

## Research Article

# The Lie Brackets on Time Scales

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The Lie derivative, which has a wide range of application in physics and geometry, is trying to be examined on time scales. Firstly, nabla Lie bracket is defined on two-dimensional time scales. Secondly, the nabla Lie multiplication and some properties are given on the time scales. Lastly, for analyzing the differences between the real Lie multiplication and the nabla Lie multiplication, a numerical example is given.

## 1. Introduction

By unifying continuous and discrete calculus a different kind of calculus was exposed which recently takes the attentions as time scale theory has been introduced by Hilger in 1988 with his doctoral dissertation. The time scale theory by means of calculus every passing day lots of new theories and different implementation parts are quickly composed. Time scale is really very important and has an useful role at a great deal of sciences which are studying with dynamical systems. Differential geometry is one of these sciences. In [1–4] some geometric notions are trying to research on the time scale. In the paper [5] the curve and surface description were made for the first time. The reference [6] direction nabla derivative and its properties were investigated. The properties of vector field, derivative mapping, and delta connection were investigated in [7, 8]. In [6] nabla covariant derivative definition was given. Also some fundamental properties about time scale can be obtained in the references [9–12].

In physics, the use of Lie derivative is based on very old time. Especially the use of the Lie brackets is quite important in nonlinear control system and field of neural networks domains. Lie derivative studies, maintained until today, always show their effects in continuous space; however, in discrete space Lie multiplication is not studied. In this study in order to eliminate this problem, we will try to survey the Lie brackets which combines discrete space and continuous space on time scale. Thus, defined nabla Lie operator

simultaneously in real terms and in different time scales, their reciprocities will be able to find easily. Additionally, for analyzing the differences between the reel Lie multiplication and the nabla Lie multiplication a numerical example is given.

## 2. Preliminaries

The following definitions and theorems will serve as a short primer on time scale calculus; they can be found in [6, 10, 11]. A time scale  $\mathbb{T}$  is any nonempty closed subset of  $\mathbb{R}$ . Within that set, define the jump operators  $\rho : \mathbb{T} \rightarrow \mathbb{T}$  by  $\rho(t) = \sup\{s \in \mathbb{T} : s < t\}$ , where  $\inf \emptyset := \sup \mathbb{T}$ , and  $\sup \emptyset := \inf \mathbb{T}$ , where  $\emptyset$  denotes the empty set. Also the graininess function is defined by  $\nu := \rho(t) - t$ . If  $f : \mathbb{T} \rightarrow \mathbb{R}$  is a function, then we define the function  $f^\rho : \mathbb{T} \rightarrow \mathbb{R}$  by  $f^\rho(t) = f(\rho(t))$  for  $t \in \mathbb{T}$ , that is,  $f^\rho = f \circ \rho$ .

**Theorem 2.1.** *If  $f, g : \mathbb{T} \rightarrow \mathbb{R}$  are  $\nabla$ -differentiable at  $t \in \mathbb{T}^k$ , then*

(i)  *$f + g$  is  $\nabla$ -differentiable at  $t$  and*

$$(f + g)^\nabla(t) = f^\nabla(t) + g^\nabla(t), \quad (2.1)$$

(ii) *for any constant  $c$ ,  $c \cdot f$  is  $\nabla$ -differentiable at  $t$  and*

$$(cf)^\nabla(t) = cf^\nabla(t), \quad (2.2)$$

(iii)  *$f \cdot g$  is  $\nabla$ -differentiable at  $t$  and*

$$(fg)^\nabla(t) = f^\nabla(t)g(t) + f(\rho(t))g^\nabla(t) = g^\nabla(t)f(t) + g(\rho(t))f^\nabla(t), \quad (2.3)$$

(iv) *if  $g(t) \cdot g(\rho(t)) \neq 0$ , then  $f/g$  is  $\nabla$ -differentiable at  $t$  and*

$$\left(\frac{f}{g}\right)^\nabla(t) = \frac{f^\nabla(t)g(t) - f(t)g^\nabla(t)}{g(t) \cdot g(\rho(t))}. \quad (2.4)$$

**Definition 2.2.** Let two vector fields  $Z$  and  $W$  be given. The covariant nabla differentiation with respect to  $W$  at the point  $P(t_1^0, t_2^0, \dots, t_n^0)$  is defined as the vector

$$D_W Z = \left( \frac{\partial Z}{\nabla W} \right)(P) = Y^\nabla(0), \quad (2.5)$$

provided it exists, where  $Y(\xi) = Z(t_1^0 + \xi w_1, \dots, t_n^0 + \xi w_n)$  for  $\xi \in \mathbb{T}$ .

**Theorem 2.3.** *Let two vector fields  $Z, W$  be given. The covariant nabla differentiation with respect to  $W$  at the point  $P(t_1^0, t_2^0, \dots, t_n^0)$  exists and is expressed by the formula*

$$\frac{\partial Z(P)}{\nabla w_P} = \sum_{i=1}^2 \frac{\partial g_i(P)}{\nabla w_P} \frac{\partial}{\nabla_i x_i}. \quad (2.6)$$

**Theorem 2.4.** Let  $a, b \in \mathbb{R}$  and two vector fields  $X$  and  $Y$  be given. For any two tangent vectors  $V_p$  and  $W_p$ , the following properties are proven:

$$\begin{aligned} \text{(i)} \quad \frac{\partial X}{\nabla(aV + bW)} &= a \frac{\partial X}{\nabla V} + b \frac{\partial X}{\nabla W}, \\ \text{(ii)} \quad \frac{\partial(aX + bY)}{\nabla V} &= a \frac{\partial X}{\nabla V} + b \frac{\partial Y}{\nabla V}. \end{aligned} \quad (2.7)$$

### 3. Nabla Lie Bracket on Time Scales

Lie multiplication and derivative are indispensable notions for algebra and geometry. Up to now we used them for discrete structures. In this study we have a chance for analyzing both discrete structures and indiscrete structures. Under both circumstances, we will research the structure of time scale. Because of covariant derivative was proved as nabla covariant derivative in the references [7, 8, 11] that is why we are defining our study as Nabla-Lie.

