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Research Article

Radially Symmetric Solutions of $\Delta w + |w|^{p-1}w = 0$

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We investigate solutions of $w'' + ((N-1)/r)w' + |w|^{p-1}w = 0$, $r > 0$ and focus on the regime $N > 2$ and $p > N/(N-2)$. Our advance is to develop a technique to efficiently classify the behavior of solutions on $(r_{\min}, r_{\max})$, their maximal positive interval of existence. Our approach is to transform the nonautonomous w equation into an autonomous ODE. This reduces the problem to analyzing the phase plane of the autonomous equation. We prove the existence of new families of solutions of the w equation and describe their asymptotic behavior. In the subcritical case $N/(N-2) < p < (N+2)/(N-2)$ there is a well-known closed-form singular solution, w_1, such that $w_1(r) \to \infty$ as $r \to 0^+$ and $w_1(r) \to 0$ as $r \to \infty$. Our advance is to prove the existence of a family of solutions of the subcritical case which satisfies $w(r_i) = w_1(r_i)$ for infinitely many values $r_i > 0$. At the critical value $p = (N+2)/(N-2)$ there is a continuum of positive singular solutions, and a continuum of sign changing singular solutions. In the supercritical regime $p > (N+2)/(N-2)$ we prove the existence of a family of "super singular" sign changing singular solutions.

1. Introduction

In this paper we investigate the behavior of solutions of

$$\Delta w + |w|^{p-1}w = 0, \tag{1.1}$$

where $w = w(x_1, \ldots, x_N)$, $N > 1$ and $p > 1$. Solutions of (1.1) are time-independent solutions of the nonlinear heat equation

$$\frac{\partial w}{\partial t} = \Delta w + |w|^{p-1}w. \tag{1.2}$$

Equation (1.1) has been widely studied as a canonical model for

$$\Delta u + f(u) = 0, \tag{1.3}$$

where $f(u) > 0$ is superlinear [1–6].

Our focus is on radially symmetric solutions of (1.1) which have the form $w = w(r)$, where $r = (x_1^2 + \cdots + x_N^2)^{1/2}$, and satisfy

$$w'' + \frac{N-1}{r} w' + |w|^{p-1} w = 0, \quad r > 0. \tag{1.4}$$

We distinguish two classes of solutions of (1.4). The first is nonsingular solutions which are bounded at $r = 0$ and satisfy $(w(0), w'(0)) = (w_0, 0)$, where w_0 is finite. The second class consists of singular solutions that are unbounded at $r = 0$. Equation (1.4) has the known positive singular solution

$$w_1(r) = \left(\frac{2(N-2)(p-1) - 4}{(p-1)^2} \right)^{1/(p-1)} r^{-2/(p-1)}, \quad N > 2,\ p > \frac{N}{N-2}. \tag{1.5}$$

Previous Results

(i) The positive singular solution $w_1(r)$ has played a central role in analyzing (1.2). For example, when appropriately chosen, similarity solution methods show how $w(x_1, \ldots, x_N, t) \to c w_1(r)$ as $t \to \infty$, where $c > 0$ is a constant [2, 5, 7]. (ii) Chen and Derrick [8] developed comparison methods to describe the time evolution of solutions of

$$\frac{\partial u}{\partial t} = \Delta u + f(u), \tag{1.6}$$

where $f(u)$ is superlinear [1–6]. Their approach is to let positive, time independent solutions act as upper and/or lower bounds for initial values of solutions of (1.6). Their comparison technique allows them to prove either global existence or finite time blowup of solutions. (iii) For the case $p = (N+2)/(N-1)$ Caffarelli et al. [9] describe the asymptotic behavior of nonnegative solutions of (1.1) that have an isolated singularity at the origin. (iv) Galaktionov [10] studied sign changing singular solutions of (1.4) on the restricted interval $0 < r \leq 1$. He set $w(r) = r^{-2/(p-1)} \phi(-\ln(r))$ and derived an ODE for $\phi(s)$, $s = -\ln(r)$. He let $\phi(0) = 0$, varied $\phi'(0)$, and gave a numerical study of sign changing solutions on $0 \leq s < \infty$. (v) Other studies of nonsingular solutions of (1.4) have used Pohozaev identities, together with integral estimates which involve the independent variable [1, 2, 4].

Specific Aims

Our goal is to develop techniques to efficiently classify the behavior of solutions of (1.4) on $(r_{\min}, r_{\max})$, their maximal positive interval of existence. We study the behavior of solutions which are positive on $(r_{\min}, r_{\max})$ and also sign changing solutions. In particular, our specific aims are the following.

Specific Aim I

Do positive singular solutions exist, other than $w_1(r)$, for which $(r_{\min}, r_{\max}) = (0, \infty)$? What is their asymptotic behavior as $r \to 0^+$, and as $r \to \infty$? In Section 2 we prove the existence of a second singular solution, $w_2(r)$, (see bottom right panel of Figure 1), which exists on $(0, \infty)$. Also, we prove the asymptotic behavior of this solutions as $r \to 0^+$ and as $r \to \infty$. This result is new and different from previous analyses. In addition, in the conclusion we suggest a possible application for the role of this new solution in analyzing the time-dependent behavior of the full PDE (1.2).

Specific Aim II

Do sign changing solutions exist for which $(r_{\min}, r_{\max}) = (0, \infty)$? What is their asymptotic behavior as $r \to 0^+$, and as $r \to \infty$? In Section 4 we prove the existence of a large amplitude sign changing solution in the (h, h') phaseplane (see top left panel of Figure 4). This solution forms a large amplitude outward spiral as the independent variable τ decreases. Such global analysis has not previously been achieved.

Our Approach

Standard methods to analyze solutions of (1.4) include Pohozaev integral estimates, or topological shooting. Obtaining global results with such methods is difficult since (1.4) is nonautonomous. Thus, to successfully address the issues in *Specific Aims I-II*, our advance is to develop a two-step approach which significantly simplifies the analysis. The first step is to transform the nonautonomous ODE (1.4) into a simpler, autonomous ODE. Let $w(r)$ solve (1.4), and define [10, 11]

$$h(\tau) = \frac{w(\exp(\tau))}{w_1(\exp(\tau))}, \quad -\infty < \tau < \infty. \tag{1.7}$$

Then $h(\tau)$ solves

$$h'' + \frac{N-2}{p-1}\left(p - \frac{N+2}{N-2}\right)h' + \frac{2(N-2)}{(p-1)^2}\left(p - \frac{N}{N-2}\right)\left(|h|^{p-1}h - h\right) = 0. \tag{1.8}$$

Because (1.8) is autonomous, we can apply phase plane techniques to analyze the behavior of solutions. The second step of our approach is to use the "inverse" formula

$$w(r) = w_1(r)h(\ln(r)), \quad 0 < r < \infty \tag{1.9}$$

to analyze corresponding solutions of the w equation (1.4). For example, in Section 2 we analyze (1.8) in the subcritical range $N/(N-2) < p < (N+2)/(N-2)$ and prove that there is a nonmonotonic heteroclinic orbit (labeled B_1 in Figure 1) leading from $(1, 0)$ to $(0, 0)$ in the (h, h') phase plane. We then use (1.9) to show that, corresponding to this heteroclinic orbit, there is an entire continuum of new positive singular solutions of (1.4). Let $w_2(r)$ denote a member of this continuum (Figure 1, 3rd row). Then $w_2(r)$ intertwines with $w_1(r)$ infinitely

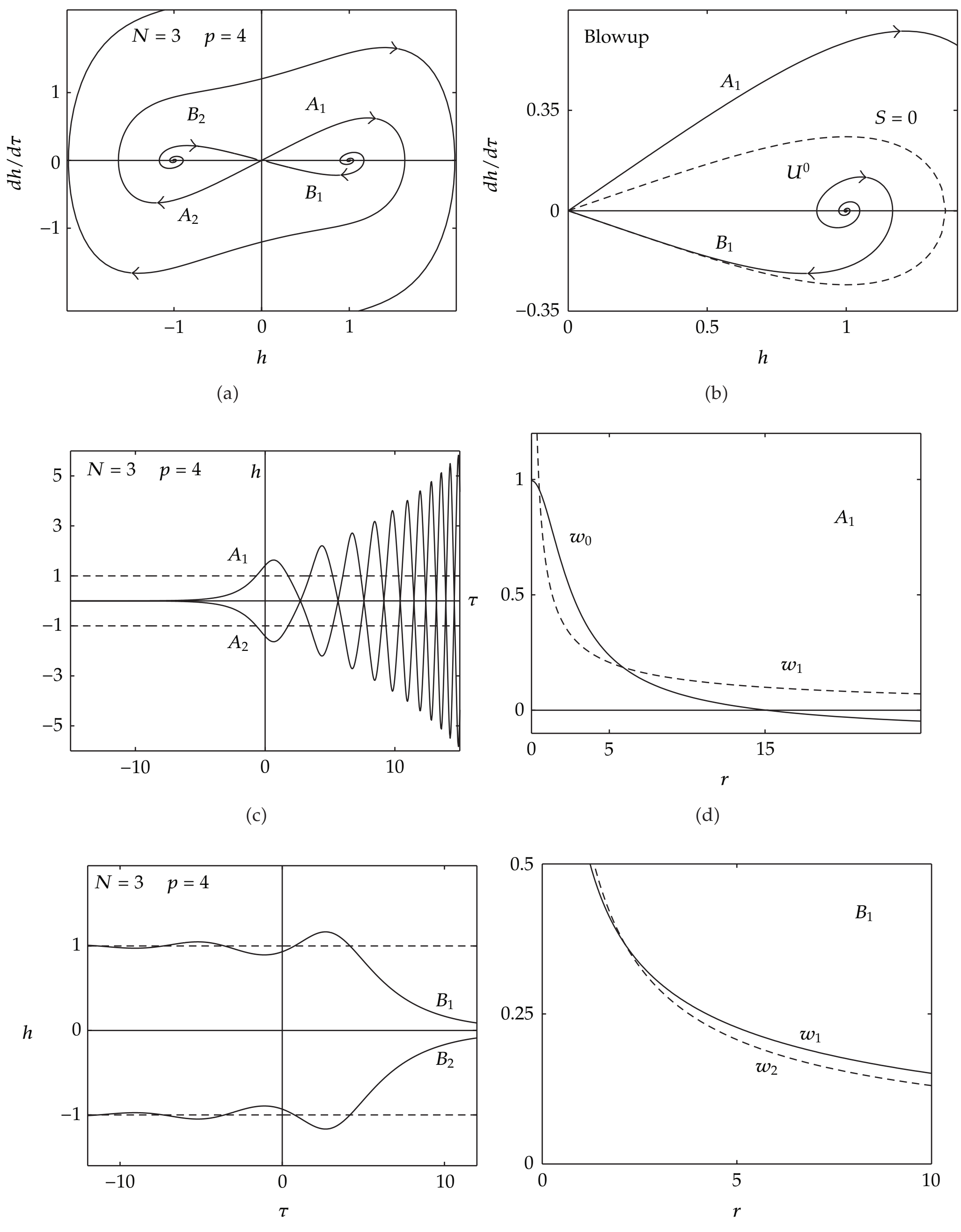

Figure 1: Subcritical example: $N = 3$, $p = 4$. ((a) and (b)) The unstable manifold (A_1 and A_2) and stable manifold (B_1 and B_2) lead from $(0, 0)$ into the (h, h') phase plane. ((c)–(f)) (c) and (e) show the h components of solutions on A_1, A_2, B_1, B_2; (d) and (f) show corresponding solutions of (1.4): w_0 is generated by A_1, $w_2(r)$ is the new positive singular solution generated by B_1, and $w_1(r) = (2/9)^{1/3} r^{-2/3}$.

often as $r \longrightarrow 0^+$. That is, there are infinitely many positive values $\{r_i\}$, with $r_i \longrightarrow 0$ as $i \longrightarrow \infty$, such that

$$w_2(r_i) = w_1(r_i), \quad i \geq 1. \tag{1.10}$$

Furthermore, there is a value $D > 0$ such that

$$\frac{w_2(r)}{w_1(r)} \longrightarrow 1 \quad \text{as } r \longrightarrow 0^+, \qquad \frac{w_2(r)}{w_1(r)} \sim D r^{-((N-2)/(p-1))(p-N/(N-2))} \longrightarrow 0 \quad \text{as } r \longrightarrow \infty. \tag{1.11}$$

Thus, $w_2(r) \sim w_1(r)$ as $r \longrightarrow 0^+$, but $w_2(r) \longrightarrow 0$ faster than $w_1(r)$ as $r \longrightarrow \infty$. To our knowledge, this family of solutions has not previously been reported.

In the conclusion, Section 5, we state an open problem which gives a conjecture for the role that $w_2(r)$ might play in the analysis of the full time-dependent PDE (1.2).

