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*Research Article*

# Exponential Extinction of Nicholson's Blowflies System with Nonlinear Density-Dependent Mortality Terms

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This paper presents a new generalized Nicholson's blowflies system with patch structure and nonlinear density-dependent mortality terms. Under appropriate conditions, we establish some criteria to guarantee the exponential extinction of this system. Moreover, we give two examples and numerical simulations to demonstrate our main results.

## 1. Introduction

To describe the population of the Australian sheep blowfly and agree well with the experimental date of Nicholson [1], Gurney et al. [2] proposed the following Nicholson's blowflies equation:

$$N'(t) = -\delta N(t) + pN(t-\tau)e^{-aN(t-\tau)}. \tag{1.1}$$

Here, $N(t)$ is the size of the population at time $t$, $p$ is the maximum per capita daily egg production, $(1/a)$ is the size at which the population reproduces at its maximum rate, $\delta$ is the per capita daily adult death rate, and $\tau$ is the generation time. There have been a large number of results on this model and its modifications (see, e.g., [3–8]). Recently, Berezansky et al. [9] pointed out that a new study indicates that a linear model of density-dependent mortality will be most accurate for populations at low densities and marine ecologists are currently in the process of constructing new fishery models with nonlinear density-dependent mortality rates. Consequently Berezansky et al. [9] presented the following Nicholson's blowflies model with a nonlinear density-dependent mortality term

$$N'(t) = -D(N(t)) + PN(t-\tau)e^{-aN(t-\tau)}, \tag{1.2}$$

where $P$ is a positive constant and function $D$ might have one of the following forms: $D(N) = aN/(N+b)$ or $D(N) = a - be^{-N}$ with positive constants $a, b > 0$.

Wang [10] studied the existence of positive periodic solutions for the model (1.2) with $D(N) = a - be^{-N}$. Hou et al. [11] investigated the permanence and periodic solutions for the model (1.2) with $D(N) = aN/(N+b)$. Furthermore, Liu and Gong [12] considered the permanence for a Nicholson-type delay systems with nonlinear density-dependent mortality terms as follows:

$$\begin{aligned} N_1'(t) &= -D_{11}(t, N_1(t)) + D_{12}(t, N_2(t)) + c_1(t)N_1(t-\tau_1(t))e^{-\gamma_1(t)N_1(t-\tau_1(t))} \\ N_2'(t) &= -D_{22}(t, N_2(t)) + D_{21}(t, N_1(t)) + c_2(t)N_2(t-\tau_2(t))e^{-\gamma_2(t)N_2(t-\tau_2(t))}, \end{aligned} \tag{1.3}$$

where

$$D_{ij}(t, N) = \frac{a_{ij}(t)N}{b_{ij}(t) + N} \quad \text{or} \quad D_{ij}(t, N) = a_{ij}(t) - b_{ij}(t)e^{-N}, \tag{1.4}$$

$a_{ij}, b_{ij}, c_i, \gamma_i : R \to (0, +\infty)$ are all continuous functions bounded above and below by positive constants, and $\tau_j(t) \geq 0$ are bounded continuous functions, $r_i = \sup_{t\in R}\tau_i(t) > 0$, and $i, j = 1, 2$.

On the other hand, since the biological species compete and cooperate with each other in real world, the growth models given by patch structure systems of delay differential equation have been provided by several authors to analyze the dynamics of multiple species (see, e.g., [13–16] and the reference therein). Moreover, the extinction phenomenon often appears in the biology, economy, and physics field and the main focus of Nicholson's blowflies model is on the scalar equation and results on patch structure of this model are gained rarely [14, 16], so it is worth studying the extinction of Nicholson's blowflies system with patch structure and nonlinear density-dependent mortality terms. Motivated by the above discussion, we shall derive the conditions to guarantee the extinction of the following Nicholson-type delay system with patch structure and nonlinear density-dependent mortality terms:

$$\begin{aligned} N_i'(t) = &- D_{ii}(t, N_i(t)) + \sum_{j=1, j\neq i}^{n} D_{ij}\left(t, N_j(t)\right) \\ &+ \sum_{j=1}^{l} c_{ij}(t)N_i\left(t-\tau_{ij}(t)\right)e^{-\gamma_{ij}(t)N_i(t-\tau_{ij}(t))}, \end{aligned} \tag{1.5}$$

where

$$D_{ij}(t, N) = \frac{a_{ij}(t)N}{b_{ij}(t) + N} \quad \text{or} \quad D_{ij}(t, N) = a_{ij}(t) - b_{ij}(t)e^{-N}, \tag{1.6}$$

$a_{ij}, b_{ij}, c_{ik}, \gamma_{ik} : R \to (0, +\infty)$ are all continuous functions bounded above and below by positive constants, and $\tau_{ik}(t) \geq 0$ are bounded continuous functions, $r_i = \max_{1\leq j\leq l}\{\sup_{t\in R}\tau_{ij}(t)\} > 0$, and $i, j = 1, 2, \ldots, n,\ k = 1, 2, \ldots, l$. Furthermore, in the case $D_{ij}(t, N) = a_{ij}(t) - b_{ij}(t)e^{-N}$, to guarantee the meaning of mortality terms we assume that $a_{ij}(t) > b_{ij}(t)$ for $t \in R$ and $i, j = 1, 2, \ldots, n$. The main purpose of this paper is to establish the conditions ensuring the exponential extinction of system (1.5).