*Definition 3.1.* Let one has the two-dimensional space which is called  $\Lambda^2 = \mathbb{T} \times \mathbb{T}$  on  $\mathbb{T}$  time scales. As  $f : M \subset \Lambda^2 \rightarrow \mathbb{R}$  function, that is, on  $P$  point, the nabla vector field  $V_p(f)$  is given with these coordinates

$$V_p(f) = \sum_{i=1}^2 V_{p_i} \frac{\partial f}{\nabla x_i} = V_{p_1} \frac{\partial f}{\nabla x_1} + V_{p_2} \frac{\partial f}{\nabla x_2}, \quad (3.1)$$

which was introduced in the reference [7].

*Definition 3.2.* Let one shows all the set of vector fields with  $\Psi(\Lambda^2)$  on  $\Lambda^2$  space. By varying the point  $P$  along the curve on  $M$ , one can obtain another smooth function  $[V(f)]_{\nabla}$  which is the nabla derivative of  $f$  along the vector field. The function

$$[V(f)]_{\nabla}(p) = V_p(f) \quad (3.2)$$

is called the *nabla Lie derivative* on time scales. It is common to denote the nabla Lie derivative of a  $f$  function which has vector field along  $V$  that is shown as  $[V(f)]_{\nabla}$ . As shown, two-dimensional nabla Lie multiplication is a function from  $\Psi(\Lambda^2) \times \Psi(\Lambda^2)$  set to  $\Psi(\Lambda^2)$  set.

**Lemma 3.3.**  $f$  function will be the completely nabla differentiable as regarded to be equal the inner multiplication vector field of  $V$  at point  $p$  with the derivative of  $f$  function at the same time interval of the nabla Lie derivation. In coordinates,

$$[V(f)]_{\nabla}(p) = \sum_{i=1}^2 \frac{\partial f}{\nabla x_i} V_i = \left( V_{p_1} \frac{\partial f}{\nabla x_1} + V_{p_2} \frac{\partial f}{\nabla x_2} \right) = \langle f^{\nabla}, V_p \rangle. \quad (3.3)$$

**Theorem 3.4.** Let one has the  $f$  function which is completely nabla differentiable;  $\rho(t)$  will be the backward jump operator of taken time scale and area of  $V$  vector is expressed by  $V^{\rho}$  notation at  $\rho(t)$  point. All smooth vector fields of the time scale space  $\Psi(\Lambda^2)$  are vector space on a  $M$  manifold.  $V_1$  and  $V_2$  are the two vector fields, given a function with Lie nabla derivative which is defined on

$M \subset \Psi(\Lambda^2)$ . Then both  $V_1[V_2f]_{\nabla}$  and  $V_2[V_1f]_{\nabla}$  will help the calculation of nabla Lie bracket. The following equations can be expressed by nabla derivations on time scales:

$$\begin{aligned} V_1[V_2f]_{\nabla}(p) = & V_{11} \left( \frac{\partial V_{21}}{\nabla x_1} \frac{\partial f}{\nabla x_1} + \frac{\partial V_{22}}{\nabla x_1} \frac{\partial f}{\nabla x_2} \right) + V_{11} \left( V_{21}^{\rho} \frac{\partial^2 f}{\nabla x_1^2} + V_{22}^{\rho} \frac{\partial^2 f}{\nabla x_1 \nabla x_2} \right) \\ & + V_{12} \left( \frac{\partial V_{21}}{\nabla x_2} \frac{\partial f}{\nabla x_1} + \frac{\partial V_{22}}{\nabla x_2} \frac{\partial f}{\nabla x_2} \right) + V_{12} \left( V_{21}^{\rho} \frac{\partial^2 f}{\nabla x_2 \nabla x_1} + V_{22}^{\rho} \frac{\partial^2 f}{\nabla x_2^2} \right), \end{aligned} \quad (3.4)$$

$$\begin{aligned} V_2[V_1f]_{\nabla}(p) = & V_{21} \left( \frac{\partial V_{11}}{\nabla x_1} \frac{\partial f}{\nabla x_1} + \frac{\partial V_{12}}{\nabla x_1} \frac{\partial f}{\nabla x_2} \right) + V_{21} \left( V_{11}^{\rho} \frac{\partial^2 f}{\nabla x_1^2} + V_{12}^{\rho} \frac{\partial^2 f}{\nabla x_1 \nabla x_2} \right) \\ & + V_{22} \left( \frac{\partial V_{11}}{\nabla x_2} \frac{\partial f}{\nabla x_1} + \frac{\partial V_{12}}{\nabla x_2} \frac{\partial f}{\nabla x_2} \right) + V_{22} \left( V_{11}^{\rho} \frac{\partial^2 f}{\nabla x_2 \nabla x_1} + V_{12}^{\rho} \frac{\partial^2 f}{\nabla x_2^2} \right). \end{aligned} \quad (3.5)$$