In Sections 3 and 4 we use similar techniques to prove the existence of new families of solutions in the critical case $p = (N+2)/(N-2)$, and the supercritical regime $p > (N+2)/(N-2)$, respectively. In particular, in Section 4 we prove the existence of a continuum of "super singular" sign changing solutions, each of which exists on an interval of the form $(r_{\min}, \infty)$. For these solutions it remains a challenging, and important, open problem (see *Open Problems I and II* in Section 4) to prove whether $r_{\min} = 0$ or $r_{\min} > 0$.

2. The Subcritical Case: $N/(N-2) < p < (N+2)/(N-2)$

In this section we consider the parameter regime $N > 2$ and $N/(N-2) < p < (N+2)/(N-2)$. In this range we first analyze solutions of the h equation (1.8) and then show how these solutions translate into corresponding solutions of the w equation (1.4). The remainder of this section consists of the following.

(I) Lemmas 2.1 and 2.2 state fundamental properties of solutions of (1.8) that satisfy $|h| < 1$ on an interval $[\tau_0, \infty)$ for some $\tau_0 \in \mathbb{R}$. These properties will be applied in the proof of Lemma 2.3 which asserts that there exists a solution h_2 of (1.8) such that $h_2 \longrightarrow 0$ monotonically as $\tau \longrightarrow \infty$. In Lemma 2.4 we show that

$$\frac{h_2'(\tau)}{h_2(\tau)} \longrightarrow -\frac{N-2}{p-1}\left(p - \frac{N}{N-2}\right) \quad \text{as } \tau \longrightarrow \infty. \tag{2.1}$$

This will be used to prove Lemma 2.5 which shows that the asymptotic behavior of h_2 is

$$h_2(\tau) \sim D \exp\left(-\frac{N-2}{p-1}\left(p - \frac{N}{N-2}\right)\tau\right) \quad \text{as } \tau \longrightarrow \infty, \tag{2.2}$$

for some constant $D > 0$.

(II) Solutions along the unstable manifold, A_1, described in part (i) of Theorem 2.9, translate into nonsingular solutions of (1.4). Of particular importance is the heteroclinic orbit solution, B_1, described in part (ii) of Theorem 2.9. Theorem 2.10 asserts that the trajectory B_1 generates an entire continuum of strictly positive singular solutions of (1.4), each of which intertwines infinitely often with $w_1(r)$ as $r \to 0^+$. To our knowledge, this family of solutions is new and has not previously been reported.

(III) *Numerical Experiments.* Figure 1 shows solutions of (1.4) and (1.8) when $(N, p) = (3, 4)$. However, it must be emphasized that it is illegitimate to claim that numerical results are rigorous proofs. Complete analytical proofs are needed to determine properties of solutions of (1.4) and (1.8).

The following two technical lemmas are used to help prove that $h_2 \to 0$ monotonically as $\tau \to \infty$.

Lemma 2.1. *Suppose that h is a nonconstant solution of* (1.8) *such that* $0 \leq h(\tau_0) < 1$ *and* $h'(\tau_0) \geq 0$ *for some* $\tau_0 \in \mathbb{R}$. *Then there exists a constant* $\Delta > 0$ *such that* $h(\tau_0 + \Delta) = 1$. *Likewise, if* $-1 < h(\tau_0) \leq 0$ *and* $h'(\tau_0) \leq 0$, *then there exists a constant* $\Delta > 0$ *such that* $h(\tau_0 + \Delta) = -1$.

Proof. The proof of this lemma relies on the property

$$0 < h(\tau) < 1, \quad h'(\tau) > 0 \quad \text{implies that}\, h''(\tau) > 0 \tag{2.3}$$

which is an immediate consequence of the case assumption $N/(N-2) < p < (N+2)/(N-2)$ and (1.8). Since h is a nonconstant solution, then uniqueness of solutions [12, Chapter 1] implies that either $h(\tau_0) > 0$ or $h'(\tau_0) > 0$. This and (1.8) imply that $h''(\tau_0) > 0$. Hence, $0 < h(\tau) < 1$ and $h'(\tau) > 0$ on an interval $(\tau_0, \tau_0 + \sigma)$ provided that $\sigma > 0$ is sufficiently small. By (2.3), $h''(\tau) > 0$ on $(\tau_0, \tau_0 + \sigma)$. The increasing values of $h'(\tau) > 0$ on $(\tau_0, \tau_0 + \sigma)$ imply the existence of $\Delta > 0$ such that $h(\tau_0 + \Delta) = 1$.

In a similar manner, $-1 < h(\tau_0) \leq 0$ and $h'(\tau_0) \leq 0$ implies the existence of a value $\Delta > 0$ such that $h(\tau_0 + \Delta) = -1$. This completes the proof. □

The following lemma is also used to show that $h_2 \to 0$ monotonically as $\tau \to \infty$.

Lemma 2.2. *Suppose that h is a nonconstant solution of* (1.8) *such that* $|h| < 1$ *on* $[\tau_0, \infty)$. *Then* $h(\tau) \neq 0$ *on* $[\tau_0, \infty)$.

Proof. Suppose that $h(\tau_1) = 0$ for some $\tau_1 \geq \tau_0$. Since h is a nonconstant solution of (1.8), then $h'(\tau_1) \neq 0$. By Lemma 2.1, there exists a constant $\Delta > 0$ such that $h(\tau_1 + \Delta) = 1$ if $h'(\tau_1) > 0$ and $h(\tau_1 + \Delta) = -1$ if $h'(\tau_1) < 0$. This contradicts the assumption that $|h| < 1$ on $[\tau_0, \infty)$ and concludes the proof of the lemma. □

Lemma 2.3. *There exists a solution* h_2 *of* (1.8) *such that*

(a) $\lim_{\tau \to \infty}(h_2(\tau), h_2'(\tau)) = (0, 0)$, *and*

(b) $h_2(\tau) > 0$ *and* $h_2'(\tau) < 0$ *on* $[0, \infty)$.

Proof. A linearization of (1.8) around the constant solution $h \equiv 0$ gives

$$h'' + \frac{N-2}{p-1}\left(p - \frac{N+2}{N-2}\right)h' - \frac{2(N-2)}{(p-1)^2}\left(p - \frac{N}{N-2}\right)h = 0. \tag{2.4}$$

The eigenvalues associated with (2.4) are

$$\lambda_1 = \frac{2}{p-1} > 0, \qquad \lambda_2 = -\frac{N-2}{p-1}\left(p - \frac{N}{N-2}\right) < 0. \tag{2.5}$$

Since $\lambda_2 < 0$, the Stable Manifold Theorem [12, Chapter 13] ensures that there exists a one-dimensional stable manifold containing $(0,0)$ in the (h,h') phase plane. Let h_2 be a solution of (1.8) such that $(h_2(0), h_2'(0))$ is a point on the stable manifold. That is,

$$\lim_{\tau\to\infty} (h_2(\tau), h_2'(\tau)) = (0,0). \tag{2.6}$$

Thus, h_2 satisfies part (a).

We now show that h_2 can be chosen to satisfy (b). Since solutions of (1.8) are translation-invariant, there is no loss in generality in assuming that $|h_2(\tau)| < 1$ on $[0,\infty)$. Combining this with Lemma 2.2 yields $h_2(\tau) \neq 0$ on $[0,\infty)$. According to the fact that h_2 and $-h_2$ are both solutions of (1.8) that satisfy (2.6), we may also assume that $h_2(\tau) > 0$ on $[0,\infty)$. Lemma 2.1 implies that $h_2'(\tau) < 0$ on $[0,\infty)$. This proves (b) and concludes the proof of this lemma. □

Description of the Stable Manifold

Throughout the remainder of this section we let h_2 denote the solution of (1.8) that satisfies

$$0 < h_2(\tau) < 1, \quad h_2'(\tau) < 0 \quad \text{on } [0,\infty). \tag{2.7}$$

Furthermore, we define

$$B_1 = \left\{ (h_2(\tau), h_2'(\tau)) \mid \tau \in (\tau_{\min}, \infty) \right\}, \tag{2.8}$$

where $(\tau_{\min}, \infty)$ denotes the maximal interval of existence of the solution h_2. In Lemma 2.6 we will show that $\tau_{\min} = -\infty$. In addition, we define the negative counterpart of B_1 by

$$B_2 = \left\{ (-h_2(\tau), -h_2'(\tau)) \mid \tau \in (\tau_{\min}, \infty) \right\}. \tag{2.9}$$

The top row of Figure 1 depicts B_1 and B_2 for $N = 3$ and $p = 4$.

Asymptotic Behavior of h_2

To state Lemma 2.4 correctly we need to derive basic properties of the functional $H_2(\tau) = \exp(-\lambda_2\tau)h_2(\tau)$. It follows that H_2 satisfies

$$H_2'' - (N-2)H_2' = -\frac{2(N-2)}{(p-1)^2}\left(p - \frac{N}{N-2}\right)|h_2|^{p-1}H_2. \tag{2.10}$$

Also, $h_2(\tau) > 0$ for all $\tau \geq 0$ implies that

$$H_2(\tau) > 0 \quad \forall \tau \geq 0. \tag{2.11}$$

It follows from (2.10) and (2.11) that

$$H_2'(\tau) > 0 \quad \forall \tau \geq 0. \tag{2.12}$$

By (2.12), all that remains is to show that $H_2(\tau)$ is bounded above. That is the purpose of the following two lemmas.

Lemma 2.4. *The solution H_2 of* (2.10) *satisfies*

$$\lim_{\tau \to \infty} \frac{H_2'(\tau)}{H_2(\tau)} = 0. \tag{2.13}$$

Moreover,

$$\lim_{\tau \to \infty} \frac{h_2'(\tau)}{h_2(\tau)} = \lambda_2. \tag{2.14}$$

Proof. Define $\rho = H_2'/H_2$. Our computations show that

$$\rho' + \rho^2 + (\lambda_2 - \lambda_1)\rho = \lambda_1 \lambda_2 |h_2|^{p-1}. \tag{2.15}$$

Alternatively, (2.15) can be written as

$$\rho' = -(\rho - \mu_-)(\rho - \mu_+), \tag{2.16}$$

where μ_- and μ_+ are defined by

$$\mu_\pm = \frac{\lambda_1 - \lambda_2 \pm \sqrt{(\lambda_1 - \lambda_2)^2 + 4\lambda_1\lambda_2 |h_2|^{p-1}}}{2}. \tag{2.17}$$

Since $h_2(\tau) \to 0$ as $\tau \to \infty$, it follows from (2.17) that

$$\lim_{\tau \to \infty} \mu_- = 0, \qquad \lim_{\tau \to \infty} \mu_+ = \lambda_1 - \lambda_2. \tag{2.18}$$

Thus, it is sufficient to show that $0 < \rho(\tau) \leq \mu_-$ for all $\tau \geq 0$. We accomplish this by process of elimination.

First, $\rho \leq 0$ is impossible due to (2.11) and (2.12). A consequence of (2.16) is that ρ increases to μ_+ whenever $\mu_- < \rho \leq \mu_+$ and ρ decreases to μ_+ whenever $\rho > \mu_+$. In either case, if $\rho(0) > \mu_-$, then

$$\lim_{\tau \to \infty} \rho = \lim_{\tau \to \infty} \mu_+ = \lambda_1 - \lambda_2. \tag{2.19}$$

Since $\rho = H_2'/H_2 = (h_2'/h_2) - \lambda_2$, then (2.19) implies that

$$\lim_{\tau\to\infty}\frac{h_2'}{h_2} = \lambda_1. \tag{2.20}$$

The fact that $\lambda_1 > 0$ contradicts (2.7). Therefore, $\rho > \mu_-$ is impossible. This leaves $0 < \rho \le \mu_-$ as the only possibility. Consequently,

$$0 \le \lim_{\tau\to\infty}\rho = \lim_{\tau\to\infty}\frac{H_2'}{H_2} = \lim_{\tau\to\infty}\mu_- = 0. \tag{2.21}$$

Combining this result with $H_2'/H_2 = (h_2'/h_2) - \lambda_2$ yields $\lim_{\tau\to\infty}(h_2'(\tau)/h_2(\tau)) = \lambda_2$. This completes the proof of the lemma. □

We now prove that $\lim_{\tau\to\infty}h_2(\tau)\exp(-\lambda_2\tau) = D$ for some constant $D > 0$.

Lemma 2.5. *The solution H_2 of* (2.10) *satisfies* $\lim_{\tau\to\infty}H_2(\tau) = D$ *for some finite constant $D > 0$.*

Proof. Because of (2.12) it is sufficient to show that $H_2(\tau)$ is bounded above on $[0,\infty)$. For a contradiction, assume that

$$\lim_{\tau\to\infty}H_2(\tau) = \infty. \tag{2.22}$$

By (2.13) there is a value $\tau_0 > 0$ such that $H_2'/H_2 < -\lambda_2(p-1)/2p$ for all $\tau \ge \tau_0$. Hence, $H_2' + (\lambda_2(p-1)/2p)H_2 < 0$ for all $\tau \ge \tau_0$ which yields

$$H_2(\tau) \le K_2\exp\left(-\frac{\lambda_2(p-1)}{2p}\tau\right) \quad \forall\tau > \tau_0, \tag{2.23}$$

where $K_2 = H_2(\tau_0)\exp(\lambda_2(p-1)\tau_0/2p)$.