For convenience, we introduce some notations. Throughout this paper, given a bounded continuous function g defined on $R$, let $g^+$ and $g^-$ be defined as

$$g^- = \inf_{t\in R} g(t), \qquad g^+ = \sup_{t\in R} g(t). \tag{1.7}$$

Let $R^n(R^n_+)$ be the set of all (nonnegative) real vectors, we will use $x = (x_1, \ldots, x_n)^T \in R^n$ to denote a column vector, in which the symbol $(^T)$ denotes the transpose of a vector. We let $|x|$ denote the absolute-value vector given by $|x| = (|x_1|, \ldots, |x_n|)^T$ and define $\|x\| = \max_{1\leq i\leq n}|x_i|$. Denote $C = \prod_{i=1}^n C([-r_i, 0], R)$ and $C_+ = \prod_{i=1}^n C([-r_i, 0], R_+)$ as Banach spaces equipped with the supremum norm defined by $\|\varphi\| = \sup_{-r_i\leq t\leq 0}\max_{1\leq i\leq n}|\varphi_i(t)|$ for all $\varphi(t) = (\varphi_1(t), \ldots, \varphi_n(t))^T \in C$ (or $\in C_+$). If $x_i(t)$ is defined on $[t_0 - r_i, \nu)$ with $t_0, \nu \in R$ and $i = 1, \ldots, n$, then we define $x_t \in C$ as $x_t = (x_t^1, \ldots x_t^n)^T$ where $x_t^i(\theta) = x_i(t+\theta)$ for all $\theta \in [-r_i, 0]$ and $i = 1, \ldots, n$.

The initial conditions associated with system (1.5) are of the form:

$$N_{t_0} = \varphi, \quad \varphi = (\varphi_1, \ldots, \varphi_n)^T \in C_+, \quad \varphi_i(0) > 0,\ i = 1, \ldots, n. \tag{1.8}$$

We write $N_t(t_0, \varphi)(N(t; t_0, \varphi))$ for a solution of the initial value problem (1.5) and (1.8). Also, let $[t_0, \eta(\varphi))$ be the maximal right-interval of existence of $N_t(t_0, \varphi)$.

*Definition 1.1.* The system (1.5) with initial conditions (1.8) is said to be exponentially extinct, if there are positive constants $M$ and $\kappa$ such that $|N_i(t; t_0, \varphi)| \leq Me^{-\kappa(t-t_0)},\ i = 1, 2\ldots, n$. Denote it as $N_i(t; t_0, \varphi) = O(e^{-\kappa(t-t_0)}),\ i = 1, 2, \ldots, n$.

The remaining part of this paper is organized as follows. In Sections 2 and 3, we shall derive some sufficient conditions for checking the extinction of system (1.5). In Section 4, we shall give two examples and numerical simulations to illustrate our results obtained in the previous sections.

## 2. Extinction of Nicholson's Blowflies System with $D_{ij}(t, N) = a_{ij}(t)N/(b_{ij}(t) + N)(i, j = 1, 2, \ldots, n)$

**Theorem 2.1.** *Suppose that there exists positive constant $K_1$ such that*

$$\frac{a_{ii}^-}{b_{ii}^+ + K_1} > \sum_{j=1, j\neq i}^n \frac{a_{ij}^+}{b_{ij}^-} + \sum_{j=1}^l \frac{c_{ij}^+}{\gamma_{ij}^- e K_1}, \quad i = 1, 2, \ldots, n. \tag{2.1}$$

*Let*

$$E^1 = \{\varphi \mid \varphi \in C_+, \varphi(0) > 0,\ 0 \leq \varphi_i(t) < K_1, \forall t \in [-r_i, 0],\ i = 1, 2, \ldots, n\}. \tag{2.2}$$

*Moreover, assume* $N(t;t_0,\varphi)$ *is the solution of* (1.5) *with* $\varphi \in E^1$ *and* $D_{ij}(t,N) = (a_{ij}(t)N/(b_{ij}(t) + N))$ $(i,j = 1,2,\dots,n)$. *Then,*

$$\begin{gathered} 0 \le N_i(t;t_0,\varphi) < K_1, \quad \forall t \in [t_0, \eta(\varphi)), \ i = 1,2,\dots,n, \\ \eta(\varphi) = +\infty. \end{gathered} \tag{2.3}$$

*Proof.* Set $N(t) = N(t;t_0,\varphi)$ for all $t \in [t_0,\eta(\varphi))$. In view of $\varphi \in C_+$, using Theorem 5.2.1 in [17, p. 81], we have $N_t(t_0,\varphi) \in C_+$ for all $t \in [t_0,\eta(\varphi))$. Assume, by way of contradiction, that (2.3) does not hold. Then, there exist $t_1 \in [t_0,\eta(\varphi))$ and $i \in \{1,2,\dots,n\}$ such that

$$N_i(t_1) = K_1, \quad 0 \le N_j(t) < K_1 \quad \forall t \in [t_0 - r_j, t_1), \ j = 1,2,\dots,n. \tag{2.4}$$

Calculating the derivative of $N_i(t)$, together with (2.1) and the fact that $\sup_{u\ge 0} ue^{-u} = 1/e$ and $a(t)N/(b(t)+N) \le a(t)N/b(t)$ for all $t \in R$, $N \ge 0$, (1.5) and (2.4) imply that

$$\begin{aligned} 0 &\le N_i'(t_1) \\ &= -D_{ii}(t_1, N_i(t_1)) + \sum_{j=1, j\ne i}^{n} D_{ij}(t_1, N_j(t_1)) \\ &\quad + \sum_{j=1}^{l} c_{ij}(t_1) N_i(t_1 - \tau_{ij}(t_1)) e^{-\gamma_{ij}(t_1)N_i(t_1-\tau_{ij}(t_1))} \\ &\le -\frac{a_{ii}(t_1)N_i(t_1)}{b_{ii}(t_1)+N_i(t_1)} + \sum_{j=1, j\ne i}^{n} \frac{a_{ij}(t_1)N_j(t_1)}{b_{ij}(t_1)} + \sum_{j=1}^{l} \frac{c_{ij}(t_1)}{\gamma_{ij}(t_1)} \frac{1}{e} \\ &\le \left( -\frac{a_{ii}^-}{b_{ii}^+ + K_1} + \sum_{j=1, j\ne i}^{n} \frac{a_{ij}^+}{b_{ij}^-} + \sum_{j=1}^{l} \frac{c_{ij}^+}{\gamma_{ij}^- e K_1} \right) K_1 \\ &< 0, \end{aligned} \tag{2.5}$$

which is a contradiction and implies that (2.3) holds. From Theorem 2.3.1 in [18], we easily obtain $\eta(\varphi) = +\infty$. This ends the proof of Theorem 2.1. □