*Proof.*  $V_1$  and  $V_2$  vector fields are written as the following:

$$V_1 = \sum_{i=1}^2 V_{1i} \frac{\partial}{\nabla x_i} = V_{11} \frac{\partial}{\nabla x_1} + V_{12} \frac{\partial}{\nabla x_2}, \quad (3.6)$$

$$V_2 = \sum_{i=1}^2 V_{2i} \frac{\partial}{\nabla x_i} = V_{21} \frac{\partial}{\nabla x_1} + V_{22} \frac{\partial}{\nabla x_2}, \quad (3.7)$$

because of Definitions 3.1 and 3.2.  $V_1[(V_2f)]_{\nabla}$  vector field can be calculated as the following by the help of nabla derivation and the definition of Lie derivation which is used on the time scales:

$$\begin{aligned} V_1[V_2]_{\nabla}(f) &= \left[ \sum_{i=1}^2 V_{1i} \frac{\partial}{\nabla x_i} \left( \sum_{j=1}^2 V_{2j} \frac{\partial f}{\nabla x_j} \right) \right]_{\nabla} \\ &= \left[ \left( V_{11} \frac{\partial}{\nabla x_1} + V_{12} \frac{\partial}{\nabla x_2} \right) \left( V_{21} \frac{\partial f}{\nabla x_1} + V_{22} \frac{\partial f}{\nabla x_2} \right) \right]_{\nabla} \\ &= V_{11} \frac{\partial}{\nabla x_1} \left( V_{21} \frac{\partial f}{\nabla x_1} \right) + V_{11} \frac{\partial}{\nabla x_1} \cdot \left( V_{22} \frac{\partial f}{\nabla x_2} \right) \\ &\quad + V_{12} \frac{\partial}{\nabla x_2} \left( V_{21} \frac{\partial f}{\nabla x_1} \right) + V_{12} \frac{\partial}{\nabla x_2} \left( V_{22} \frac{\partial f}{\nabla x_2} \right) \\ &= V_{11} \left( \frac{\partial V_{21}}{\nabla x_1} \cdot \frac{\partial f}{\nabla x_1} + V_{21}^{\rho} \frac{\partial^2 f}{\nabla x_1^2} \right) + V_{11} \left( \frac{\partial V_{22}}{\nabla x_1} \cdot \frac{\partial f}{\nabla x_2} + V_{22}^{\rho} \frac{\partial^2 f}{\nabla x_1 \nabla x_2} \right) \\ &\quad + V_{12} \left( \frac{\partial V_{21}}{\nabla x_2} \cdot \frac{\partial f}{\nabla x_1} + V_{21}^{\rho} \frac{\partial^2 f}{\nabla x_2 \nabla x_1} \right) + V_{12} \left( \frac{\partial V_{22}}{\nabla x_2} \cdot \frac{\partial f}{\nabla x_2} + V_{22}^{\rho} \frac{\partial^2 f}{\nabla x_2^2} \right) \end{aligned}$$

$$\begin{aligned}
&= V_{11} \left( \frac{\partial V_{21}}{\nabla x_1} \cdot \frac{\partial f}{\nabla x_1} + \frac{\partial V_{22}}{\nabla x_1} \frac{\partial f}{\nabla x_2} \right) + V_{11} \left( V_{21}^\rho \frac{\partial^2 f}{\nabla x_1^2} + V_{22}^\rho \frac{\partial^2 f}{\nabla x_1 \cdot \nabla x_2} \right) \\
&\quad + V_{12} \left( \frac{\partial V_{21}}{\nabla x_2} \cdot \frac{\partial f}{\nabla x_1} + \frac{\partial V_{22}}{\nabla x_2} \cdot \frac{\partial f}{\nabla x_2} \right) + V_{12} \left( V_{21}^\rho \frac{\partial^2 f}{\nabla x_2 \nabla x_1} + V_{22}^\rho \frac{\partial^2 f}{\nabla x_2^2} \right).
\end{aligned} \tag{3.8}$$

With the similar idea, the following equation:

$$\begin{aligned}
V_2[V_1]_\nabla(f) &= \left[ \sum_{i=1}^2 V_{2i} \frac{\partial}{\nabla x_i} \left( \sum_{j=1}^2 V_{1j} \frac{\partial f}{\nabla x_j} \right) \right] \\
&= \left[ \left( V_{21} \frac{\partial}{\nabla x_1} + V_{22} \frac{\partial}{\nabla x_2} \right) \left( V_{11} \frac{\partial f}{\nabla x_1} + V_{12} \frac{\partial f}{\nabla x_2} \right) \right] \\
&= V_{21} \frac{\partial}{\nabla x_1} \left( V_{11} \frac{\partial f}{\nabla x_1} \right) + V_{21} \frac{\partial}{\nabla x_1} \left( V_{12} \frac{\partial f}{\nabla x_2} \right) \\
&\quad + V_{22} \frac{\partial}{\nabla x_2} \left( V_{11} \frac{\partial f}{\nabla x_1} \right) + V_{22} \frac{\partial}{\nabla x_2} \left( V_{12} \frac{\partial f}{\nabla x_2} \right) \\
&= V_{21} \left( \frac{\partial V_{11}}{\nabla x_1} + V_{11}^\rho \frac{\partial^2 f}{\nabla x_1^2} \right) + V_{21} \left( \frac{\partial V_{12}}{\nabla x_1} \frac{\partial f}{\nabla x_2} + V_{12}^\rho \frac{\partial^2 f}{\nabla x_1 \nabla x_2} \right) \\
&\quad + V_{22} \left( \frac{\partial V_{11}}{\nabla x_2} \frac{\partial f}{\nabla x_1} + V_{11}^\rho \frac{\partial^2 f}{\nabla x_2 \nabla x_1} \right) + V_{22} \left( \frac{\partial V_{12}}{\nabla x_2} \frac{\partial f}{\nabla x_2} + V_{12}^\rho \frac{\partial^2 f}{\nabla x_2^2} \right) \\
&= V_{21} \left( \frac{\partial V_{11}}{\nabla x_1} \frac{\partial f}{\nabla x_1} + \frac{\partial V_{12}}{\nabla x_1} \frac{\partial f}{\nabla x_2} \right) + V_{21} \left( V_{11}^\rho \frac{\partial^2 f}{\nabla x_1^2} + V_{12}^\rho \frac{\partial^2 f}{\nabla x_1 \nabla x_2} \right) \\
&\quad + V_{22} \left( \frac{\partial V_{11}}{\nabla x_2} \frac{\partial f}{\nabla x_1} + \frac{\partial V_{12}}{\nabla x_2} \right) + V_{22} \left( V_{11}^\rho \frac{\partial^2 f}{\nabla x_2 \nabla x_1} + V_{12}^\rho \frac{\partial^2 f}{\nabla x_2^2} \right)
\end{aligned} \tag{3.9}$$