Integrating (2.10) over $[\tau_0,\tau]$ gives

$$H_2' - (N-2)H_2 = K_1 - \frac{2(N-2)}{(p-1)^2}\left(p - \frac{N}{N-2}\right)\int_{\tau_0}^{\tau}|h_2|^{p-1}H_2\,d\eta, \tag{2.24}$$

where $K_1 = H_2'(\tau_0) - (N-2)H_2(\tau_0)$. Next, we use (2.23) to show that $\int_{\tau_0}^{\infty}|h_2|^{p-1}H_2\,d\eta$ exists and is finite. Subsequently, we divide both sides of (2.24) by H_2 and let $\tau \to \infty$.

Combining (2.23) with the fact that $h_2(\tau) = \exp(\lambda_2\tau)H_2(\tau) > 0$ for all $\tau \ge 0$ yields

$$\begin{aligned}|h_2(\tau)|^{p-1}H_2(\tau) &= \exp(\lambda_2(p-1)\tau)H_2^p(\tau)\\ &\le \exp(\lambda_2(p-1)\tau)K_2^p\exp\left(-\frac{\lambda_2(p-1)}{2}\tau\right)\\ &= K_2^p\exp\left(\frac{\lambda_2(p-1)}{2}\tau\right).\end{aligned} \tag{2.25}$$

This implies that

$$\int_{\tau_0}^{\infty} |h_2|^{p-1} H_2 d\eta < \infty. \tag{2.26}$$

Dividing both sides of (2.24) by H_2 and letting $\tau \to \infty$ we obtain $-(N-2) = 0$ as a consequence of (2.13), (2.22), and (2.26). This is an obvious contradiction. Thus, there exists a finite constant $D > 0$ such that

$$\lim_{\tau\to\infty} h_2(\tau)\exp(-\lambda_2\tau) = \lim_{\tau\to\infty} H_2(\tau) = D. \tag{2.27}$$

This concludes the proof of the lemma. □

Conclusion. It follows from (2.27) that the asymptotic behavior of h_2 described in (2.2) is now proved.

The next lemma shows that h_2 exists on $\mathbb{R}$ and that $h_2(\tau) \to 1$ as $\tau \to -\infty$.

Lemma 2.6. *The solution h_2 of* (1.8) *is defined on $\mathbb{R}$ and satisfies* $\lim_{\tau\to-\infty}(h_2(\tau), h_2'(\tau)) = (1,0)$. *Furthermore, there is a decreasing sequence $\{\tau_k\}$, with $\lim_{k\to\infty}\tau_k = -\infty$, such that*

$$h(\tau_k) = 1 \quad \forall k \geq 1, \qquad h'(\tau_k) < 0 \quad \text{if } k \text{ is odd}, \qquad h'(\tau_k) > 0 \quad \text{if } k \text{ is even}. \tag{2.28}$$

Proof. Let $(\tau_{\min}, \infty)$ denote the maximal interval of existence of h_2. We claim that $\tau_{\min} = -\infty$. To prove this, we make use of the functional

$$S = \frac{1}{2}(h')^2 + \frac{2(N-2)}{(p-1)^2}\left(p - \frac{N}{N-2}\right)\left(\frac{|h|^{p+1}}{p+1} - \frac{h^2}{2}\right), \tag{2.29}$$

and the region U defined by

$$U = \{(h, h') \mid h \geq 0, S \leq 0\}. \tag{2.30}$$

Let U^0 denote the interior of U (Figure 1, upper right). Note that the constant solution $(h(\tau), h'(\tau)) \equiv (1,0) \in U^0$. Since $p < (N+2)/(N-2)$, a differentiation of (2.29) gives

$$S' = -\frac{N-2}{p-1}\left(p - \frac{N+2}{N-2}\right)(h')^2 \geq 0 \quad \forall \tau \in (\tau_{\min}, \infty). \tag{2.31}$$

We conclude from (2.2), (2.6), (2.14), (2.29), and (2.31) that, when $h = h_2$,

$$S(\infty) = 0, \qquad S'(\tau) \geq 0, \quad S(\tau) < 0 \quad \forall \tau \in (\tau_{\min}, \infty). \tag{2.32}$$

Therefore, $(h_2(\tau), h_2'(\tau))$ is uniformly bounded on $(\tau_{\min}, \infty)$. From this and standard theory it follows that $\tau_{\min} = -\infty$. Thus, for the solution h_2, we conclude that $S(\tau) < 0$ for all $\tau \in \mathbb{R}$ and that

$$(h_2(\tau), h_2'(\tau)) \in U^0 \quad \forall \tau \in (-\infty, \infty). \tag{2.33}$$

From (2.31) we also conclude that (1.8) has no periodic solutions. In addition, the constant solution $(h, h') \equiv (1, 0)$ is the only constant solution in U^0. Thus, it follows from (2.33) and standard phase plane arguments that

$$h_2(\tau) > 0 \quad \forall \tau \in \mathbb{R}, \qquad \lim_{\tau \to -\infty} (h_2(\tau), h_2'(\tau)) = (1, 0). \tag{2.34}$$

Finally, we need to determine precisely how solutions approach $(h, h') = (1, 0)$ as $\tau \to -\infty$. For this a linearization of (1.8) around the constant solution $h \equiv 1$ gives

$$h'' + \frac{N-2}{p-1}\left(p - \frac{N+2}{N-2}\right)h' + \frac{2(N-2)}{p-1}\left(p - \frac{N}{N-2}\right)h = 0. \tag{2.35}$$

The eigenvalues associated with (2.35) are complex, with positive real parts. Thus, solutions with initial values on the curve B_1 must spiral into the constant solution $(h, h') = (1, 0)$ as $\tau \to -\infty$. Property (2.28) follows as a consequence. This completes the proof of the lemma. □

In Lemma 2.3, the eigenvalue $\lambda_2 < 0$ defined in (2.5) led to the existence of the solution h_2 of (1.8). The decay rate of h_2 as $\tau \to \infty$ is described in Lemmas 2.3–2.5. The methods used to prove Lemmas 2.3–2.5 can be applied to the eigenvalue $\lambda_1 > 0$ defined in (2.5) to result in the following lemma.

Lemma 2.7. *There exists a solution h_1 of* (1.8) *such that*

(a) $\lim_{\tau \to -\infty} (h_1(\tau), h_1'(\tau)) = (0, 0)$,

(b) $h_1(\tau) > 0$ *and* $h_1'(\tau) > 0$ *on* $(-\infty, 0]$,

(c) $\lim_{\tau \to -\infty} (h_1'(\tau)/h_1(\tau)) = \lambda_1$, *and*

(d) $\lim_{\tau \to -\infty} h_1(\tau) \exp(-\lambda_1 \tau) = c$ *for some constant* $c > 0$.

Description of the Unstable Manifold

As shown in the proof of Lemma 2.3, the solution h_1 is chosen so that $(h_1(0), h_1'(0))$ is a point on the unstable manifold of the constant solution $(h, h') \equiv (0, 0)$. As depicted in Figure 1 we let A_1 denote the component of the unstable manifold in the $h > 0$ and $h' > 0$ quadrant. Also, we denote the component of the unstable manifold in the $h < 0$ and $h' < 0$ component by A_2. Precisely,

$$A_1 = \{(h_1(\tau), h_1'(\tau)) \mid \tau \in (-\infty, \tau_{\max})\}, \tag{2.36}$$

where $(-\infty, \tau_{\max})$ denotes the maximal interval of existence of h_1. Noting that $-h_1$ is also a solution of (1.8) such that $\lim_{\tau\to-\infty}(-h_1(\tau), -h_1'(\tau)) = (0,0)$ we can define

$$A_2 = \{(-h_1(\tau), -h_1'(\tau)) \mid \tau \in (-\infty, \tau_{\max})\}. \tag{2.37}$$

In the next lemma, we continue our analysis of the solution h_1 as $\tau \to \tau_{\max}^-$.

Lemma 2.8. *There is an increasing, positive sequence* $\{\tau_k\}$ *such that*

$$h_1'(\tau_k) = 0 \quad \forall k \geq 1, \qquad h_1(\tau_k) > 1 \quad \text{if } k \text{ is odd}, \qquad h_1(\tau_k) < -1 \quad \text{if } k \text{ is even}. \tag{2.38}$$

Proof. The corresponding solution $w_0(r)$ of (1.4) satisfies

$$w_0(r) = h_1(\ln(r))\left(\frac{2(N-2)(p-1)-4}{(p-1)^2}\right)^{1/(p-1)} r^{-2/(p-1)}, \quad r > 0, \tag{2.39}$$

and it follows from parts (b) and (d) of Lemma 2.7 and (2.39) that $w_0(r) > 0$ for all $r > 0$, and

$$w_0(r) \longrightarrow w_0 = c\left(\frac{2(N-2)(p-1)-4}{(p-1)^2}\right)^{1/(p-1)} > 0 \quad \text{as } r \longrightarrow 0. \tag{2.40}$$

It follows from standard theory that solutions that are bounded at $r = 0$ must satisfy $w'(0) = 0$. Thus, for solutions of (1.8) such that $(h(0), h'(0)) \in A_1$, and satisfying parts (b) and (d) of Lemma 2.7, the corresponding solution $w_0(r)$ of (1.4) is nonsingular and satisfies $w_0(0) = w_0 > 0$, $w_0'(0) = 0$. Haraux and Weissler [2] showed that $w_0(r)$ has at least one positive zero. Chen et al. [1] proved that $w_0(r)$ has infinitely many positive zeros. These results, and the fact that $h_1 = w/w_1$, imply that solutions of (1.8) satisfying $(h_1(0), h_1'(0)) \in A_1$ have infinitely many positive zeros. Thus, there is an increasing, positive sequence $\{\tau_k\}$, where h_1 attains a positive relative maximum when $k \geq 1$ is odd, and a negative relative minimum when k is even. It follows from (1.8) that

$$h_1(\tau_k) > 1 \quad \text{when } k \text{ is odd}, \qquad h_1(\tau_k) < -1 \quad \text{when } k \text{ is even}. \tag{2.41}$$

This concludes the proof of the lemma. □

The following theorem summarizes our results obtained thus far. In particular, part (i) of the following theorem summarizes the results of Lemmas 2.7 and 2.8 regarding the solution h_1. Part (ii) summarizes the results of Lemmas 2.3–2.6 regarding the solution h_2.

Theorem 2.9. *Let* $N > 2$ *and* $N/(N-2) < p < (N+2)/(N-2)$.

(i) *There is a one-dimensional unstable manifold of solutions of* (1.8) *leading from* $(0,0)$ *in the* (h, h') *phase plane. One component,* A_1*, points into the positive quadrant, and its negative* *counterpart, A_2, points into the negative quadrant. If $(h(0), h'(0)) \in A_1$, and $h(0) > 0$ is sufficiently small, then there is a value $c > 0$ such that*

$$h(\tau) > 0 \quad \forall \tau \in (-\infty, 0], \qquad h(\tau) \sim c \exp\left(\frac{2\tau}{p-1}\right) \quad \text{as}\, \tau \longrightarrow -\infty. \tag{2.42}$$

Furthermore, there is an increasing, positive sequence $\{\tau_k\}$ such that

$$h'(\tau_k) = 0 \quad \forall k \geq 1, \qquad h(\tau_k) > 1 \quad \text{if } k \text{ is odd}, \qquad h(\tau_k) < -1 \quad \text{if } k \text{ is even}. \tag{2.43}$$

(ii) *There is a one-dimensional stable manifold of solutions leading to $(0, 0)$ in the (h, h') phase plane. One component, B_1, points into the $h > 0$, $h' < 0$ quadrant of the phase plane, and its negative counterpart, B_2, points into the $h < 0$, $h' > 0$ quadrant. If $(h(0), h'(0)) \in B_1$, then*

$$h(\tau) > 0 \quad \forall \tau \in \mathbb{R}, \quad \lim_{\tau \to -\infty} (h(\tau), h'(\tau)) = (1, 0), \quad \lim_{\tau \to \infty} (h(\tau), h'(\tau)) = (0, 0). \tag{2.44}$$

Also, there is a decreasing sequence $\{\tau_k\}$, with $\lim_{k \to \infty} \tau_k = -\infty$, such that

$$h(\tau_k) = 1 \quad \forall k \geq 1, \qquad h'(\tau_k) < 0 \quad \text{if } k \text{ is odd}, \qquad h'(\tau_k) > 0 \quad \text{if } k \text{ is even}. \tag{2.45}$$

Finally, there is a constant $D > 0$ such that

$$h(\tau) \sim D \exp\left(-\frac{N-2}{p-1}\left(p - \frac{N}{N-2}\right)\tau\right) \quad \text{as}\, \tau \longrightarrow \infty. \tag{2.46}$$

Solutions of the w Equation

Below, in Theorem 2.10, we show how to combine part (ii) of Theorem 2.9 with the formula

$$w(r) = h(\ln(r)) w_1(r), \tag{2.47}$$

to prove the existence and asymptotic behavior of a new family of singular solutions of the w equation (1.4). Our approach is to let $(h(0), h'(0))$ be an arbitrarily chosen element of the continuous curve B_1. Since $r = e^\tau$, the initial conditions for the corresponding solution of (1.4) are given at $r = e^0 = 1$ and satisfy

$$w(1) = h(0) w_1(1), \qquad w'(1) = h'(0) w(1) + h(0) w'(1). \tag{2.48}$$

Since $(h(0), h'(0)) \in B_1$, and B_1 is a continuous curve, then (2.48) generates an entire continuum of solutions of the w equation. We show how these solutions intertwine with $w_1(r)$ infinitely often as $r \to 0^+$. In addition, we show how to prove the limiting behavior of each solution at both ends of $(r_{\min}, r_{\max})$, its maximal positive interval of existence.