**Theorem 2.2.** *Suppose that there exists positive constant* $K_1$ *satisfying* (2.1) *and*

$$\frac{a_{ii}^-}{b_{ii}^+ + K_1} > \sum_{j=1, j\ne i}^{n} \frac{a_{ij}^+}{b_{ij}^-} + \sum_{j=1}^{l} c_{ij}^+, \quad i = 1,2,\dots,n. \tag{2.6}$$

*Then the solution* $N(t;t_0,\varphi)$ *of* (1.5) *with* $\varphi \in E^1$ *and* $D_{ij}(t,N) = (a_{ij}(t)N/(b_{ij}(t)+N))$ $(i,j = 1,2,\dots,n)$ *is exponentially extinct as* $t \to +\infty$.

*Proof.* Define continuous functions $\Gamma_{\mathrm{i}}(\omega)$ by setting

$$\Gamma_{\mathrm{i}}(\omega) = \omega - \frac{a_{ii}^{-}}{b_{ii}^{+} + K_1} + \sum_{j=1, j\neq i}^{n} \frac{a_{ij}^{+}}{b_{ij}^{-}} + \sum_{j=1}^{l} c_{ij}^{+} e^{\omega r_i}, \quad i = 1, 2, \ldots, n. \tag{2.7}$$

Then, from (2.6), we obtain

$$\Gamma_i(0) = -\frac{a_{ii}^{-}}{b_{ii}^{+} + K_1} + \sum_{j=1, j\neq i}^{n} \frac{a_{ij}^{+}}{b_{ij}^{-}} + \sum_{j=1}^{l} c_{ij}^{+} < 0, \quad i = 1, 2, \ldots, n. \tag{2.8}$$

The continuity of $\Gamma_i(\omega)$ implies that there exists $\lambda > 0$ such that

$$\Gamma_i(\lambda) = \lambda - \frac{a_{ii}^{-}}{b_{ii}^{+} + K_1} + \sum_{j=1, j\neq i}^{n} \frac{a_{ij}^{+}}{b_{ij}^{-}} + \sum_{j=1}^{l} c_{ij}^{+} e^{\lambda r_i} < 0, \quad i = 1, 2, \ldots, n. \tag{2.9}$$

Let

$$y_i(t) = N_i(t) e^{\lambda(t-t_0)}, \quad i = 1, 2, \ldots, n. \tag{2.10}$$

Calculating the derivative of $y(t)$ along the solution $N(t)$ of system (1.5) with $\varphi \in E^1$, we have

$$\begin{aligned} y_i'(t) &= \lambda y_i(t) + e^{\lambda(t-t_0)} N_i'(t) \\ &= \lambda y_i(t) - \frac{a_{ii}(t) y_i(t)}{b_{ii}(t) + N_i(t)} + \sum_{j=1, j\neq i}^{n} \frac{a_{ij}(t) y_j(t)}{b_{ij}(t) + N_j(t)} \\ &\quad + \sum_{j=1}^{l} c_{ij}(t) e^{\lambda \tau_{ij}(t)} y_i(t - \tau_{ij}(t)) e^{-\gamma_{ij}(t) N_i(t - \tau_{ij}(t))}, \quad i = 1, 2, \ldots, n. \end{aligned} \tag{2.11}$$

Let $M_1$ denote an arbitrary positive number and set

$$M_1 > y_i(t), \quad \forall t \in [t_0 - r_i, t_0],\ i = 1, 2, \ldots, n. \tag{2.12}$$

We claim that

$$y_i(t) < M_1, \quad \forall t \in [t_0, +\infty),\ i = 1, 2, \ldots, n. \tag{2.13}$$

If this is not valid, there must exist $t_2 \in (t_0, +\infty)$ and $i \in \{1, 2, \ldots, n\}$ such that

$$y_i(t_2) = M_1, \quad y_j(t) < M_1, \quad \forall t < t_2,\ j = 1, 2, \ldots, n. \tag{2.14}$$

Then, from (2.3) and (2.11), we have

$$\begin{aligned}
0 &\le y_i'(t_2) \\
&= \lambda y_i(t_2) - \frac{a_{ii}(t_2)y_i(t_2)}{b_{ii}(t_2)+N_i(t_2)} + \sum_{j=1,j\neq i}^{n} \frac{a_{ij}(t_2)y_j(t_2)}{b_{ij}(t_2)+N_j(t_2)} \\
&\quad + \sum_{j=1}^{l} c_{ij}(t_2)e^{\lambda\tau_{ij}(t_2)} y_i\big(t_2-\tau_{ij}(t_2)\big)e^{-\gamma_{ij}(t_2)N_i(t_2-\tau_{ij}(t_2))} \\
&\le \lambda M_1 - \frac{a_{ii}(t_2)M_1}{b_{ii}(t_2)+K_1} + \sum_{j=1,j\neq i}^{n} \frac{a_{ij}(t_2)M_1}{b_{ij}(t_2)} + \sum_{j=1}^{l} c_{ij}(t_2)e^{\lambda r_i}M_1 \\
&\le \left(\lambda - \frac{a_{ii}^-}{b_{ii}^+ + K_1} + \sum_{j=1,j\neq i}^{n} \frac{a_{ij}^+}{b_{ij}^-} + \sum_{j=1}^{l} c_{ij}^+ e^{\lambda r_i}\right) M_1 \\
&< 0.
\end{aligned} \tag{2.15}$$

This contradiction implies that (2.13) holds. Thus,

$$N_i(t) = y_i(t)e^{-\lambda(t-t_0)} \le M_1 e^{-\lambda(t-t_0)} \quad \forall t \in [t_0 - r_i, +\infty),\ i = 1,2,\ldots,n. \tag{2.16}$$