can be found. Thus, obtained equations are desired at the theorem.  $\square$

*Definition 3.5.*  $V_1, V_2 \in \Psi(\Lambda^2)$  are vector fields. The equation

$$[V_1, V_2]_\nabla := V_1[V_2]_\nabla - V_2[V_1]_\nabla \tag{3.10}$$

is called *nabla Lie bracket* on time scales. Here  $V_1[V_2]_\nabla$  and  $V_2[V_1]_\nabla$  are nabla Lie derivations.

**Lemma 3.6.** *Let one has the two completely nabla-differentiable vector fields of  $f$  functions and  $V_1, V_2 \in \Psi(\Lambda^2)$  vector fields. Due to the definitions of  $V_1[V_2f]_{\nabla}$  and  $V_2[V_1f]_{\nabla}$ , nabla Lie is equal to the subtraction of the following equation:*

$$\begin{aligned}
[V_1, V_2]_{\nabla}(f) &:= V_1[(V_2f)]_{\nabla} - V_2[(V_1f)]_{\nabla} \\
&= V_{11} \left( \frac{\partial V_{21}}{\nabla x_2} \frac{\partial f}{\nabla x_1} + \frac{\partial V_{22}}{\nabla x_1} \frac{\partial f}{\nabla x_2} \right) - V_{21} \left( \frac{\partial V_{11}}{\nabla x_1} \frac{\partial f}{\nabla x_1} + \frac{\partial V_{12}}{\nabla x_1} \frac{\partial f}{\nabla x_2} \right) \\
&\quad + V_{12} \left( \frac{\partial V_{21}}{\nabla x_2} \frac{\partial f}{\nabla x_1} + \frac{\partial V_{22}}{\nabla x_2} \frac{\partial f}{\nabla x_2} \right) - V_{22} \left( \frac{\partial V_{11}}{\nabla x_2} \frac{\partial f}{\nabla x_1} + \frac{\partial V_{12}}{\nabla x_2} \frac{\partial f}{\nabla x_2} \right) \\
&\quad + \left( V_{21}^{\rho} V_{11} - V_{11}^{\rho} V_{21} \right) \frac{\partial^2 f}{\nabla x_1^2} + \left( V_{21}^{\rho} V_{12} - V_{12}^{\rho} V_{21} \right) \frac{\partial^2 f}{\nabla x_1 \nabla x_2} \\
&\quad + \left( V_{22}^{\rho} V_{12} - V_{22} V_{12}^{\rho} \right) \frac{\partial^2 f}{\nabla x_2^2} + \left( V_{22}^{\rho} V_{11} - V_{11}^{\rho} V_{22} \right) \frac{\partial^2 f}{\nabla x_1 \nabla x_2} \tag{3.11} \\
&= \sum_{i=1}^2 \left[ \sum_{j=1}^2 \left( \frac{\partial V_{2j}}{\nabla x_i} V_{1i} - \frac{\partial V_{1j}}{\nabla x_i} V_{2i} \right) \right] \frac{\partial}{\nabla x_j} \\
&\quad + \left( V_{21}^{\rho} V_{11} - V_{11}^{\rho} V_{21} \right) \frac{\partial^2 f}{\nabla x_1^2} + \left( V_{21}^{\rho} V_{12} - V_{12}^{\rho} V_{21} \right) \frac{\partial^2 f}{\nabla x_1 \nabla x_2} \\
&\quad + \left( V_{22}^{\rho} V_{12} - V_{22} V_{12}^{\rho} \right) \frac{\partial^2 f}{\nabla x_2^2} + \left( V_{22}^{\rho} V_{11} - V_{11}^{\rho} V_{22} \right) \frac{\partial^2 f}{\nabla x_1 \nabla x_2}.
\end{aligned}$$

**Theorem 3.7.** *Let one has the two completely nabla differentiable vector fields as  $V_1$  and  $V_2$ . Nabla Lie multiplication can be defined as nabla covariant derivation because of the definition of nabla covariant derivative at [8] regarding N graininess function on time scales. The above representation gives a different geometrical dimension to Lie bracket. Lie parenthesis operator has an expression*

$$[V_1, V_2]_{\nabla}(f) = D_{V_1} V_2 - D_{V_2} V_1 + \left[ \frac{\partial V_2}{\nabla x_1} V_1[f] - \frac{\partial V_1}{\nabla x_1} V_2[f] \right] v, \tag{3.12}$$

which is like that in the nabla covariant defined with Definition 2.2.