Theorem 2.10 (a continuum of new singular solutions of (1.4)). *Let $N > 2$ and $N/(N-2) < p < (N+2)/(N-2)$. Let $w_1(r)$ denote the positive singular solution of (1.4) defined in (1.5), and let $h_2(\tau)$ be a solution of (1.8) which satisfies $(h_2(0), h_2'(0)) \in B_1$ in part (ii) of Theorem 2.9. The corresponding solution $w_2(r) = h_2(\ln(r))w_1(r)$ of (1.4) has initial values*

$$w_2(1) = h_2(0)w_1(1), \qquad w_2'(1) = h_2'(0)w_1(1) + h_2(0)w_1'(1). \tag{2.49}$$

Furthermore, $(0,\infty)$ is the maximal interval of existence of $w_2(r)$, and there is a decreasing positive sequence, $\{r_k\}$, with $\lim_{k\to\infty} r_k = 0$, such that

$$0 < w_2(r) < w_1(r) \quad \forall r > r_1, \qquad w_2(r_k) = w_1(r_k) \quad \forall k \geq 1, \tag{2.50}$$

$$\lim_{r\to 0^+} \frac{w_2(r)}{w_1(r)} = 1, \quad \frac{w_2(r)}{w_1(r)} \sim Dr^{-((N-2)/(p-1))(p-N/(N-2))} \quad \text{as } r \longrightarrow \infty. \tag{2.51}$$

Numerical Example

In Figure 1 we let $(N,p) = (3,4)$ so that $\lambda_2 = -1/3$. The stable manifold B_1 (third row, left panel) is generated by solution $h_2(\tau)$ of (1.8) with $(h_2(0), h_2'(0)) = (.93, .07)$. The right panel shows the corresponding solution $w_2(r)$ of (1.4). For this example the asymptotic properties (2.51) become

$$w_2(r) \sim w_1(r) = \left(\frac{2}{9}\right)^{1/3} r^{-2/3} \quad \text{as } r \longrightarrow 0^+, \qquad w_2(r) \sim \frac{.05}{r} \quad \text{as } r \longrightarrow \infty. \tag{2.52}$$

Proof of Theorem 2.10. Let h_2 denote a solution of (1.8) which satisfies part (ii) of Theorem 2.9. The solution of (1.4) corresponding to h_2 is

$$w_2(r) = h_2(\ln(r))w_1(r). \tag{2.53}$$

It follows from (2.45) in Theorem 2.9 that the sequence $\{r_k\}$ defined by

$$r_k = \exp(\tau_k) \quad \forall k \geq 1 \tag{2.54}$$

is positive and decreasing in k, with $\lim_{k\to\infty} r_k = 0$, and

$$w_2(r_k) = w_1(r_k) \quad \forall k \geq 1. \tag{2.55}$$

Next, it follows from (2.53), and the definition $\tau = \ln(r)$, that

$$\frac{w_2(r)}{w_1(r)} = h_2(\ln(r)) = h_2(\tau). \tag{2.56}$$

Since $r = e^{\tau} \to 0$ as $\tau \to -\infty$, and since $h_2(\tau) \to 1$ as $\tau \to -\infty$, it follows from (2.56) that

$$\lim_{r\to 0^+} \frac{w_2(r)}{w_1(r)} = 1. \tag{2.57}$$

This proves the first part of (2.51). It remains to prove the asymptotic behavior of the solutions as $r \to \infty$. For this we combine property (2.46) in Theorem 2.9 with (2.56) and substitute $\tau = \ln(r)$ to obtain

$$\frac{w_2(r)}{w_1(r)} \sim D r^{-((N-2)/(p-1))(p-N/(N-2))} \quad \text{as } r \longrightarrow \infty. \tag{2.58}$$

This completes the proof of Theorem 2.10. □

3. The Critical Case: $p = (N+2)/(N-2)$

In this section we investigate the behavior of solutions of (1.4) and (1.8) when $N > 2$ and $p = (N+2)/(N-2)$. In this case (1.4) and (1.8) become

$$w'' + \frac{N-1}{r} w' + |w|^{4/(N-2)} w = 0, \quad r > 0, \tag{3.1}$$

$$\frac{d^2 h}{d\tau^2} + \left(\frac{N-2}{2}\right)^2 \left(|h|^{4/(N-2)} h - h\right) = 0, \tag{3.2}$$

and (1.5) reduces to

$$w_1(r) = \left(\frac{N-2}{2}\right)^{(N-2)/2} r^{-(N-2)/2}, \quad r > 0. \tag{3.3}$$

The remainder of this section consists of the following.

(I) Theorem 3.1 gives a complete classification of solutions of (3.2).

(II) In Theorem 3.2 we show how to combine the results of Theorem 3.1 with the formula $w(r) = h(\ln(r)) w_1(r)$ to obtain a continuum of new positive singular solutions of (3.1), and also a continuum of new sign changing singular solutions.

(III) Figures 2 and 3 illustrate our results when $(N, p) = (3, 5)$.

Theorem 3.1. *Let $N > 2$ and $p = (N+2)/(N-2)$. Each solution of (3.2) satisfies*

$$\frac{1}{2}(h')^2 + \left(\frac{N-2}{2}\right)^2 \left(\frac{N-2}{2N} |h|^{2N/(N-2)} - \frac{h^2}{2}\right) = E, \tag{3.4}$$

where E is a constant. Define $E_* = -(N-2)^2/4N$.

(i) *If* $E < E_*$*, then there are no real solutions of* (3.2) *which satisfy* (3.4).

(ii) *If* $E = E_*$*, then solutions of* (3.2) *are constant, and either* $(h(\tau), h'(\tau)) = (1, 0)$ *or* $(h(\tau), h'(\tau)) = (-1, 0)$ *for all* $\tau \in \mathbb{R}$.

(iii) *If* $E_* < E < 0$*, then solutions of* (3.2) *are nonconstant, periodic, they have one sign, and the interior of their trajectories in the* (h, h') *phase plane contains one of the constant solutions* $(\pm 1, 0)$.

(iv) *If* $E = 0$*, then there is a one parameter family of solutions*

$$h_\kappa(\tau) = \kappa \left(\frac{2N \exp(\tau)}{N(N-2) + |\kappa|^{4/(N-2)} \exp(2\tau)} \right)^{(N-2)/2}, \quad \kappa \neq 0, \ \tau \in \mathbb{R}. \tag{3.5}$$

Depending on the sign of κ*, these solutions are either strictly positive or negative. Their trajectories form homoclinic orbits in the* (h, h') *phase plane, with one of the constant solutions* $(\pm 1, 0)$ *in their interior, and*

$$\lim_{\tau \to \pm\infty} (h(\tau), h'(\tau)) = (0, 0). \tag{3.6}$$

(v) *If* $E > 0$ *solutions of* (3.2) *are nonconstant, periodic, they change sign, and the interior of their trajectories in the* (h, h') *phase plane contains all three constant solutions* $(\pm 1, 0)$ *and* $(0, 0)$.

Proof. Let $Q = (1/2)(h')^2 + ((N-2)/2)^2(((N-2)/2N)|h|^{2N/(N-2)} - h^2/2)$. Then $Q' = 0$ for all τ, hence

$$\frac{1}{2}(h')^2 + \left(\frac{N-2}{2}\right)^2 \left(\frac{N-2}{2N}|h|^{2N/(N-2)} - \frac{h^2}{2}\right) = E \quad \forall \tau, \tag{3.7}$$

where E is a constant, and (3.4) is proved. Properties (i)–(v) follow from (3.7). □

Numerical Experiments

Figure 2 illustrates homoclinic orbits solutions, and also periodic solutions, of (3.2) in the (h, h') plane when $(N, p) = (3, 5)$. Graphs of the h components of these solutions are shown in the left column of Figure 3. The corresponding solutions of the w equation (3.1) are shown in the right column of Figure 3. Proofs of their existence are given below in Theorem 3.2.

Solutions of the w *Equation*

We now show how to combine the results of Theorem 3.1 with the formula

$$w(r) = h(\ln(r)) w_1(r), \tag{3.8}$$

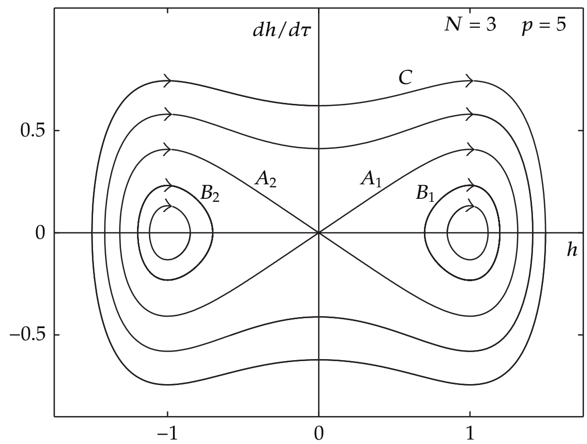

Figure 2: Critical case: $N = 3$, $p = 5$. Periodic and homoclinic orbits of (3.2) proved in Theorem 3.1.

to prove the existence and asymptotic behavior of solutions of (3.1). First, part (ii) of Theorem 3.1 shows that when $E = E_* = -(N-2)^2/4N$, then $h \equiv \pm 1$. This and (3.8) imply that the corresponding solutions of (3.1) are $w(r) = \pm w_1(r)$. Below, in Theorem 3.2, we show how parts (iii)–(v) of Theorem 3.1 generate continuous families of *strictly positive* solutions of (3.1), and also a family of sign changing singular solutions.

Theorem 3.2. *Let* $N > 2$ *and* $p = (N+2)/(N-2)$.