This completes the proof. □

## 3. Extinction of Nicholson's Blowflies System with $D_{ij}(t,N) = a_{ij}(t) - b_{ij}(t)e^{-N}\,(i,j = 1,2,\ldots,n)$

**Theorem 3.1.** *Suppose that there exists positive constant $K_2$ such that*

$$a_{ii}^- > \sum_{j=1,j\neq i}^{n} a_{ij}^+ + \left(b_{ii}^+ - \sum_{j=1,j\neq i}^{n} b_{ij}^-\right)e^{-K_2} + \sum_{j=1}^{l} \frac{c_{ij}^+}{\gamma_{ij}^- e}, \quad i = 1,2,\ldots,n, \tag{3.1}$$

$$-a_{ii}(t) + b_{ii}(t) + \sum_{j=1,j\neq i}^{n} \big(a_{ij}(t) - b_{ij}(t)\big) \ge 0, \quad i = 1,2,\ldots,n. \tag{3.2}$$

*Let*

$$E^2 = \{\varphi \mid \varphi \in C_+,\ \varphi(0) > 0,\ 0 \le \varphi_i(t) < K_2, \forall t \in [-r_i, 0],\ i = 1,2,\ldots,n\}. \tag{3.3}$$

*Moreover, assume $N(t;t_0,\varphi)$ is the solution of* (1.5) *with $\varphi \in E^2$ and $D_{ij}(t,N) = a_{ij}(t) - b_{ij}(t) e^{-N}$ $(i,j=1,2,\ldots,n)$. Then,*

$$0 \le N_i(t;t_0,\varphi) < K_2, \quad \forall t \in [t_0,\eta(\varphi)), \; i=1,2,\ldots,n, \tag{3.4}$$

$$\eta(\varphi) = +\infty. \tag{3.5}$$

*Proof.* Set $N(t) = N(t;t_0,\varphi)$ for all $t \in [t_0,\eta(\varphi))$. Rewrite the system (1.5) as

$$N'(t) = f(t,N_t), \tag{3.6}$$

where $f(t,\phi) = (f_1(t,\phi), f_2(t,\phi),\ldots,f_n(t,\phi))^T$ and

$$\begin{aligned} f_i(t,\phi) = & -a_{ii}(t) + b_{ii}(t)e^{-\phi_i(0)} + \sum_{j=1, j\neq i}^{n} \left(a_{ij}(t) - b_{ij}(t)e^{-\phi_j(0)}\right) \\ & + \sum_{j=1}^{l} c_{ij}(t)\phi_i(-\tau_{ij}(t))e^{-\gamma_{ij}(t)\phi_i(-\tau_{ij}(t))}, \quad i=1,2,\ldots,n, \; \phi \in C. \end{aligned} \tag{3.7}$$

In view of (3.2), whenever $\phi \in C$ satisfies $\phi \ge 0$, $\phi_i(0) = 0$ for some $i$ and $t \in R$, then

$$\begin{aligned} f_i(t,\phi) &= -a_{ii}(t) + b_{ii}(t) + \sum_{j=1, j\neq i}^{n} \left(a_{ij}(t) - b_{ij}(t)e^{-\phi_j(0)}\right) \\ &\quad + \sum_{j=1}^{l} c_{ij}(t)\phi_i(-\tau_{ij}(t))e^{-\gamma_{ij}(t)\phi_i(-\tau_{ij}(t))} \\ &\ge -a_{ii}(t) + b_{ii}(t) + \sum_{j=1, j\neq i}^{n} (a_{ij}(t) - b_{ij}(t)) \\ &\ge 0. \end{aligned} \tag{3.8}$$

Thus, using Theorem 5.2.1 in [17, p. 81], we have $N_t(t_0,\varphi) \in C_+$ for all $t \in [t_0,\eta(\varphi))$ and $\varphi \in E^2 \subset C_+$. Assume, by way of contradiction, that (3.4) does not hold. Then, there exist $t_3 \in [t_0,\eta(\varphi))$ and $i \in \{1,2,\ldots,n\}$ such that

$$N_i(t_3) = K_2, \quad 0 \le N_j(t) < K_2 \quad \forall t \in [t_0 - r_j, t_3), \; j=1,2,\ldots,n. \tag{3.9}$$

Calculating the derivative of $N_i(t)$, together with (3.1) and the fact that $\sup_{u\geq 0} ue^{-u} = 1/e$, (1.5) and (3.9) imply that

$$\begin{aligned}
0 &\leq N_i'(t_3) \\
&= -D_{ii}(t_3, N_i(t_3)) + \sum_{j=1, j\neq i}^{n} D_{ij}(t_3, N_j(t_3)) + \sum_{j=1}^{l} c_{ij}(t_3) N_i(t_3 - \tau_{ij}(t_3)) \\
&\quad \times e^{-\gamma_{ij}(t_3) N_i(t_3 - \tau_{ij}(t_3))} \\
&\leq -a_{ii}(t_3) + b_{ii}(t_3)e^{-K_2} + \sum_{j=1, j\neq i}^{n} \left(a_{ij}(t_3) - b_{ij}(t_3)e^{-K_2}\right) + \sum_{j=1}^{l} \frac{c_{ij}(t_3)}{\gamma_{ij}(t_3)} \frac{1}{e} \\
&\leq -a_{ii}^{-} + \sum_{j=1, j\neq i}^{n} a_{ij}^{+} + \left(b_{ii}^{+} - \sum_{j=1, j\neq i}^{n} b_{ij}^{-}\right) e^{-K_2} + \sum_{j=1}^{l} \frac{c_{ij}^{+}}{\gamma_{ij}^{-} e} \\
&< 0,
\end{aligned} \tag{3.10}$$

which is a contradiction and implies that (3.4) holds. From Theorem 2.3.1 in [18], we easily obtain $\eta(\varphi) = +\infty$. This ends the proof of Theorem 3.1. □