*Proof.* Let us briefly write the subtraction equations of nabla lie derivations by the help of equations that take place at Lemma 3.6. Then, we will, respectively, add and remove the expressions in brackets. In the next step, we will try to reach nabla covariant derivations from the nabla derivation definition by multiplying and dividing with  $(\rho(x_1) - x_1)$ :

$$\begin{aligned}
[V_1, V_2]_{\nabla}(f) &= V_1[(V_2f)]_{\nabla} - V_2[(V_1f)]_{\nabla} \\
&= V_{11} \left( \frac{\partial V_{21}}{\nabla x_1} \frac{\partial f}{\nabla x_1} + \frac{\partial V_{22}}{\nabla x_1} \frac{\partial f}{\nabla x_2} \right) - V_{21} \left( \frac{\partial V_{11}}{\nabla x_1} \frac{\partial f}{\nabla x_1} + \frac{\partial V_{12}}{\nabla x_1} \frac{\partial f}{\nabla x_2} \right)
\end{aligned}$$

$$\begin{aligned}
& + V_{12} \left( \frac{\partial V_{21}}{\nabla x_2} \frac{\partial f}{\nabla x_1} + \frac{\partial V_{22}}{\nabla x_2} \frac{\partial f}{\nabla x_2} \right) - V_{22} \left( \frac{\partial V_{11}}{\nabla x_2} \frac{\partial f}{\nabla x_1} + \frac{\partial V_{12}}{\nabla x_2} \frac{\partial f}{\nabla x_2} \right) \\
& + \left( V_{21}^\rho V_{11} - V_{11}^\rho V_{21} \right) \frac{\partial^2 f}{\nabla x_1^2} + \left( V_{21}^\rho V_{12} - V_{12}^\rho V_{21} \right) \frac{\partial^2 f}{\nabla x_1 \nabla x_2} \\
& + \left( V_{22}^\rho V_{12} - V_{22} V_{12}^\rho \right) \frac{\partial^2 f}{\nabla x_2^2} + \left( V_{22}^\rho V_{11} - V_{11}^\rho V_{22} \right) \frac{\partial^2 f}{\nabla x_1 \nabla x_2} \\
& = \frac{\partial f}{\nabla x_1} \left( V_{11} \frac{\partial V_{21}}{\nabla x_1} + V_{12} \frac{\partial V_{21}}{\nabla x_2} \right) + \frac{\partial f}{\nabla x_2} \left( V_{11} \frac{\partial V_{22}}{\nabla x_1} + V_{12} \frac{\partial V_{22}}{\nabla x_2} \right) \\
& - \frac{\partial f}{\nabla x_1} \left( V_{21} \frac{\partial V_{11}}{\nabla x_1} + V_{22} \frac{\partial V_{11}}{\nabla x_2} \right) - \frac{\partial f}{\nabla x_2} \left( V_{21} \frac{\partial V_{12}}{\nabla x_1} + V_{22} \frac{\partial V_{12}}{\nabla x_2} \right) \\
& + \left( V_{21}^\rho V_{11} - V_{11}^\rho V_{21} + V_{21} V_{11} - V_{21} V_{11} \right) \frac{\partial^2 f}{\nabla x_1^2} \\
& + \left( V_{21}^\rho V_{12} - V_{12}^\rho V_{21} + V_{21} V_{12} - V_{21} V_{12} \right) \frac{\partial^2 f}{\nabla x_1 \nabla x_2} \\
& + \left( V_{22}^\rho V_{12} - V_{22} V_{12}^\rho + V_{22} V_{11} - V_{22} V_{12} \right) \frac{\partial^2 f}{\nabla x_2^2} \\
& + \left( V_{22}^\rho V_{11} - V_{11}^\rho V_{22} + V_{22} V_{11} - V_{22} V_{11} \right) \frac{\partial^2 f}{\nabla x_1 \nabla x_2} \\
& = \frac{\partial f}{\nabla x_1} V_1[V_{21}] + \frac{\partial f}{\nabla x_2} V_1[V_{22}] - \frac{\partial f}{\nabla x_1} V_2[V_{11}] - \frac{\partial f}{\nabla x_2} V_2[V_{12}] \\
& + \frac{V_{11} \left( V_{21}^\rho - V_{21} \right) - V_{21} \left( V_{11}^\rho - V_{11} \right)}{\rho(x_1) - x_1} \cdot (\rho(x_1) - x_1) \frac{\partial^2 f}{\nabla x_1^2} \\
& + \frac{V_{12} \left( V_{21}^\rho - V_{21} \right) - V_{21} \left( V_{12}^\rho - V_{12} \right)}{\rho(x_1) - x_1} \cdot (\rho(x_1) - x_1) \cdot \frac{\partial^2 f}{\nabla x_1 \nabla x_2} \\
& + \frac{V_{12} \left( V_{22}^\rho - V_{22} \right) - V_{22} \left( V_{12}^\rho - V_{12} \right)}{\rho(x_1) - x_1} \cdot (\rho(x_1) - x_1) \cdot \frac{\partial^2 f}{\nabla x_2^2} \\
& + \frac{V_{11} \left( V_{22}^\rho - V_{22} \right) - V_{22} \left( V_{11}^\rho - V_{11} \right)}{\rho(x_1) - x_1} \cdot (\rho(x_1) - x_1) \cdot \frac{\partial^2 f}{\nabla x_1 \nabla x_2} \\
& = \frac{\partial f}{\nabla x_1} V_1[V_{21}] + \frac{\partial f}{\nabla x_2} V_1[V_{22}] - \frac{\partial f}{\nabla x_1} V_2[V_{11}] - \frac{\partial f}{\nabla x_2} V_2[V_{12}] \\
& + \left( V_{11} \frac{\partial V_{21}}{\nabla x_1} - V_{21} \frac{\partial V_{11}}{\nabla x_1} \right) \frac{\partial^2 f}{\nabla x_1^2} + \left( V_{12} \frac{\partial V_{21}}{\nabla x_1} - V_{21} \frac{\partial V_{12}}{\nabla x_1} \right) \frac{\partial^2 f}{\nabla x_1 \nabla x_2} \\
& + \left( V_{12} \frac{\partial V_{22}}{\nabla x_1} - V_{22} \frac{\partial V_{12}}{\nabla x_1} \right) \frac{\partial^2 f}{\nabla x_2^2} + \left( V_{11} \frac{\partial V_{22}}{\nabla x_1} - V_{22} \frac{\partial V_{11}}{\nabla x_1} \right) \frac{\partial^2 f}{\nabla x_1 \nabla x_2}
\end{aligned}$$