(a) A Continuum of Positive Nonsingular Solutions. For each $\kappa > 0$ (3.1) *has the nonsingular solution*

$$w_0(r) = \kappa \left(\frac{N(N-2)}{N(N-2) + |\kappa|^{4/(N-2)} r^2} \right)^{(N-2)/2}, \quad r > 0. \tag{3.9}$$

(b) A Continuum of Positive "Interlacing" Singular Solutions. Let $h_2(\tau)$ *be a member of the continuum of positive periodic solutions of* (3.2) *which satisfy part* (iii) *of Theorem 3.1. The corresponding solution* $w_2(r) = h_2(\ln(r))w_1(r)$ *of* (3.1) *has initial values*

$$w_2(1) = h_2(0)w_1(1), \qquad w_2'(1) = h_2'(0)w_1(1) + h_2(0)w_1'(1), \tag{3.10}$$

and its interval of existence is $(0, \infty)$. *Furthermore, the solution* $w_2(r)$ *interlaces with* $w_1(r)$*; that is, there is a positive increasing sequence,* $\{r_k\}$, *with* $\lim_{k\to-\infty} r_k = 0$ *and* $\lim_{k\to\infty} r_k = \infty$ *such that*

$$w_2(r_k) = w_1(r_k), \quad -\infty < k < \infty, \tag{3.11}$$

$$\lim_{r\to 0^+} w_2(r) = \infty, \qquad \lim_{r\to\infty} w_2(r) = 0. \tag{3.12}$$

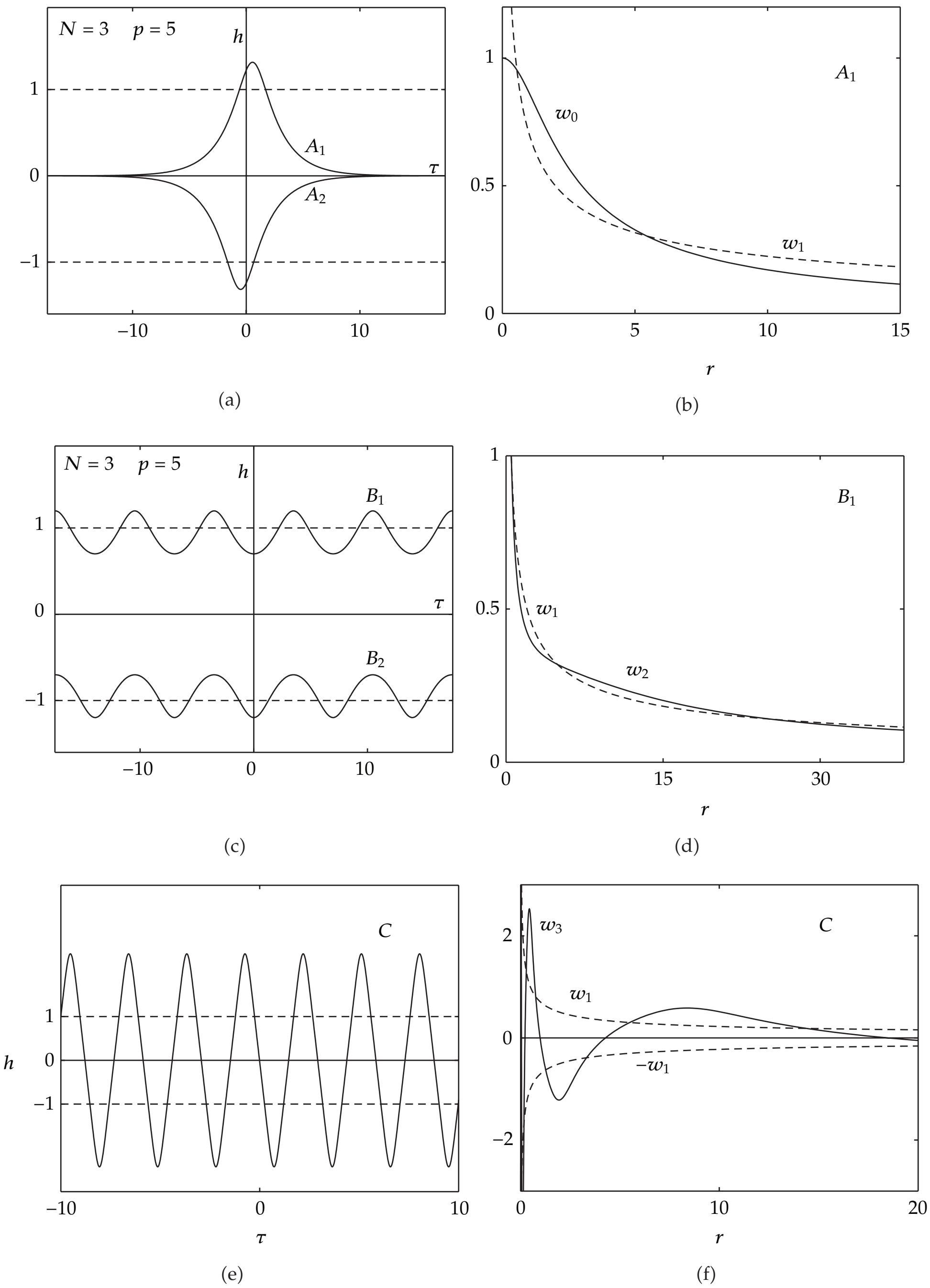

Figure 3: Critical case: $N = 3$, $p = 5$. ((a), (c), and (e)) Homoclinic and periodic orbit solutions of (3.2) proved in Theorem 3.1. ((b), (d), and (f)) The corresponding solutions of (3.1) proved in Theorem 3.2: $w_1(r) = 1/\sqrt{2r}$ is the known singular solution, $w_2(r)$ is a new positive singular solution, and $w_3(r)$ is a new sign changing singular solution.

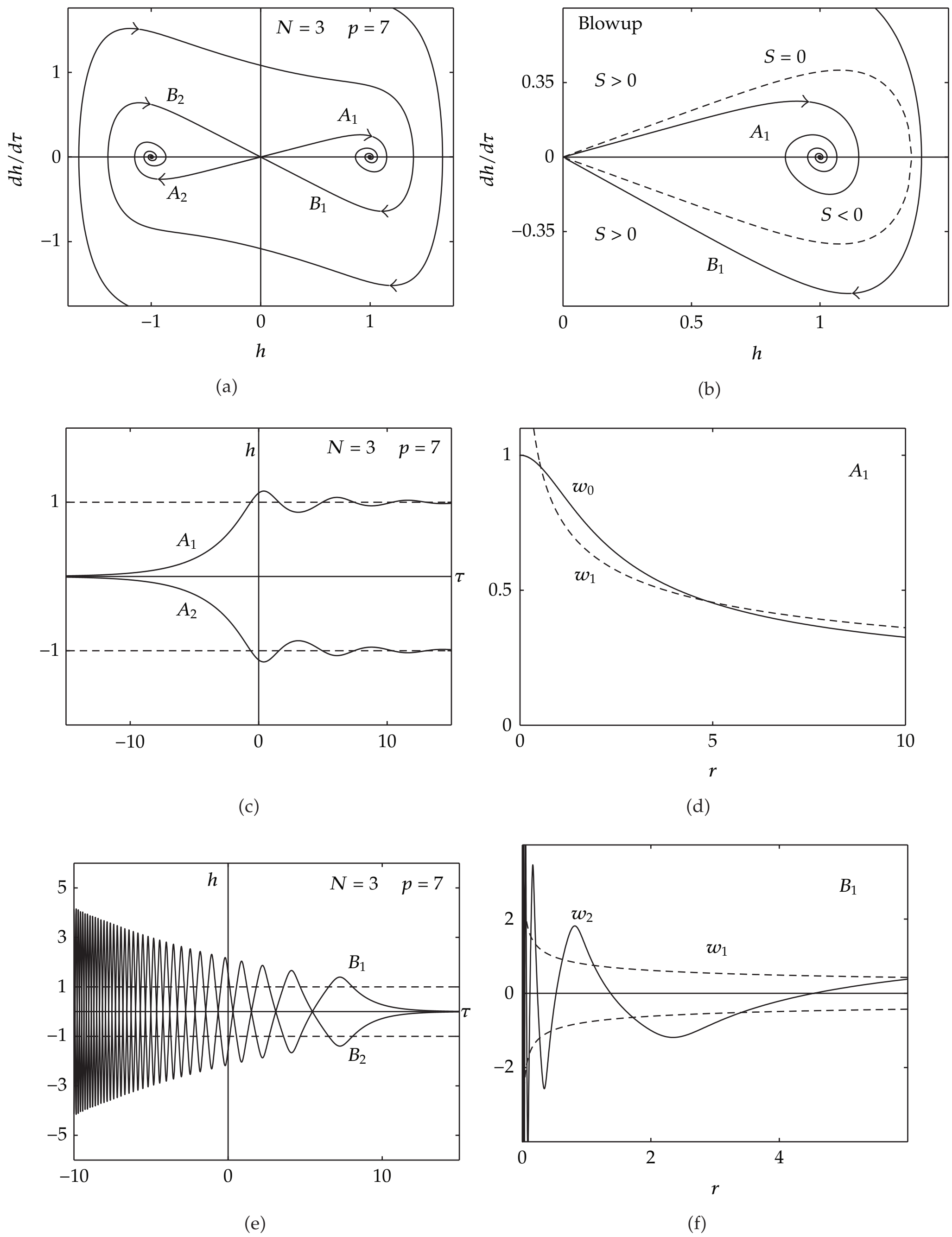

Figure 4: Supercritical example: $N = 3$, $p = 7$. ((a) and (b)) Trajectories of the unstable manifold (A_1 and A_2) and stable manifold (B_1 and B_2) leading from $(0,0)$ into the (h, h') plane. ((c)–(f)) The h components of solutions along A_1, A_2, B_1, B_2 (left) and corresponding w components along A_1 and B_1 (right): $w_0(r)$ is bounded at $r = 0$, $w_1(r) = (2/9)^{1/6} r^{-1/3}$ is the known singular solution, and $w_2(r)$ is the new sign changing singular solution corresponding to B_1.

(c) A Continuum of Sign Changing Singular Solutions. Let $h_3(\tau)$ be a member of the family of positive, sign changing periodic solutions of (3.2) *which satisfy part* (v) *of Theorem 3.1. The corresponding solution $w_3(r) = h_3(\ln(r))w_1(r)$ of* (3.1) *has initial values*

$$w_3(1) = h_3(0)w_1(1), \qquad w_3'(1) = h_3'(0)w_1(1) + h_3(0)w_1'(1), \tag{3.13}$$

and its interval of existence is $(0,\infty)$. Furthermore, the solution $w_3(r)$ changes sign infinitely often as follows: there is an positive increasing sequence, $\{r_i\}$, with $\lim_{i\to-\infty} r_i = 0$ and $\lim_{i\to\infty} r_i = \infty$ such that

$$w_3(r_i) = w_1(r_i) \quad \text{if } i \text{ is even}, \qquad w_3(r_i) = -w_1(r_i) \quad \text{if } i \text{ is odd}, \tag{3.14}$$

$$\limsup_{r\to 0^+} w_3(r) = \infty, \qquad \liminf_{r\to 0^+} w_3(r) = -\infty, \qquad \lim_{r\to\infty} w_3(r) = 0. \tag{3.15}$$

Remarks. (i) The solutions given in (3.9) were first derived by Joseph and Lundgren [3]. (ii) To our knowledge, the singular solutions described in parts (b) and (c) have not previously been reported.

Proof of Theorem 3.2.

Part (a). For each $\kappa > 0$, let $h_\kappa(\tau)$ denote the solution given in (3.5) in Theorem 3.1. Setting $\tau = \ln(r)$ in (3.5) gives

$$h_\kappa(\ln(r)) = \kappa\left(\frac{2Nr}{N(N-2) + |\kappa|^{4/(N-2)} r^2}\right)^{(N-2)/2}, \quad r > 0. \tag{3.16}$$

Next, substitute (3.3) and (3.16) into (3.8) and obtain

$$w_0(r) = \kappa\left(\frac{N(N-2)}{N(N-2) + |\kappa|^{4/(N-2)} r^2}\right)^{(N-2)/2}, \quad r > 0. \tag{3.17}$$

Part (b). Let $h_2(\tau)$ be a member of the continuum of positive periodic solutions of (3.2) which satisfy part (iii) of Theorem 3.1. The trajectory of (h_2, h_2') lies in the positive quadrant of the (h, h') plane and surrounds the constant solution $(h, h') = (1, 0)$. Thus, there are values $L_2 > L_1 > 0$ and a positive increasing sequence $\{\tau_k\}$, such that

$$\lim_{k\to-\infty} \tau_k = -\infty, \qquad \lim_{k\to\infty} \tau_k = \infty, \tag{3.18}$$

$$0 < L_1 < h_2(\tau) < L_2 \quad \forall \tau \in \mathbb{R}, \qquad h_2(\tau_k) = 1 \quad \forall k. \tag{3.19}$$

The solution of the w equation (3.1) corresponding to h_2 is

$$w_2(r) = h_2(\ln(r))w_1(r) \quad r > 0. \tag{3.20}$$

Define $r_k = e^{\tau_k}$ for all k. It follows from (3.18)-(3.19)-(3.20) that $\lim_{k\to-\infty} r_k = 0$ and $\lim_{k\to\infty} r_k = \infty$, and

$$w_2(r_k) = h_2(\tau_k)w_1(r_k) = w_1(r_k) \quad \forall k. \tag{3.21}$$

This proves property (3.11). It remains to prove property (3.12). For this we combine (3.19) with (3.20), and the fact that $\tau = \ln(r)$, to conclude that

$$0 < L_1 w_1(r) < w_2(r) < L_2 w_1(r) \quad \forall r > 0. \tag{3.22}$$

It follows from (3.3) and (3.22) that

$$\lim_{r\to 0^+} w_2(r) = \infty, \qquad \lim_{r\to\infty} w_2(r) = 0. \tag{3.23}$$

This completes the proof of property (3.12).