**Theorem 3.2.** *Let* (3.1) *and* (3.2) *hold. Moreover, suppose that there exist two positive constants* $\widetilde{\lambda}$ *and* $\widetilde{M}$ *such that*

$$-a_{ii}(t) + b_{ii}(t) + \sum_{j=1, j\neq i}^{n} \left(a_{ij}(t) - b_{ij}(t)\right) \leq \widetilde{M} e^{-\widetilde{\lambda}(t-t_0)}, \quad t \in R, \ i = 1, 2, \ldots, n, \tag{3.11}$$

$$b_{ii}^{-} > 1 + \frac{K_2}{2} b_{ii}^{+} + \sum_{j=1, j\neq i}^{n} b_{ij}^{+} + \sum_{j=1}^{l} c_{ij}^{+}, \quad i = 1, 2, \ldots, n. \tag{3.12}$$

*Then the solution* $N(t; t_0, \varphi)$ *of* (1.5) *with* $\varphi \in E^2$ *and* $D_{ij}(t, N) = a_{ij}(t) - b_{ij}(t)e^{-N}$ $(i, j = 1, 2, \ldots, n)$, *is exponentially extinct as* $t \to +\infty$.

*Proof.* Define continuous functions $\Gamma_i(\omega)$ by setting

$$\Gamma_i(\omega) = \omega - b_{ii}^{-} + 1 + \frac{K_2}{2} b_{ii}^{+} + \sum_{j=1, j\neq i}^{n} b_{ij}^{+} + \sum_{j=1}^{l} c_{ij}^{+} e^{\omega r_i}, \quad i = 1, 2, \ldots, n. \tag{3.13}$$

Then, from (3.12), we obtain

$$\Gamma_i(0) = -b_{ii}^{-} + 1 + \frac{K_2}{2} b_{ii}^{+} + \sum_{j=1, j\neq i}^{n} b_{ij}^{+} + \sum_{j=1}^{l} c_{ij}^{+} < 0, \quad i = 1, 2, \ldots, n. \tag{3.14}$$

The continuity of $\Gamma_i(\omega)$ implies that there exists $0 < \mu < \tilde{\lambda}$ such that

$$\Gamma_i(\mu) = \mu - b_{ii}^{-} + 1 + \frac{K_2}{2} b_{ii}^{+} + \sum_{j=1, j\neq i}^{n} b_{ij}^{+} + \sum_{j=1}^{l} c_{ij}^{+} e^{\mu r_i} < 0, \quad i = 1, 2, \ldots, n. \tag{3.15}$$

Let

$$x_i(t) = N_i(t) e^{\mu(t-t_0)}, \quad i = 1, 2, \ldots, n. \tag{3.16}$$

Calculating the derivative of $x(t)$ along the solution $N(t)$ of system (1.5) with $\varphi \in E^2$, in view of (3.4) and (3.11), we have

$$\begin{aligned}
x_i'(t) &= \mu x_i(t) + e^{\mu(t-t_0)} N_i'(t) \\
&= \mu x_i(t) + e^{\mu(t-t_0)} \left[ -a_{ii}(t) + b_{ii}(t) e^{-N_i(t)} + \sum_{j=1, j\neq i}^{n} \left( a_{ij}(t) - b_{ij}(t) e^{-N_j(t)} \right) \right] \\
&\quad + \sum_{j=1}^{l} c_{ij}(t) e^{\mu \tau_{ij}(t)} x_i\left(t - \tau_{ij}(t)\right) e^{-\gamma_{ij}(t) N_i(t - \tau_{ij}(t))} \\
&\leq \mu x_i(t) + e^{\mu(t-t_0)} \left[ -a_{ii}(t) + b_{ii}(t) \left( 1 - N_i(t) + \frac{1}{2} N_i^2(t) \right) \right. \\
&\qquad \left. + \sum_{j=1, j\neq i}^{n} \left( a_{ij}(t) - b_{ij}(t) \left( 1 - N_j(t) \right) \right) \right] + \sum_{j=1}^{l} c_{ij}^{+} e^{\mu r_i} x_i\left(t - \tau_{ij}(t)\right) \\
&= \mu x_i(t) + e^{\mu(t-t_0)} \left[ -a_{ii}(t) + b_{ii}(t) + \sum_{j=1, j\neq i}^{n} \left( a_{ij}(t) - b_{ij}(t) \right) \right] - b_{ii}(t) x_i(t) \\
&\quad + \frac{1}{2} b_{ii}(t) N_i(t) x_i(t) + \sum_{j=1, j\neq i}^{n} b_{ij}(t) x_j(t) + \sum_{j=1}^{l} c_{ij}^{+} e^{\mu r_i} x_i\left(t - \tau_{ij}(t)\right) \\
&\leq \mu x_i(t) + \widetilde{M} e^{(\mu - \tilde{\lambda})(t-t_0)} - b_{ii}^{-} x_i(t) + \frac{K_2}{2} b_{ii}^{+} x_i(t) \\
&\quad + \sum_{j=1, j\neq i}^{n} b_{ij}^{+} x_j(t) + \sum_{j=1}^{l} c_{ij}^{+} e^{\mu r_i} x_i\left(t - \tau_{ij}(t)\right).
\end{aligned} \tag{3.17}$$

Let $M_2$ denote an arbitrary positive number and set

$$M_2 > \max\left\{ x_i(t), \widetilde{M} \right\} \quad \forall t \in [t_0 - r_i, t_0], \; i = 1, 2, \ldots, n. \tag{3.18}$$

We claim that

$$x_i(t) < M_2, \quad \forall t \in [t_0, +\infty), \ i = 1, 2, \ldots, n. \tag{3.19}$$