$$\begin{aligned}
& + \frac{\partial f}{\nabla x_1} V_1[V_{21}] + \frac{\partial f}{\nabla x_2} V_1[V_{22}] - \frac{\partial f}{\nabla x_1} V_2[V_{11}] - \frac{\partial f}{\nabla x_2} V_2[V_{12}] \\
& + \nu \frac{\partial V_{21}}{\nabla x_1} \frac{\partial f}{\nabla x_1} \left( V_{11} \frac{\partial f}{\nabla x_1} + V_{12} \frac{\partial f}{\nabla x_2} \right) + \nu \frac{\partial V_{22}}{\nabla x_1} \frac{\partial f}{\nabla x_2} \left( V_{12} \frac{\partial f}{\nabla x_2} + V_{11} \frac{\partial f}{\nabla x_1} \right) \\
& - \nu \frac{\partial V_{11}}{\nabla x_1} \frac{\partial f}{\nabla x_1} \left( V_{21} \frac{\partial f}{\nabla x_1} + V_{22} \frac{\partial f}{\nabla x_2} \right) - \nu \frac{\partial V_{12}}{\nabla x_1} \frac{\partial f}{\nabla x_2} \left( V_{21} \frac{\partial f}{\nabla x_1} + V_{22} \frac{\partial f}{\nabla x_2} \right) \\
& = \frac{\partial f}{\nabla x_1} V_1[V_{21}] + \frac{\partial f}{\nabla x_2} V_1[V_{22}] - \frac{\partial f}{\nabla x_1} V_2[V_{11}] - \frac{\partial f}{\nabla x_2} V_2[V_{12}] \\
& + \nu \frac{\partial V_{21}}{\nabla x_1} \frac{\partial f}{\nabla x_1} V_1[f] + \nu \frac{\partial V_{22}}{\nabla x_1} \frac{\partial f}{\nabla x_2} V_1[f] \\
& - \nu \frac{\partial V_{11}}{\nabla x_1} \frac{\partial f}{\nabla x_1} V_2[f] - \nu \frac{\partial V_{12}}{\nabla x_1} \frac{\partial f}{\nabla x_2} V_2[f] \\
& = \frac{\partial V_2}{\nabla V_1} - \frac{\partial V_1}{\nabla V_2} + \left( \frac{\partial V_{21}}{\nabla x_1} \frac{\partial f}{\nabla x_1} + \frac{\partial V_{22}}{\nabla x_1} \frac{\partial f}{\nabla x_2} \right) V_1[f] \nu \\
& - \left( \frac{\partial V_{11}}{\nabla x_1} \frac{\partial f}{\nabla x_1} + \frac{\partial V_{12}}{\nabla x_1} \frac{\partial f}{\nabla x_2} \right) V_2[f] \nu \\
& = \frac{\partial V_2}{\nabla V_1} - \frac{\partial V_1}{\nabla V_2} + \left[ \frac{\partial V_2}{\nabla x_1} V_1[f] - \frac{\partial V_1}{\nabla x_1} V_2[f] \right] \nu \\
& = D_{V_1} V_2 - D_{V_2} V_1 + \left[ \frac{\partial V_2}{\nabla x_1} V_1[f] - \frac{\partial V_1}{\nabla x_1} V_2[f] \right] \nu.
\end{aligned} \tag{3.13}$$

Thus, it has been proven that may be thinking of the subtraction of nabla covariant vectors which are belonging to two vector fields of nabla Lie derivatives. When the time scale is  $\mathbb{R}$ , the grannies function  $\nu$  is zero, so it is seen that nabla Lie bracket is transformed to Lie bracket at real. Thus, as a result of the equation it was proved to achieve with nabla lie bracket for both different time scales and its equivalents at real.  $\square$

**Theorem 3.8.** *The vector fields  $V_1, V_2, V_3 \in \Xi(\Lambda^2)$  are completely nabla differentiable. From the following equations, it is seen that the nabla Lie bracket is a Lie bracket operator:*

$$\begin{aligned}
& \text{(i)} \quad [V_1, V_2]_{\nabla} = -[V_2, V_1]_{\nabla}, \\
& \text{(ii)} \quad [\alpha V_1 + \beta V_2, V_3]_{\nabla} = \alpha [V_1, V_3]_{\nabla} + \beta [V_2, V_3]_{\nabla}, \\
& \text{(iii)} \quad [V_1, [V_2, V_3]_{\nabla}]_{\nabla} + [V_2, [V_3, V_1]_{\nabla}]_{\nabla} + [V_3, [V_1, V_2]_{\nabla}]_{\nabla} = 0.
\end{aligned} \tag{3.14}$$

*Proof.* The following proofs are obtained by the definition of Lie bracket and the results of nabla covariant derivation.  $\square$

(i) Here we will proof that the nabla Lie bracket is not commutative:

$$[V_1, V_2]_{\nabla} = V_1[V_2]_{\nabla} - V_2[V_1]_{\nabla} = -[V_2[V_1]_{\nabla} - V_1[V_2]_{\nabla}] = -[V_2, V_1]_{\nabla}. \tag{3.15}$$