Part (c). Let $h_3(\tau)$ be a member of the continuum of positive periodic solutions of (3.2) which satisfy part (v) of Theorem 3.1. The trajectory of (h_3, h_3') surrounds the constant solutions $(h, h') = (0, 0)$ and $(h, h') = (\pm 1, 0)$ in the (h, h') plane. Thus, there exists a value $L > 0$, and a positive increasing sequence $\{\tau_i\}$, such that

$$-L < h_3(\tau) < L \quad \forall \tau \in \mathbb{R}, \tag{3.24}$$

$$\lim_{i\to-\infty} \tau_i = -\infty, \qquad \lim_{i\to\infty} \tau_i = \infty, \tag{3.25}$$

$$h_3(\tau_i) = -1 \quad \text{if } i \text{ is odd}, \qquad h_3(\tau_i) = 1 \quad \text{if } i \text{ is even}. \tag{3.26}$$

The solution of the w equation (3.1) corresponding to h_3 is

$$w_3(r) = h_3(\ln(r))w_1(r) \quad r > 0. \tag{3.27}$$

Define $r_i = e^{\tau_i}$ for all i. It follows from (3.24)-(3.25)-(3.27) that $\lim_{i\to-\infty} r_i = 0$, $\lim_{i\to\infty} r_i = \infty$, and

$$w_3(r_i) = -w_1(r_i) \quad \text{if } i \text{ is odd}, \qquad w_3(r_i) = w_1(r_i) \quad \text{if } i \text{ is even}. \tag{3.28}$$

This proves property (3.14). It remains to prove property (3.15). For this we combine (3.24) with (3.27), and the fact that $\tau = \ln(r)$, to conclude that

$$-Lw_1(r) < w_3(r) < Lw_1(r) \quad \forall r > 0. \tag{3.29}$$

It follows from (3.3), (3.28), and (3.29) that

$$\limsup_{r\to 0^+} w_3(r) = \infty, \qquad \liminf_{r\to 0^+} w_3(r) = -\infty, \qquad \lim_{r\to\infty} w_3(r) = 0. \tag{3.30}$$

This completes the proof of property (3.15) and of Theorem 3.2. □

4. The Supercritical Case: $p > (N+2)/(N-2)$

In this section we investigate the behavior of solutions of (1.4) and (1.8) when $N > 2$ and $p > (N+2)/(N-2)$. The remainder of the section consists of the following.

(I) Theorem 4.1 classifies the behavior of solutions of (1.8). Again, we focus on solutions whose trajectories in the (h, h') phase plane form the stable and unstable manifolds of solutions associated with the constant solution $(0,0)$. Part (ii) of Theorem 4.1 gives a detailed proof that solutions on the stable manifold form an outgoing spiral in the (h, h') phase plane as τ decreases from $\tau = \infty$. The proof is sufficiently general as to include Galaktionov's numerical observation of large amplitude oscillations [10]. For these spiraling solutions it remains a challenging open problem to prove their asymptotic behavior at the left endpoint $\tau_{\min}$ of their interval of existence (see *Open Problem I* after the statement of Theorem 4.1).

(II) Theorem 4.5 shows how to combine the results of Theorem 4.1 with the formula $w(r) = h(\ln(r))w_1(r)$ to obtain a continuum of positive nonsingular solutions of (3.1). In addition, we prove the existence of a continuum of new sign changing, "super singular" solutions which, to our knowledge, have not previously been reported. For these sign changing solutions it remains a challenging open problem to prove their asymptotic behavior at the left endpoint $r_{\min}$ of their interval of existence (see *Open Problem II* after the statement of Theorem 4.5).

(III) Figure 4 illustrates the behavior of solutions when $(N, p) = (3, 7)$.

Theorem 4.1. *Let $N > 2$ and $p > (N+2)/(N-2)$.*

(i) *There is a one-dimensional unstable manifold of solutions of* (1.8) *leading from* $(0,0)$ *into the* (h, h') *phase plane. One component, A_1, points into the positive quadrant, and its negative counterpart, A_2, points into the negative quadrant (Figure 4, upper left). If $(h(0), h'(0)) \in A_1$, then $h(\tau) > 0$ for all $\tau \in \mathbb{R}$, and there is a constant $c > 0$ such that*

$$h(\tau) \sim c\exp\left(\frac{2\tau}{p-1}\right) \quad \text{as}\ \tau \longrightarrow -\infty, \qquad \lim_{\tau\to\infty}\left(h(\tau), h'(\tau)\right) = (1,0). \tag{4.1}$$

(ii) *There is a one-dimensional stable manifold of solutions leading to* $(0,0)$ *in the* (h, h') *phase plane. One component, B_1, points into the $h > 0$, $h' < 0$ quadrant of the phase plane, and its negative counterpart, B_2, points into the $h < 0$, $h' > 0$ quadrant (Figure 4, upper left). Additionally, if $h(0) > 0$ is sufficiently small and $(h(0), h'(0)) \in B_1$, then there is a $D > 0$ such that*

$$h(\tau) > 0 \quad \forall \tau \geq 0, \qquad h(\tau) \sim D\exp\left(-\frac{N-2}{p-1}\left(p - \frac{N}{N-2}\right)\tau\right) \quad \text{as}\ \tau \longrightarrow \infty. \tag{4.2}$$

Let $\overline{h} = ((p+1)/2)^{1/(p-1)}$*. There is a negative decreasing sequence* $\{\tau_N\}$ *such that*

$$h'(\tau_N) = 0 \quad \forall N \geq 1, \qquad h(\tau_N) > \overline{h} \quad \text{if } N \text{ is odd}, \qquad h(\tau_N) < -\overline{h} \quad \text{if } N \text{ is even}. \tag{4.3}$$

Moreover, $|h(\tau_N)|$ *increases as* N *increases.*

Remarks. The proof of part (i) of Theorem 4.1 uses straightforward phase plane type arguments. The proof of (ii) is admittedly more technical. Our numerical experiments (Figure 4, lower left) indicate that the amplitudes of the oscillations of the solutions described in part (ii) grow without bound as τ decreases. It remains a challenging open problem to determine whether these solutions exist on the entire interval $(-\infty, \infty)$, or if they exist only on a semi-infinite interval of the form $(\tau_{\min}, \infty)$. These fundamental theoretical questions are summarized in the following.

Open Problem I (Super Singular Solutions). Let $\{\tau_N\}$ denote the decreasing sequence described in part (ii) of Theorem 4.1. Prove whether $\tau_{\min} = -\infty$, or $\tau_{\min} > -\infty$. Second, prove whether $\limsup_{N\to\infty}|h(\tau_N)|$ is finite or infinite. Our numerical study suggests that $\limsup_{N\to\infty}|h(\tau_N)| = \infty$.

Proof of Theorem 4.1.

Part (i). First, note that properties of solutions on the component A_1 of the unstable manifold leading from $(0,0)$ into the positive quadrant of the (h, h') phase plane are the same as those seen in Lemma 2.7. From these properties it again follows that if $(h(0), h'(0)) \in A_1$, and $h(0) > 0$ is sufficiently small, then $h(\tau) > 0$ for all $\tau \leq 0$, and

$$h(\tau) \sim c\exp(\lambda_1 \tau), \quad \frac{h'(\tau)}{h(\tau)} \longrightarrow \lambda_1 \quad \text{as } \tau \longrightarrow -\infty \tag{4.4}$$

for some $c > 0$, where $\lambda_1 = 2/(p-1)$. This proves the first part of (4.1). To complete the proof of (4.1) recall that the functional S and the region U defined in the proof of Lemma 2.6 are

$$S = \frac{1}{2}(h')^2 + \frac{2(N-2)}{(p-1)^2}\left(p - \frac{N}{N-2}\right)\left(\frac{|h|^{p+1}}{p+1} - \frac{h^2}{2}\right), \tag{4.5}$$

$$U = \{(h, h') \mid h \geq 0, S \leq 0\}. \tag{4.6}$$

Again, let U^0 denote the interior of U, and note that U^0 contains one constant solution, $(h, h') = (1, 0)$. A differentiation of (4.5) gives

$$S' = -\frac{N-2}{p-1}\left(p - \frac{N+2}{N-2}\right)(h')^2 \leq 0. \tag{4.7}$$

Thus, if $(h(0), h'(0)) \in A_1$, and $h(0) > 0$ is sufficiently small, we conclude from (4.4)–(4.7) (Figure 4, upper right) that $h(\tau)$ exists for all $\tau \in \mathbb{R}$, and

$$S(-\infty) = 0, \qquad S'(\tau) < 0 \quad \forall \tau \leq 0, \qquad S(\tau) < 0 \quad \forall \tau \in \mathbb{R}. \tag{4.8}$$

Therefore,

$$(h(\tau), h'(\tau)) \in U^0 \quad \forall \tau \in \mathbb{R}. \tag{4.9}$$

From (4.7) we conclude that (1.8) has no periodic solutions. Also, a linearization of (1.8) around the constant solution $(h, h') = (1, 0)$ shows that $(1, 0)$ is an asymptotically stable equilibrium point in the (h, h') phase plane. Thus, if $h(\tau)$ is a solution of (1.8) with initial condition $(h(0), h'(0)) \in A_1$, it follows from (4.7)–(4.9), and standard phase plane arguments, that

$$h(\tau) > 0 \quad \forall \tau \in \mathbb{R}, \quad \lim_{\tau \to \infty} (h(\tau), h'(\tau)) = (1, 0). \tag{4.10}$$

Part (ii). It follows from (2.5) that there is a one-dimensional stable manifold of solutions leading to $(0, 0)$ in the (h, h') phase plane. As in Theorem 2.9, one component, B_1, points into the quadrant $h > 0$, $h' < 0$, and along B_1 solutions satisfy

$$\lim_{\tau \to \infty} (h(\tau), h'(\tau)) = (0, 0), \qquad \lim_{\tau \to \infty} \frac{h'(\tau)}{h(\tau)} = \lambda_2 < 0, \tag{4.11}$$

where $\lambda_2 = -((N-2)/(p-1))(p-N/(N-2))$. Thus, if $h_2(\tau)$ is a solution of (1.8) with $h_2(0) > 0$ sufficiently small and $(h_2(0), h_2'(0)) \in B_1$, there is a $D > 0$ such that

$$h_2(\tau) > 0, \quad h_2'(\tau) < 0 \quad \forall \tau \geq 0, \tag{4.12}$$

$$h_2(\tau) \sim D \exp(\lambda_2 \tau) \quad \text{as } \tau \longrightarrow \infty. \tag{4.13}$$

Along this solution S (Figure 4, upper right) satisfies

$$S(\infty) = 0, \qquad S'(\tau) < 0, \quad S(\tau) > 0 \quad \forall \tau \geq 0. \tag{4.14}$$

To complete the proof of (ii) let $\overline{h} = ((p+1)/2)^{1/(p-1)}$ be the unique positive h value where $S = h' = 0$. We need to prove that $(h_2(\tau), h_2'(\tau))$ rotates counterclockwise around $(1, 0)$ and that $(h_2(\tau), h_2'(\tau))$ generates an outwardly growing spiral as τ decreases. For this we show that there is a decreasing sequence of negative values $\{\tau_N\}$ such that

$$h_2'(\tau_N) = 0, \quad |h_2(\tau_{N+1})| > |h_2(\tau_N)| \quad \forall N \geq 1, \tag{4.15}$$

and that

$$h_2(\tau_N) > \overline{h} \quad \text{if } N \text{ is odd}, \qquad h_2(\tau_N) < -\overline{h} \quad \text{if } N \text{ is even}. \tag{4.16}$$

The proof of (4.15)-(4.16) is in two steps. First, we prove three technical results, Lemmas 4.2–4.4. Secondly, we use these lemmas to follow $(h_2(\tau), h_2'(\tau))$ as τ decreases. The proofs of Lemmas 4.2 and 4.4 are straightforward. The proof of Lemma 4.3 is admittedly technical. □

Lemma 4.2. *Let* $\hat{\tau} \in (-\infty, 0)$. (a) *If a solution of* (1.8) *satisfies*

$$0 \leq h(\hat{\tau}) < \overline{h} = \left(\frac{p+1}{2}\right)^{1/(p-1)}, \qquad h'(\hat{\tau}) < 0, \qquad S(\hat{\tau}) > 0, \tag{4.17}$$

then there is a $\tilde{\tau} < \hat{\tau}$ *such that*

$$h'(\tau) < 0 \quad \forall \tau \in [\tilde{\tau}, \hat{\tau}], \qquad h(\tilde{\tau}) = \overline{h}. \tag{4.18}$$

(b) *If a solution of* (1.8) *satisfies*

$$-\overline{h} < h(\hat{\tau}) \leq 0, \qquad h'(\hat{\tau}) > 0, \qquad S(\hat{\tau}) > 0, \tag{4.19}$$

then there is a $\tilde{\tau} < \hat{\tau}$ *such that*

$$h'(\tau) > 0 \quad \forall \tau \in [\tilde{\tau}, \hat{\tau}], \qquad h(\tilde{\tau}) = -\overline{h}. \tag{4.20}$$