If this is not valid, there must exist $t_4 \in (t_0, +\infty)$ and $i \in \{1, 2, \ldots, n\}$ such that

$$x_i(t_4) = M_2, \quad x_j(t) < M_2, \quad \forall t < t_4, \ j = 1, 2, \ldots, n. \tag{3.20}$$

Then, from (3.15) and (3.17), we have

$$\begin{aligned}
0 &\leq x_i'(t_4) \\
&\leq \mu x_i(t_4) + \widetilde{M} e^{(\mu - \widetilde{\lambda})(t_4 - t_0)} - b_{ii}^- x_i(t_4) + \frac{K_2}{2} b_{ii}^+ x_i(t_4) \\
&\quad + \sum_{j=1, j \neq i}^{n} b_{ij}^+ x_j(t_4) + \sum_{j=1}^{l} c_{ij}^+ e^{\mu r_i} x_i\left(t_4 - \tau_{ij}(t_4)\right) \\
&\leq \left[\mu + 1 - b_{ii}^- + \frac{K_2}{2} b_{ii}^+ + \sum_{j=1, j \neq i}^{n} b_{ij}^+ + \sum_{j=1}^{l} c_{ij}^+ e^{\mu r_i}\right] M_2 \\
&< 0.
\end{aligned} \tag{3.21}$$

This contradiction implies that (3.19) holds. Thus,

$$N_i(t) = x_i(t) e^{-\mu(t - t_0)} \leq M_2 e^{-\mu(t - t_0)} \quad \forall t \in [t_0 - r_i, +\infty), \ i = 1, 2, \ldots, n. \tag{3.22}$$

This completes the proof. □

## 4. Numerical Examples

In this section, we give two examples and numerical simulations to demonstrate the results obtained in previous sections.

*Example 4.1.* Consider the following Nicholson's blowflies system with patch structure and nonlinear density-dependent mortality terms:

$$\begin{aligned}
N_1'(t) = &-\frac{(25+|\cos 3t|)N_1(t)}{5+|\sin 2t|+N_1(t)}+\frac{(1+|\sin 2t|)N_2(t)}{3+|\cos 3t|+N_2(t)}+\frac{(1+|\cos 2t|)N_3(t)}{3+|\sin 3t|+N_3(t)}\\
&+\frac{1}{4}\left(1+\cos^2 t\right)N_1(t-2|\sin t|)e^{-4N_1(t-2|\sin t|)}\\
&+\frac{1}{4}\left(1+\sin^2 t\right)N_1(t-2|\cos t|)e^{-4N_1(t-2|\cos t|)}\\
N_2'(t) = &-\frac{(25+|\sin 3t|)N_2(t)}{5+|\cos 2t|+N_2(t)}+\frac{(1+|\cos 2t|)N_1(t)}{3+|\sin 3t|+N_1(t)}+\frac{(1+|\sin 2t|)N_3(t)}{3+|\cos 3t|+N_3(t)}\\
&+\frac{1}{4}\left(1+\sin^2 t\right)N_2(t-2|\cos t|)e^{-4N_2(t-2|\cos t|)}\\
&+\frac{1}{4}\left(1+\cos^2 t\right)N_2(t-2|\sin t|)e^{-4N_2(t-2|\sin t|)}\\
N_3'(t) = &-\frac{(25+|\sin 5t|)N_3(t)}{5+|\cos 6t|+N_3(t)}+\frac{(1+|\cos 3t|)N_1(t)}{3+|\sin 2t|+N_1(t)}+\frac{(1+|\sin 3t|)N_2(t)}{3+|\cos 2t|+N_2(t)}\\
&+\frac{1}{4}\left(1+\sin^2 2t\right)N_3(t-2|\cos 2t|)e^{-4N_3(t-2|\cos 2t|)}\\
&+\frac{1}{4}\left(1+\cos^2 2t\right)N_3(t-2|\sin 2t|)e^{-4N_3(t-2|\sin 2t|)}.
\end{aligned} \tag{4.1}$$

Obviously, $a_{ii}^- = 25$, $b_{ii}^+ = 6$, $(i = 1,2,3)$, $a_{ij}^+ = 2$, $b_{ij}^- = 3$, $(i,j = 1,2,3, i \neq j)$, $c_{ij}^+ = 1/2$, $\gamma_{ij}^- = 4$, $(i = 1,2,3,\ j = 1,2)$. Let $K_1 = e$, then we have

$$\begin{aligned}
\frac{25}{6+\mathrm{e}} = \frac{a_{ii}^-}{b_{ii}^+ + K_1} &> \sum_{j=1, j\neq i}^{3}\frac{a_{ij}^+}{b_{ij}^-}+\sum_{j=1}^{2}\frac{c_{ij}^+}{\gamma_{ij}^- e K_1} = \frac{4}{3}+\frac{1}{4e^2},\\
\frac{25}{6+e} = \frac{a_{ii}^-}{b_{ii}^+ + K_1} &> \sum_{j=1, j\neq i}^{3}\frac{a_{ij}^+}{b_{ij}^-}+\sum_{j=1}^{2}c_{ij}^+ = \frac{7}{3}.
\end{aligned} \tag{4.2}$$

Then (4.2) imply that the system (4.1) satisfies (2.1) and (2.6). Hence, from Theorems 2.1 and 2.2, the solution $N(t)$ of system (4.1) with $D_{ij}(t,N) = a_{ij}(t)N/(b_{ij}(t)+N)(i,j = 1,2,3)$ and $\varphi \in E^1 = \{\varphi \mid \varphi \in C_+,\ \varphi(0) > 0$ and $0 \le \varphi_i(t) < e,$ for all, $t \in [-2,0],\ i = 1,2,3\}$ is exponentially extinct as $t \to +\infty$ and $N(t) = N(t,0,\varphi) = O(e^{-\kappa t})$, $\kappa \approx 0.0001$. The fact is verified by the numerical simulation in Figure 1.