(ii) Let us denote the sum  $\alpha V_1 + \beta V_2$  with  $W$  in the following equations:

$$\begin{aligned}
 [\alpha V_1 + \beta V_2, V_3]_{\nabla} &= D_{V_3} W - D_W V_3 + \left[ \frac{\partial V_3}{\nabla x_1} W - \frac{\partial W}{\nabla x_1} V_3 \right] \nu \\
 &= \alpha \frac{\partial V_3}{\nabla V_1} + \beta \frac{\partial V_3}{\nabla V_2} - \alpha \frac{\partial V_1}{\nabla V_3} - \beta \frac{\partial V_2}{\nabla V_3} \\
 &\quad + \left[ \alpha \frac{\partial V_3}{\nabla x_1} V_1 - \beta \frac{\partial V_3}{\nabla x_1} V_2 - \alpha \frac{\partial V_1}{\nabla x_1} V_3 - \beta \frac{\partial V_2}{\nabla x_1} V_3 \right] \nu \\
 &= \alpha \left[ \frac{\partial V_3}{\nabla V_1} - \frac{\partial V_1}{\nabla V_3} + \left[ \frac{\partial V_3}{\nabla x_1} V_1 - \frac{\partial V_1}{\nabla x_1} V_3 \right] \nu \right] \\
 &\quad + \beta \left[ \frac{\partial V_3}{\nabla V_2} - \frac{\partial V_2}{\nabla V_3} + \left[ \frac{\partial V_3}{\nabla x_1} V_2 - \frac{\partial V_2}{\nabla x_1} V_3 \right] \nu \right] \\
 &= \alpha [V_1, V_3]_{\nabla} + \beta [V_2, V_3]_{\nabla}.
 \end{aligned} \tag{3.16}$$

(iii) From the definition of nabla Lie derivative we can obtain the following equations:

$$\begin{aligned}
 [V_1, [V_2, V_3]_{\nabla}]_{\nabla} &= V_1([V_2, V_3]_{\nabla}) - [V_2, V_3]_{\nabla}(V_1) \\
 &= V_1[V_2[V_3]_{\nabla}]_{\nabla} - V_1[V_3[V_2]_{\nabla}]_{\nabla} - V_2[V_3[V_1]_{\nabla}]_{\nabla} + V_3[V_2[V_1]_{\nabla}]_{\nabla}, \\
 [V_2, [V_3, V_1]_{\nabla}]_{\nabla} &= V_2([V_3, V_1]_{\nabla}) - [V_3, V_1]_{\nabla}(V_2) \\
 &= V_2[V_3[V_1]_{\nabla}]_{\nabla} - V_2[V_1[V_3]_{\nabla}]_{\nabla} - V_3[V_1[V_2]_{\nabla}]_{\nabla} + V_1[V_3[V_2]_{\nabla}]_{\nabla}, \\
 [V_3, [V_1, V_2]_{\nabla}]_{\nabla} &= V_3([V_1, V_2]_{\nabla}) - [V_1, V_2]_{\nabla}(V_3) \\
 &= V_3[V_1[V_2]_{\nabla}]_{\nabla} - V_3[V_2[V_1]_{\nabla}]_{\nabla} - V_1[V_2[V_3]_{\nabla}]_{\nabla} + V_2[V_1[V_3]_{\nabla}]_{\nabla}.
 \end{aligned} \tag{3.17}$$

It is easy to see that by the addition of three equations above, we can obtain the result as zero:

$$[V_1, [V_2, V_3]_{\nabla}]_{\nabla} + [V_2, [V_3, V_1]_{\nabla}]_{\nabla} + [V_3, [V_1, V_2]_{\nabla}]_{\nabla} = 0. \tag{3.18}$$

#### 4. A Numeric Example

Let us give a function  $f(x_1, x_2) = (x_1 x_2 + 3x_2)(\partial/\nabla x_1) + (2x_1 - x_2)(\partial/\nabla x_2)$  and the vector fields on  $\Lambda^2$  as the following:

$$\begin{aligned}
 V_1 &= (x_1 - x_2^2) \frac{\partial}{\nabla x_1} + (x_1^2) \frac{\partial}{\nabla x_2}, \\
 V_2 &= (x_2 + 5) \frac{\partial}{\nabla x_1} + (x_1 - 4x_2) \frac{\partial}{\nabla x_2}.
 \end{aligned} \tag{4.1}$$

Here we will calculate the nabla Lie multiplication, then we will try to obtain  $[V_1, V_2]_{\nabla}$  with using different time scales:

$$\begin{aligned}
[V_1, V_2]_{\nabla}(f) &:= V_1[(V_2 f)] - V_2[(V_1 f)] \\
&= (x_1 - x_2^2) \left( 0 \frac{\partial f}{\nabla x_1} + 1 \frac{\partial f}{\nabla x_2} \right) - (x_2 + 5) \left( 1 \frac{\partial f}{\nabla x_1} + (\rho(x_1) + x_1) \frac{\partial f}{\nabla x_2} \right) \\
&\quad + x_1^2 \left( 1 \frac{\partial f}{\nabla x_1} - 4 \frac{\partial f}{\nabla x_2} \right) - (x_1 - 4x_2) \left( -(\rho(x_2) + x_2) \frac{\partial f}{\nabla x_1} + 0 \frac{\partial f}{\nabla x_2} \right) \\
&\quad + \left( (x_2 + 5)^{\rho} (x_1 - x_2^2) - (x_1 - x_2^2)^{\rho} (x_2 + 5) \right) \frac{\partial^2 f}{\nabla x_1^2} \\
&\quad + \left( (x_2 + 5)^{\rho} x_1^2 - (x_1^2)^{\rho} (x_2 + 5) \right) \frac{\partial^2 f}{\nabla x_1 \nabla x_2} \\
&\quad + \left( (x_1 - 4x_2)^{\rho} (x_1^2) - (x_1 - 4x_2) (x_1^2)^{\rho} \right) \frac{\partial^2 f}{\nabla x_2^2} \\
&\quad + \left( (x_2 + 5)^{\rho} (x_1 - x_2^2) - (x_1 - x_2^2)^{\rho} (x_2 + 5) \right) \frac{\partial^2 f}{\nabla x_1 \nabla x_2} \\
&= \left( x_1 + 3 - x_2^2 - 5x_2 + (x_2 + 5)(\rho(x_1) + x_1)(x_1 + 3) \right. \\
&\quad \left. - x_1^2 x_2 - 4x_1^2 (x_1 + 3) + (x_1 - 4x_2)(\rho(x_2) + x_2)x_2 \right. \\
&\quad \left. + \rho(x_2 + 5)x_1^2 - \rho(x_1^2)(x_2 + 5) \right) \frac{\partial}{\nabla x_1} \\
&\quad + \left( -x_1 + x_2^2 - (x_2 + 5)(\rho(x_1) + x_1) + 4x_1^2 \right) \frac{\partial}{\nabla x_2}.
\end{aligned} \tag{4.2}$$