Proof. (a) Let $\delta = 2S(\hat{\tau}) > 0$. Since (4.7) implies that $S'(\tau) \leq 0$ for all τ, then S increases monotonically as τ decreases. It follows from the definition of S in (4.5) that $h' \leq -\sqrt{\delta}$ for $\tau \leq \hat{\tau}$ as long as $0 \leq h(\tau) < \overline{h}$. An integration of $h' \leq -\sqrt{\delta}$ shows that $h(\tilde{\tau}) = \overline{h}$ at some first $\tilde{\tau} \in [\hat{\tau} + (h(\hat{\tau}) - \overline{h})/\sqrt{\delta}, \hat{\tau})$. This proves (4.18). The proof of part (b) is the same and is omitted. □

Lemma 4.3. *Let* $\tilde{\tau} \in (-\infty, 0)$. (a) *If a solution of* (1.8) *satisfies*

$$h(\tilde{\tau}) = \overline{h} = \left(\frac{p+1}{2}\right)^{1/(p-1)}, \qquad h'(\tilde{\tau}) < 0, \tag{4.21}$$

then there is a $\tau^* < \tilde{\tau}$ *such that*

$$h'(\tau) < 0 \quad \forall \tau \in (\tau^*, \tilde{\tau}], \qquad h'(\tau^*) = 0, \qquad h(\tau^*) > \overline{h}. \tag{4.22}$$

(b) *If a solution of* (1.8) *satisfies*

$$h(\tilde{\tau}) = -\overline{h}, \qquad h'(\tilde{\tau}) > 0, \tag{4.23}$$

then there is a $\tau^* < \tilde{\tau}$ *such that*

$$h'(\tau) > 0 \quad \forall \tau \in (\tau^*, \tilde{\tau}], \qquad h'(\tau^*) = 0, \qquad h(\tau^*) < -\overline{h}. \tag{4.24}$$

Proof. (a) The first step is to assume, for contradiction, that τ^* does not exist and that

$$h'(\tau) < 0 \quad \forall \tau \in (T, \tilde{\tau}), \tag{4.25}$$

where $(T, \tilde{\tau})$ denotes the maximal left interval of existence of the solution. We claim that $T = -\infty$. Suppose that $T > -\infty$. Then standard theory implies that h' must be unbounded on $(T, \tilde{\tau})$. To show that this cannot happen we use the function $H = (h')^2$. Then H satisfies

$$H' - 2(\lambda_1 + \lambda_2)H = \lambda_1\lambda_2\left(|h|^{p-1} - 1\right)hh' > 0 \quad \forall\tau \in (T, \tilde{\tau}), \tag{4.26}$$

since $\lambda_1\lambda_2 < 0$, $h(\tau) > 1$ and $h' < 0$ for all $\tau \in (T, \tilde{\tau})$. Integrating (4.26) gives

$$0 < H(\tau) \leq H(\tilde{\tau}) \exp(-2(\lambda_1 + \lambda_2)(\tilde{\tau} - \tau)) \quad \forall\tau \in (T, \tilde{\tau}). \tag{4.27}$$

From (4.27) it follows that h' is bounded on $(T, \tilde{\tau})$, contradicting the previous conclusion that h' is unbounded on $(T, \tilde{\tau})$. We conclude that $T = -\infty$. Therefore, (4.25) becomes

$$h'(\tau) < 0 \quad \forall\tau \in (-\infty, \tilde{\tau}). \tag{4.28}$$

Next, to obtain a contradiction of (4.28) we need to prove two technical properties. The first is that

$$h''(\tau) \geq 0, \quad h'(\tau) \leq h'(\tilde{\tau}) < 0 \quad \forall\tau \leq \tilde{\tau}, \qquad \lim_{\tau \to -\infty} h(\tau) = \infty. \tag{4.29}$$

To prove the first part of (4.29), differentiate (1.8) and get

$$h''' + \frac{N-2}{p-1}\left(p - \frac{N+2}{N-2}\right)h'' = \frac{2N - 2p(N-2)}{(p-1)^2}\left(p|h|^{p-1} - 1\right)h' > 0 \tag{4.30}$$

for $\tau \leq \tilde{\tau}$ as long as $h(\tau) > \overline{h} > 1$. If there is an $a \leq \tilde{\tau}$, where $h''(a) < 0$, then an integration of (4.30) gives

$$h''(\tau) \leq h''(a) \exp\left(\frac{N-2}{p-1}\left(p - \frac{N+2}{N-2}\right)(a - \tau)\right) < 0 \quad \forall\tau \leq a. \tag{4.31}$$

An integration of (4.31) gives $h'(\tau) > 0$ for $\tau \ll a$, contradicting (4.28). We conclude that $h''(\tau) \geq 0$ for all $\tau \leq \tilde{\tau}$, and the first part of (4.29) is proved. In turn, this implies that

$$h'(\tau) \leq h'(\tilde{\tau}) < 0 \quad \forall\tau \leq \tilde{\tau}, \qquad \lim_{\tau \to -\infty} h(\tau) = \infty, \tag{4.32}$$

and the proof of (4.29) is complete. The second property we need is

$$\rho(\tau) = \frac{h'(\tau)}{h(\tau)} \geq -m = \min(\lambda_2, \rho(\tilde{\tau})) \quad \forall\tau \leq \tilde{\tau}, \tag{4.33}$$

where $\rho(\tau)$ satisfies the ODE

$$\rho' + (\rho - \lambda_1)(\rho - \lambda_2) = \lambda_1\lambda_2|h|^{p-1}. \tag{4.34}$$

Property (4.33) follows immediately from (4.34), since (4.34) implies that $\rho' < 0$ whenever $\rho \le \lambda_2$.

We now show how to use properties (4.29) and (4.33) to obtain a contradiction of (4.28). First, an integration of (4.34) gives

$$\left((\rho(\tau) - \lambda_2)e^{-\int_\tau^{\tilde{\tau}}(\rho-\lambda_1)ds}\right)' = \lambda_1\lambda_2|h|^{p-1}e^{-\int_\tau^{\tilde{\tau}}(\rho-\lambda_1)ds}, \quad \forall\tau \le \tilde{\tau}. \tag{4.35}$$

Integrating both sides of (4.35) from τ to $\tilde{\tau}$, we obtain

$$\rho(\tau) = \lambda_2 + \mathrm{I}_1(\tau) + \mathrm{I}_2(\tau), \tag{4.36}$$

where

$$\mathrm{I}_1(\tau) = (\rho(\tilde{\tau}) - \lambda_2)e^{\int_\tau^{\tilde{\tau}}(\rho-\lambda_1)ds}, \tag{4.37}$$

$$\mathrm{I}_2(\tau) = -e^{\int_\tau^{\tilde{\tau}}(\rho-\lambda_1)ds}\lambda_1\lambda_2\int_\tau^{\tilde{\tau}}|h|^{p-1}e^{-\int_\eta^{\tilde{\tau}}(\rho-\lambda_1)ds}d\eta. \tag{4.38}$$

Our goal is to prove that

$$\lim_{\tau\to-\infty}\mathrm{I}_1 = 0, \qquad \lim_{\tau\to-\infty}\mathrm{I}_2 = \infty. \tag{4.39}$$

Once we prove these properties, we combine (4.39) with (4.36) and conclude that $\rho = h'/h > 0$ when $\tau \ll \tilde{\tau}$. Since $h(\tau) > 0$, this implies that $h' > 0$ when $\tau \ll \tilde{\tau}$, which contradicts (4.28).

To prove the first part of (4.39), evaluate the right side of (4.37) and get

$$\mathrm{I}_1(\tau) = (\rho(\tilde{\tau}) - \lambda_2)\frac{h(\tilde{\tau})}{h(\tau)}e^{-\lambda_1(\tilde{\tau}-\tau)} \longrightarrow 0 \quad \text{as}\,\tau \longrightarrow -\infty, \tag{4.40}$$

since $\lambda_1 > 0$ and $0 < h(\tilde{\tau})/h(\tau) \le 1$ for all $\tau \le \tilde{\tau}$. To prove the second part of (4.39), recall from (4.32) that $\lim_{\tau\to-\infty}h(\tau) = \infty$. Thus, since $\rho < 0$ and $\lambda_1 > 0$, we conclude that

$$\int_\tau^{\tilde{\tau}}|h|^{p-1}e^{-\int_\eta^{\tilde{\tau}}(\rho-\lambda_1)ds}d\eta \ge \int_\tau^{\tilde{\tau}}|h|^{p-1}d\eta \longrightarrow \infty \quad \text{as}\,\tau \longrightarrow -\infty. \tag{4.41}$$

Next, it follows from (4.33) that $-\lambda_1 > \rho - \lambda_1 \ge -m - \lambda_1$. This, (4.41), and the facts that $\lambda_1\lambda_2 < 0$ and $e^{\int_\tau^{\tilde{\tau}}(\rho-\lambda_1)ds} \to 0$ as $\tau \to -\infty$ allow us to apply L'Hospital's Rule to $\mathrm{I}_2(\tau)$. This gives

$$\lim_{\tau\to-\infty}\mathrm{I}_2(\tau) = \lim_{\tau\to-\infty}\left(\frac{\lambda_1\lambda_2|h(\tau)|^{p-1}}{\rho(\tau) - \lambda_1}\right) = \infty. \tag{4.42}$$

This proves part (a). The proof of part (b) is the same and is omitted. □

Lemma 4.4. *Let* $\tau^* \in (-\infty, 0)$. (a) *If a solution of* (1.8) *satisfies*

$$h(\tau^*) > \overline{h} = \left(\frac{p+1}{2}\right)^{1/(p-1)}, \qquad h'(\tau^*) = 0, \tag{4.43}$$

then there is a $\tau^{**} < \tau^*$ *such that*

$$h'(\tau) > 0 \quad \forall \tau \in [\tau^{**}, \tau^*), \qquad h(\tau^{**}) = 0. \tag{4.44}$$

(b) *If a solution of* (1.8) *satisfies*

$$h(\tau^*) < -\overline{h}, \qquad h'(\tau^*) = 0, \tag{4.45}$$

then there is a $\tau^{**} < \tau^*$ *such that*

$$h'(\tau) < 0 \quad \forall \tau \in [\tau^{**}, \tau^*), \qquad h(\tau^{**}) = 0. \tag{4.46}$$

Proof. (a) Since $\overline{h} > 1$, it follows from (1.8) that $h''(\tau^*) < 0$. Thus, there is an $a < \tau^*$ such that

$$h'(\tau) > 0 \quad \forall \tau \in [a, \tau^*), \qquad \overline{h} < h(a) < h(\tau^*). \tag{4.47}$$

Since $\overline{h} < h(a)$, a calculation shows that

$$\frac{|h|^{p+1}}{p+1} - \frac{h^2}{2} \leq \frac{|h|^{p+1}(a)}{p+1} - \frac{h^2(a)}{2} \quad \text{when}\, h \in [0, h(a)]. \tag{4.48}$$

Recall from (4.7) that $S'(\tau) \leq 0$ for all τ. Thus, $S(\tau) \geq S(a)$ for all $\tau \leq a$, and (4.48) implies that

$$\frac{(h'(\tau))^2}{2} - \frac{(h'(a))^2}{2} \geq -\lambda_1\lambda_2\left(\frac{|h|^{p+1}(a)}{p+1} - \frac{h^2(a)}{2} - \frac{|h|^{p+1}(\tau)}{p+1} + \frac{h^2(\tau)}{2}\right) \geq 0 \tag{4.49}$$

for $\tau \leq a$ as long as $0 \leq h(\tau) \leq h(a)$. Integrating $h'(\tau) \geq h'(a) > 0$ shows that there is a $\tau^{**} \in [a - h'(a)/h(a), a)$ such that

$$h'(\tau) \geq h'(a) > 0 \quad \forall \tau \in [\tau^{**}, a), \qquad h(\tau^{**}) = 0. \tag{4.50}$$

This proves part (a). The proof of part (b) is the same and is omitted.

We now return to the proof of Theorem 4.1. It remains to prove that $(h_2(\tau), h_2'(\tau))$ rotates counterclockwise around $(0,0)$ in the (h, h') plane as τ decreases from $\tau = 0$.