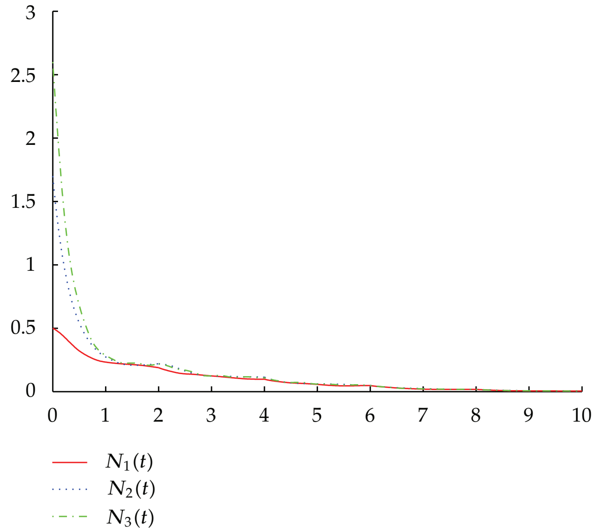


**Figure 1:** Numerical solution $N(t) = (N_1(t), N_2(t), N_3(t))^T$ of system (4.1) for initial value $\varphi(t) \equiv (0.5, 1.7, 2.6)^T$.

*Example 4.2.* Consider the following Nicholson's blowflies system with patch structure and nonlinear density-dependent mortality terms:

$$\begin{aligned}
N_1'(t) = &-(12+|\sin t|)+(11+|\cos t|)e^{-N_1(t)}+\left(1+\frac{1}{2}|\sin t|\right)-\left(\frac{1}{2}+\frac{1}{2}|\cos t|\right)e^{-N_2(t)}\\
&+\left(1+\frac{1}{2}|\sin t|\right)-\left(\frac{1}{2}+\frac{1}{2}|\cos t|\right)e^{-N_3(t)}+\frac{1}{4}\left(1+\cos^2 t\right)N_1(t-2|\sin t|)e^{-N_1(t-2|\sin t|)}\\
&+\frac{1}{4}\left(1+\sin^2 t\right)N_1(t-2|\cos t|)e^{-N_1(t-2|\cos t|)}\\
N_2'(t) = &-(12+|\cos t|)+(11+|\sin t|)e^{-N_2(t)}+\left(1+\frac{1}{2}|\cos t|\right)-\left(\frac{1}{2}+\frac{1}{2}|\sin t|\right)e^{-N_1(t)}\\
&+\left(1+\frac{1}{2}|\cos t|\right)-\left(\frac{1}{2}+\frac{1}{2}|\sin t|\right)e^{-N_3(t)}+\frac{1}{4}\left(1+\sin^2 t\right)N_2(t-2|\cos t|)e^{-N_2(t-2|\cos t|)}\\
&+\frac{1}{4}\left(1+\cos^2 t\right)N_2(t-2|\sin t|)e^{-N_2(t-2|\sin t|)}\\
N_3'(t) = &-(12+|\sin 2t|)+(11+|\cos 2t|)e^{-N_3(t)}+\left(1+\frac{1}{2}|\sin 2t|\right)-\left(\frac{1}{2}+\frac{1}{2}|\cos 2t|\right)e^{-N_1(t)}\\
&+\left(1+\frac{1}{2}|\sin 2t|\right)-\left(\frac{1}{2}+\frac{1}{2}|\cos 2t|\right)e^{-N_2(t)}\\
&+\frac{1}{4}\left(1+\cos^2 2t\right)N_3(t-2|\sin 2t|)e^{-N_3(t-2|\sin 2t|)}\\
&+\frac{1}{4}\left(1+\sin^2 2t\right)N_3(t-2|\cos 2t|)e^{-N_3(t-2|\cos 2t|)}.
\end{aligned} \tag{4.3}$$

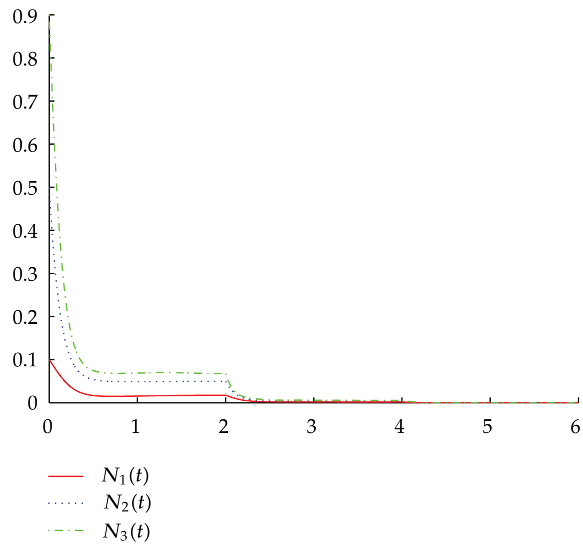


**Figure 2:** Numerical solution $N(t) = (N_1(t), N_2(t), N_3(t))^T$ of system (4.3) for initial value $\varphi(t) \equiv (0.1, 0.5, 0.9)^T$.

Obviously, $a_{ii}^- = 12$, $b_{ii}^- = 11$, $b_{ii}^+ = 12$, $(i = 1,2,3)$, $a_{ij}^+ = 3/2$, $b_{ij}^- = 1/2$, $b_{ij}^+ = 1$, $(i,j = 1,2,3,\ i \neq j)$, $c_{ij}^+ = 1/2$, $\gamma_{ij}^- = 1$, $(i = 1,2,3,\ j = 1,2)$. Let $K_2 = 1$, then we have

$$12 = a_{ii}^- > \sum_{j=1, j\neq i}^{3} a_{ij}^+ + \left( b_{ii}^+ - \sum_{j=1, j\neq i}^{n} b_{ij}^- \right) e^{-K_2} + \sum_{j=1}^{2} \frac{c_{ij}^+}{\gamma_{ij}^- e} = 3 + \frac{12}{e}, \quad i = 1,2,3,$$

$$-a_{ii}(t) + b_{ii}(t) + \sum_{j=1, j\neq i}^{3} \left( a_{ij}(t) - b_{ij}(t) \right) = 0, \quad i = 1,2,3, \tag{4.4}$$

$$11 = b_{ii}^- > 1 + \frac{K_2}{2} b_{ii}^+ + \sum_{j=1, j\neq i}^{3} b_{ij}^+ + \sum_{j=1}^{2} c_{ij}^+ = 10, \quad i = 1,2,3.$$

