline{D}, \\ v \times (E^s - V) &= -v \times E_m(\cdot, z_n), \quad \text{on } \partial D, \\ v \times \operatorname{curl}(E^s - V) &= ik\lambda_1(E_T^s + E_m(\cdot, z_n)) - v \times \operatorname{curl} E_m(\cdot, z_n), \quad \text{on } \Gamma_1 \cap \partial D. \end{aligned} \quad (3.40)$$

Here  $E^s$  satisfies the Silver-Müller radiation condition at infinity, and  $E_m$  denotes the magnetic dipole defined by

$$E_m(x, z_n) = \operatorname{curl}_x \left( v(z) \frac{e^{ik_b|x-z_n|}}{|x-z_n|} \right). \quad (3.41)$$

Define

$$F_1(x, z_n) = \begin{cases} V(x), & \text{if } x \in D, \\ E^s(x) + E_m(x, z_n), & \text{if } x \in \mathbb{R}^3 \setminus (\overline{D} \cup z_n). \end{cases} \quad (3.42)$$

It can be proved that  $F_1(x, z_n)$  is a solution of Maxwell's equations with homogeneous conductive boundary value conditions on  $\partial D$  in any domain  $\Omega \subset \mathbb{R}^3$  with  $\overline{D}_0 \subset \Omega$  and  $z_n \notin \Omega$ .

Define

$$\kappa_1(x) = \begin{cases} k_{1*}^2(x) & \text{if } x \in \mathbb{R}^3 \setminus (\overline{D_1} \setminus D), \\ k_b^2 & \text{if } x \in D_1 \setminus D, \end{cases} \quad (3.43)$$

and  $\tilde{\lambda}_1(x) = \lambda_1$ .

In view of the above definitions of  $\kappa_1(x)$  and  $\tilde{\lambda}_1(x)$ , it follows that  $F_1(x, z_n)$  satisfies the variational equation (3.15) in Lemma 3.2 for the obstacle  $D$ . The well posedness of the direct problem for (CBP) and the fact that  $z_n$  is bounded away from  $\partial D$  imply that the solution  $(V, E^s)$  of (3.40) is uniformly bounded in  $\tilde{X}$ . We now define another singular solution with respect to  $D_2$  by

$$F_2(x, z_n) = \begin{cases} \tilde{V}(x), & \text{if } x \in D_2, \\ \tilde{E}^s(x) + E_m(x, z_n), & \text{if } x \in \mathbb{R}^3 \setminus (\overline{D_2} \cup z_n), \end{cases} \quad (3.44)$$

where  $E_m$  is a magnetic dipole defined in (3.41), and  $(\tilde{V}, \tilde{E}^s)$  is a solution of the problem

$$\begin{aligned} \operatorname{curl} \operatorname{curl} \tilde{V} - k_{D_2}^2 \tilde{V} &= 0, \quad \text{in } D_2, \\ \operatorname{curl} \operatorname{curl} \tilde{E}^s - k_b^2 \tilde{E}^s &= 0, \quad \text{in } \mathbb{R}^3 \setminus (\overline{D_2} \cup z_n), \\ \nu \times (\tilde{E}^s - \tilde{V}) &= -\nu \times E_m(\cdot, z_n), \quad \text{on } \partial D, \\ \nu \times \operatorname{curl}(\tilde{E}^s - \tilde{V}) &= ik\lambda_2(\tilde{E}_T^s + (E_m(\cdot, z_n))_T) - \nu \times \operatorname{curl} E_m(\cdot, z_n), \quad \text{on } \Gamma_2. \end{aligned} \quad (3.45)$$

Here  $\tilde{E}^s$  satisfies the Silver-Müller radiation condition at infinity. Noting that  $F_2(x, z_n)$  satisfies the variational equation (3.15) in Lemma 3.2 with  $\kappa_2(x) = k_{2*}^2$  and  $\tilde{\lambda}_2 = \lambda_2$ , it follows that both  $F_1(x, z_n)$  and  $F_2(x, z_n)$  satisfy the relationship (3.14), then we obtain

$$\begin{aligned} &k^2 \left( (n_D - n_b) F_1, \overline{F_2} \right)_D + ik \left\langle \lambda_1 (F_1)_T, (\overline{F_2})_T \right\rangle_{\partial D} \\ &= k^2 \left( (n_{D_2} - n_b) F_1, \overline{F_2} \right)_{D_2} + ik \left\langle \lambda_2 (F_2)_T, (\overline{F_1})_T \right\rangle_{\Gamma_2}. \end{aligned} \quad (3.46)$$

For case (i), by the fact that  $z \in \Gamma_1 \setminus D_2$  and the singularities of the magnetic dipole  $E_m$  defined in (3.41), it can be proved that  $|k^2((n_D - n_b)F_1, \overline{F_2})_D + ik\langle \lambda_1(F_1)_T, (\overline{F_2})_T \rangle_{\partial D}| \rightarrow \infty$  as  $n \rightarrow \infty$ ,

this, together with the fact that the other terms in the right hand of (3.46) are bounded, leads to a contradiction. Hence we have  $D_1 \subset D_2$ . By choosing  $z \in \Gamma_2 \setminus D_1$  and using the similar analysis as in the proof above, one can prove that  $D_2 \subset D_1$ . Finally, we obtain that  $D_1 = D_2$ . For other cases, due to the singularities of  $F_j$  ( $j = 1, 2$ ), a contradiction also arises in (3.46) as  $n \rightarrow \infty$ . This proves the theorem.  $\square$

**Theorem 3.4.** *Assume  $D_1 = D_2$  with parameters  $\lambda_j \in C(\partial\bar{D}_j)$  and the scattered fields  $E_{j,x_0}^s$ , ( $j = 1, 2$ ) satisfy  $\nu \times E_{1,x_0}^s|_{\partial\Omega} = \nu \times E_{2,x_0}^s|_{\partial\Omega}$  for all  $x_0 \in S_{R_0}$ , then we have  $\lambda_1 = \lambda_2$  on  $\partial D$ .*

*Proof.* From the proof of Theorem 3.3, it follows that there exists two singular solutions  $F_1, F_2$  of the conductive boundary problem with respect to the obstacle  $D$  for some  $z \in \partial D$ . By Lemma 3.2 and the identity  $D_1 = D_2$ , it can be checked that

$$\int_D (k_{1*}^2 - k_{2*}^2) F_1(x, z_n) \cdot F_2(x, z_n) dx + ik \int_{\partial D} (\lambda_1 - \lambda_2) F_1(x, z_n) \cdot F_2(x, z_n) ds(x) = 0. \quad (3.47)$$

The singularities of  $F_1, F_2$  ensure that  $\lambda_1 = \lambda_2$ . This completes the proof of the theorem.  $\square$

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