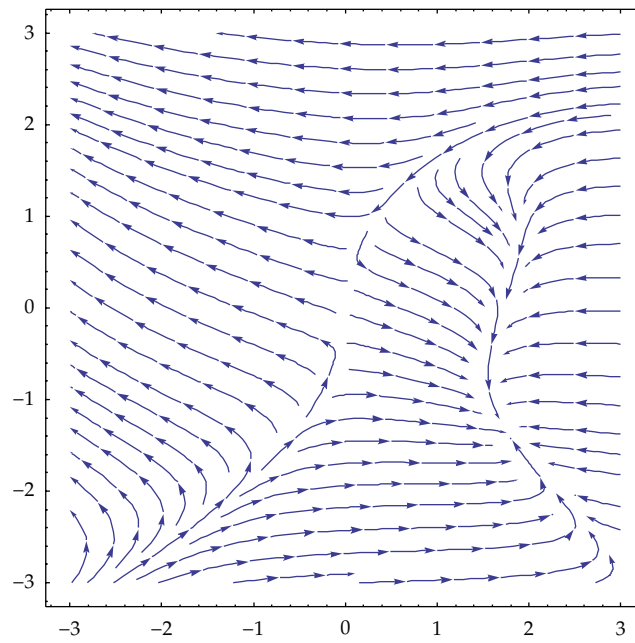
Let us firstly identify the  $\mathbb{T} = \mathbb{R}$  situation that obtained for nabla Lie bracket:

$$\begin{aligned}
[V_1, V_2]_{\nabla}(f) &= \left( -4x_1^3 + 3x_1^2 x_2 - 12x_1^2 + 31x_1 + 2x_1 x_2^2 + 6x_1 x_2 - 8x_2^3 - x_2^2 - 5x_2 \right) \frac{\partial}{\nabla x_1} \\
&\quad + \left( 4x_1^2 - 11x_1 - 2x_1 x_2 + x_2^2 \right) \frac{\partial}{\nabla x_2}.
\end{aligned} \tag{4.3}$$

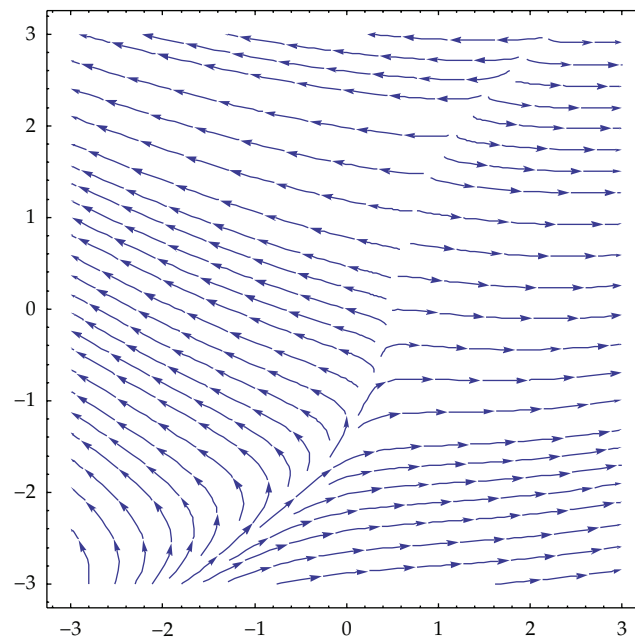
From these equations, when the time scale is  $\mathbb{T} = \mathbb{R}$ , we have seen that we could obtain Lie brackets which are known from the geometry, see in Figure 1.

When the time scale is  $\mathbb{T} = \mathbb{Z}$  as an example, we have obtained a different equation from to have known Lie bracket at real:

$$\begin{aligned}
[V_1, V_2]_{\nabla}(f) &= \left( 36x_1 - 9x_2 - 3x_1^2 - x_2^2 + 3x_2 x_1^2 + 6x_1 x_2 - 4x_1^3 + 2x_1 x_2^2 - 17 \right) \frac{\partial}{\nabla x_1} \\
&\quad + \left( -11x_1 - 2x_1 x_2 + 4x_1^2 + x_2^2 + x_2 + 5 \right) \frac{\partial}{\nabla x_2}.
\end{aligned} \tag{4.4}$$



**Figure 1:** The graph of the vector fields  $[V_1, V_2]_v$  on  $\mathbb{T} = \mathbb{R}$ .



**Figure 2:** The graph of the vector fields  $[V_1, V_2]_v$  on  $\mathbb{T} = \mathbb{Z}$ .

This is seen as a difference of natural consequence of the continuous and discrete structures, see in Figure 2.

## 5. Conclusion

In this study, it is seen that the Lie derivative which is frequently used in physics handles instead of continuous derivative in the way that the nabla Lie multiplication which is created by using time scale both continuous and discrete spaces that are obtained at the same time. This is possible to observe. This is the way to perform the easier and smoother transmission from the continuous space to discrete space. If we consider that the Lie multiplication cannot be analyzed yet in differential geometry discrete space and we can understand how important practice transmission to discrete space with nabla Lie multiplication. It is possible to use nabla Lie multiplication theoretically and practically in many fields of physics with this study.

For instance, with the leading of our work it will be possible to use nabla Lie derivative in nonlinear control systems, field of neural networks, and periodic orbits of a dynamical system, which are important fields.

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