Then (4.4) imply that the system (4.3) satisfies (3.1), (3.2), (3.11), and (3.12). Hence, from Theorems 3.1 and 3.2, the solution $N(t)$ of system (4.1) with $D_{ij}(t,N) = a_{ij}(t) - b_{ij}(t)e^{-N}(i,j = 1,2,3)$ and $\varphi \in E^2 = \{\varphi \mid \varphi \in C_+,\ \varphi(0) > 0 \text{ and } 0 \leq \varphi_i(t) < 1, \text{ for all}, t \in [-2,0],\ i = 1,2,3\}$ is exponentially extinct as $t \to +\infty$ and $N(t) = N(t,0,\varphi) = O(e^{-\kappa t})$, $\kappa \approx 0.0001$. The fact is verified by the numerical simulation in Figure 2.

*Remark 4.3.* To the best of our knowledge, few authors have considered the problems of the extinction of Nicholson's blowflies model with patch structure and nonlinear density-dependent mortality terms. Wang [10] and Hou et al. [11] have researched the permanence and periodic solution for scalar Nicholson's blowflies equation with a nonlinear density-dependent mortality term. Liu and Gong [12] have considered the permanence for Nicholson-type delay systems with nonlinear density-dependent mortality terms and Takeuchi et al. [13] have investigated the global stability of population model with patch structure. Faria [14], Liu [15], and Berzansky et al. [16] have, respectively, studied the local and global stability of positive equilibrium for constant coefficients of Nicholson's blowflies model with patch structure. It is clear that all the results in [10–16] and the references therein cannot be applicable to prove the extinction of (4.1) and (4.3). This implies that the results of this paper are new.

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## References

[1] A. J. Nicholson, "The self adjustment of population to change," *Cold Spring Harbor Symposia on Quantitative Biology*, vol. 22, pp. 153–173, 1957.
[2] W. S. C. Gurney, S. P. Blythe, and R. M. Nisbet, "Nicholson's blowflies revisited," *Nature*, vol. 287, pp. 17–21, 1980.
[3] B. Liu, "Global stability of a class of Nicholson's blowflies model with patch structure and multiple time-varying delays," *Nonlinear Analysis: Real World Applications*, vol. 11, no. 4, pp. 2557–2562, 2010.
[4] M. R. S. Kulenović, G. Ladas, and Y. G. Sficas, "Global attractivity in Nicholson's blowflies," *Applicable Analysis*, vol. 43, no. 1-2, pp. 109–124, 1992.
[5] J. W.-H. So and J. S. Yu, "Global attractivity and uniform persistence in Nicholson's blowflies," *Differential Equations and Dynamical Systems*, vol. 2, no. 1, pp. 11–18, 1994.
[6] M. Li and J. Yan, "Oscillation and global attractivity of generalized Nicholson's blowfly model," in *Differential Equations and Computational Simulations*, pp. 196–201, World Scientific, River Edge, NJ, USA, 2000.
[7] Y. Chen, "Periodic solutions of delayed periodic Nicholson's blowflies models," *The Canadian Applied Mathematics Quarterly*, vol. 11, no. 1, pp. 23–28, 2003.
[8] J. Li and C. Du, "Existence of positive periodic solutions for a generalized Nicholson's blowflies model," *Journal of Computational and Applied Mathematics*, vol. 221, no. 1, pp. 226–233, 2008.
[9] L. Berezansky, E. Braverman, and L. Idels, "Nicholson's blowflies differential equations revisited: main results and open problems," *Applied Mathematical Modelling*, vol. 34, no. 6, pp. 1405–1417, 2010.
[10] W. Wang, "Positive periodic solutions of delayed Nicholson's blowflies models with a nonlinear density-dependent mortality term," *Applied Mathematical Modelling*, vol. 36, no. 10, pp. 4708–4713, 2012.
[11] X. Hou, L. Duan, and Z. Huang, "Permanence and periodic solutions for a class of delay Nicholson's blowflies models," *Applied Mathematical Modelling*, vol. 37, no. 3, pp. 1537–1544, 2012.
[12] B. Liu and S. Gong, "Permanence for Nicholson-type delay systems with nonlinear density-dependent mortality terms," *Nonlinear Analysis: Real World Applications*, vol. 12, no. 4, pp. 1931–1937, 2011.
[13] Y. Takeuchi, W. Wang, and Y. Saito, "Global stability of population models with patch structure," *Nonlinear Analysis: Real World Applications*, vol. 7, no. 2, pp. 235–247, 2006.
[14] T. Faria, "Global asymptotic behaviour for a Nicholson model with patch structure and multiple delays," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 74, no. 18, pp. 7033–7046, 2011.
[15] B. Liu, "Global stability of a class of delay differential systems," *Journal of Computational and Applied Mathematics*, vol. 233, no. 2, pp. 217–223, 2009.
[16] L. Berezansky, L. Idels, and L. Troib, "Global dynamics of Nicholson-type delay systems with applications," *Nonlinear Analysis: Real World Applications*, vol. 12, no. 1, pp. 436–445, 2011.
[17] H. L. Smith, *Monotone Dynamical Systems: An Introduction to the Theory of Competitive and Cooperative System*, vol. 41 of *Mathematical Surveys and Monographs*, American Mathematical Society, Providence, RI, USA, 1995.

[18] J. K. Hale and S. M. Verduyn Lunel, *Introduction to Functional-Differential Equations*, vol. 99 of *Applied Mathematical Sciences*, Springer, New York, NY, USA, 1993.