To accomplish this we use Lemmas 4.2–4.4 to show that $(h_2(\tau), h_2'(\tau))$ passes infinitely often through the sets

$$
\begin{aligned}
R_1 &= \left\{(h,h') \mid 0 \le h < \overline{h}, h' < 0, S > 0\right\} \\
R_2 &= \left\{(h,h') \mid h = \overline{h}, h' < 0\right\} \\
R_3 &= \left\{(h,h') \mid h > \overline{h}, h' = 0\right\} \\
R_4 &= \left\{(h,h') \mid h = 0, h' > 0\right\} \\
R_5 &= \left\{(h,h') \mid h = -\overline{h}, h' > 0\right\} \\
R_6 &= \left\{(h,h') \mid h < -\overline{h}, h' = 0\right\} \\
R_7 &= \left\{(h,h') \mid h = 0, h' < 0\right\}.
\end{aligned}
\tag{4.51}
$$

Recall that $(h_2(0), h_2'(0)) \in B_1$ and that (4.12) is satisfied, consequently $(h(0), h'(0)) \in R_1$. Lemma 4.2 implies that there is a first $\tilde{\tau} < 0$ such that $(h(\tilde{\tau}), h'(\tilde{\tau})) \in R_2$, that is, $h(\tilde{\tau}) = \overline{h}$ and $h'(\tilde{\tau}) < 0$. This and Lemma 4.3 imply that there is a $\tau_1 < \tilde{\tau}$ such that $h_2'(\tau) < 0$ for all $\tau \in (\tau_1, \tilde{\tau}]$, $h_2'(\tau_1) = 0$, and $h_2(\tau_1) > \overline{h}$. Thus, $(h(\tau_1), h'(\tau_1)) \in R_3$. It follows from Lemma 4.4 that there is a $\tau^{**} < \tau_1$ such that $h_2'(\tau) > 0$ for all $\tau \in [\tau^{**}, \tau_1)$ and $h_2(\tau^{**}) = 0$, hence $(h_2(\tau^{**}), h_2'(\tau^{**})) \in R_4$. This and part (b) of Lemma 4.2 imply that there is a $b < \tau^{**}$ such that $h_2'(\tau) > 0$ for all $\tau \in [b, \tau^{**}]$ and $h_2(b) = -\overline{h}$, hence $(h(b), h'(b)) \in R_5$. This and part (b) of Lemma 4.3 imply that there is a $\tau_2 < b$ such that $h_2'(\tau) > 0$ for all $\tau \in (\tau_2, b]$, $h_2'(\tau_2) = 0$, and $h_2(\tau_2) < -\overline{h}$, hence $(h_2(\tau_2), h_2'(\tau_2)) \in R_6$. This and part (b) of Lemma 4.4 imply that there is a $c < \tau_2$ such that $h_2'(\tau) < 0$ for all $\tau \in [c, \tau_2)$ and $h_2(c) = 0$, hence $(h_2(c), h_2'(c)) \in R_7$. We have shown how $(h_2(\tau), h_2'(\tau))$ passes sequentially through $R_1, \ldots, R_7$ as τ decreases. Since R_7 is contained in R_1 it follows from a repetition of the steps given above, and mathematical induction, that $(h_2(\tau), h_2'(\tau))$ passes sequentially through $R_1, \ldots, R_7$ infinitely often as τ decreases from $\tau = 0$. This produces a decreasing sequence $\{\tau_N\}$ where $h_2'(\tau_N) = 0$ for all $N \ge 1$,

$$
h_2(\tau_N) > \overline{h} \quad \text{when } N \text{ is odd}, \qquad h_2(\tau_N) < -\overline{h} \quad \text{when } N \text{ is even}. \tag{4.52}
$$

Finally, since S increases as τ decreases, it follows that $|h(\tau_N)|$ increases as N increases. This completes the proof of Theorem 4.1. □

Solutions of the w Equation

Below, in Theorem 4.5, we show how to combine part (ii) of Theorem 4.1 with the formula

$$
w(r) = h(\ln(r)) w_1(r), \tag{4.53}
$$

to prove the existence and asymptotic behavior of families of nonsingular and singular solutions of the w equation (1.4). In particular, in part (b) of Theorem 4.5 we show how a new family of "super singular" solutions is generated. *Open Problem II* stated after Theorem 4.5 describes important, and as yet unproven, properties of this continuum of singular solutions.

Theorem 4.5. *Let* $N > 2$ *and* $p > (N+2)/(N-2)$.

(a) A Continuum of Positive Nonsingular Solutions. Let $h_0(\tau)$ *denote a solution of* (1.8) *which satisfies* $(h_0(0), h_0'(0)) \in A_1$ *in part* (i) *of Theorem 4.1. The corresponding solution* $w_0(r) = h_0(\ln(r))w_1(r)$ *of* (1.4) *is strictly positive and satisfies*

$$w_0(1) = h_0(0)w_1(1), \qquad w_0'(1) = h_0'(0)w_1(1) + h_0(0)w_1'(1). \tag{4.54}$$

Furthermore, its interval of existence is $(0, \infty)$,

$$0 < w_0(0) < \infty, \qquad w_0'(0) = 0, \qquad \lim_{r\to\infty}\frac{w_0(r)}{w_1(r)} = 1. \tag{4.55}$$

(b) A Continuum of Sign Changing Singular Solutions. Let $h_2(\tau)$ *be a member of the family of "spiraling" solutions of* (3.2) *which satisfy* $(h_2(0), h_2'(0)) \in B_1$ *in part* (ii) *of Theorem 4.1, and let* $\{\tau_N\}$ *denote the decreasing sequence of* τ *values which satisfy property* (4.3)*. The corresponding solution* $w_2(r) = h_2(\ln(r))w_1(r)$ *of* (3.1) *satisfies*

$$w_2(1) = h_2(0)w_1(1), \qquad w_2'(1) = h_2'(0)w_1(1) + h_2(0)w_1'(1). \tag{4.56}$$

Its interval of existence is of the form $(r_{\min}, \infty)$*. As* $r \to \infty$, $w_2(r) \to 0$ *faster than* $w_1(r)$*. That is, there exists* $D > 0$ *such that.*

$$\frac{w_2(r)}{w_1(r)} \sim r^{-((N-2)/(p-1))(p-(N/(N-2)))} \longrightarrow 0 \quad \text{as } r \longrightarrow \infty. \tag{4.57}$$

As r *decreases from* ∞, $w_2(r)$ *changes sign infinitely often. That is, along the decreasing sequence* $\{r_N = \exp(\tau_N)\}$*, one has* $\lim_{N\to-\infty} r_N = r_{\min}$,

$$w_2(r_N) > w_1(r_N) \quad \text{if } N \text{ is odd}, \qquad w_2(r_N) < -w_1(r_N) \quad \text{if } N \text{ is even}. \tag{4.58}$$

Open Problem II (Super Singular Solutions). Let $\{r_N\}$ denote the decreasing sequence in part (b), which satisfies $\lim_{N\to\infty} r_N = r_{\min}$. Prove whether $r_{\min} = 0$ or $r_{\min} > 0$. Secondly, prove whether $\limsup_{N\to\infty}|w_2(r_N)|$ is finite or infinite. Our numerical experiments suggest that $\limsup_{N\to\infty}|w_2(r_N)| = \infty$. As in Open Problem I, our analytical methods have not allowed us to resolve these fundamental theoretical issues.

Proof of Theorem 4.5.

Part (a). Let $h_0(\tau)$ be a solution of (1.8) satisfying $(h_0(0), h_0'(0)) \in A_1$ in part (i) of Theorem 4.1. The corresponding solution of (1.4) is

$$w_0(r) = h_0(\ln(r))w_1(r), \tag{4.59}$$

where we recall that

$$w_1(r) = \left(\frac{2(N-2)(p-1)-4}{(p-1)^2}\right)^{1/(p-1)} r^{-2/(p-1)}, \quad r > 0. \tag{4.60}$$

Because $r = \exp(\tau)$, the initial point $\tau = 0$ for $h_0(\tau)$ translates to $r = 1$ for $w_0(r)$. This, (4.59) and (4.60) give (4.54). Next, it follows from property (4.1) and the fact that $\tau = \ln(r)$ that there exists $c > 0$ such that

$$h_0(\ln(r)) \sim cr^{2/(p-1)} \quad \text{as } r \longrightarrow 0^+, \qquad \lim_{r\to\infty} h_0(\ln(r)) = 1. \tag{4.61}$$

We conclude from (4.59), (4.60), and the first part of (4.61) that

$$w_0(r) \longrightarrow c\left(\frac{2(N-2)(p-1)-4}{(p-1)^2}\right)^{1/(p-1)} > 0 \quad \text{as } r \longrightarrow 0^+. \tag{4.62}$$

It then follows from standard theory that $w_0'(0) = 0$. This agrees with a result of Haraux and Weissler (see Theorem 4 in [2]). Finally, we conclude from (4.59), (4.60), and the second part of (4.61) that

$$\lim_{r\to\infty} \frac{w_0(r)}{w_1(r)} = \lim_{r\to\infty} h_0(\ln(r)) = 1. \tag{4.63}$$

This completes the proof of part (a).

Part (b). Let $h_2(\tau)$ be a solution of (1.8) satisfying $(h_2(0), h_2'(0)) \in B_1$ in part (ii) of Theorem 4.1. The corresponding solution of (1.4) is

$$w_2(r) = h_2(\ln(r))w_1(r). \tag{4.64}$$

It follows from (4.2) that there exists $D > 0$ such that

$$h_2(\ln(r)) \sim Dr^{-((N-2)/(p-1))(p-N/(N-2))} \quad \text{as } r \longrightarrow \infty. \tag{4.65}$$

Combining (4.60), (4.64), and (4.65) gives

$$\frac{w_2(r)}{w_1(r)} \sim Dr^{-((N-2)/(p-1))(p-N/(N-2))} \quad \text{as } r \longrightarrow \infty. \tag{4.66}$$

This proves the first part of property (4.57). Next, let $\{\tau_N\}$ denote a sequence of τ values satisfying property (4.3) in Theorem 4.1. Then $\{\tau_N\}$ decreases as N increases, with $\lim_{N\to\infty} \tau_N = \tau_{\min}$, and

$$h_2(\tau_N) > \overline{h} \quad \text{if } N \text{ is odd}, \qquad h_2(\tau_N) < -\overline{h} \quad \text{if } N \text{ is even}. \tag{4.67}$$

Define $r_N = \exp(\tau_N)$ for all $N \geq 1$. Setting $\tau_N = \ln(r_N)$ in (4.67) gives

$$h_2(\ln(r_N)) > \overline{h} \quad \text{if } N \text{ is odd}, \qquad h_2(\ln(r_N)) < -\overline{h} \quad \text{if } N \text{ is even}. \tag{4.68}$$

Finally, we combine with (4.64) and (4.68) and obtain

$$w_2(r_N) > \overline{h}w_1(r_N) \quad \text{if } N \text{ is odd}, \qquad w_2(r_N) < -\overline{h}w_1(r_N) \quad \text{if } N \text{ is even}. \tag{4.69}$$

Since $\overline{h} > 1$, this completes the proof of (4.58) and of Theorem 4.5. □

5. Conclusions

In this paper we have developed an analytical method to classify the behavior of radially symmetric, time-independent solutions of the nonlinear heat equation (1.2). These solutions satisfy the ODE (1.4). We have studied solutions which remain strictly positive on their entire intervals of existence, and also solutions which change sign. There have been few analyses in the literature of sign changing solutions. Our analytical method follows a three-step approach:

Step 1. Transform the nonautonomous w equation (1.4) into the autonomous h equation (1.8).

Step 2. Analyze (1.8) using phase plane methods.

Step 3. Use the inverse transformation

$$w(r) = h(\ln(r))w_1(r), \tag{5.1}$$

to translate results for (1.8) into new results for the w equation (1.4).

Our Advance

This approach has allowed us to prove the existence and asymptotic behavior of several new families of solutions of (1.4). In particular, we mention two important classes of solutions which, to our knowledge, have not previously been reported.

(I) When $N > 2$ and $N/(N-2) < p < (N+2)/(N-2)$, we proved (see part (ii) of Theorem 2.10) the existence and asymptotic behavior of a continuum of positive, singular solutions which "interlace" with the known singular solutions $w_1(r)$.

(II) When $N > 2$ and $N/(N-2) < p < (N+2)/(N-2)$, we proved (see part (ii) of Theorem 4.1) the existence of sign changing solutions of the h equation which form outward spirals in the (h, h') phase plane. These solutions transform, by means of (5.1), into "super singular" sign changing solutions of the w equation (1.4). *Open Problems I and II* (see Section 4) summarize important issues for these solutions which have not yet been resolved.

Below, we describe challenging problems for further research.

Open Problem III. When $N/(N-2) < p < (N+2)/(N-2)$, do the new positive singular solutions, which interlace with $w_1(r)$, play an important role similar to that of $w_1(r)$ (e.g., see [5]) in proving the large time behavior of solutions of the time-dependent problem (1.2)?

Open Problem IV. Equation (1.1) is a canonical model for the general equation

$$\Delta u + f(u) = 0, \tag{5.2}$$

where $f(u) > 0$ is positive and superlinear [1–5]. A natural extension of our investigation is to use our new singular solutions of (1.4) as a guide in analyzing (5.2) for the existence of new classes of solutions.

Open Problem V. Gazzola and Grunau [13] investigate the behavior of solutions of the biharmonic equation

$$-\Delta^2 u + |u|^{p-1} u = 0. \tag{5.3}$$

This equation has the singular solution $w_1 = Ar^{-4/(p-1)}$, $r > 0$. It is hoped that our approach can be used to look for new classes of solutions of (5.3).

Open Problem VI. Develop a comparison technique which allows one to use the new singular solution $w_2(r)$ to establish blowup of solutions of the full time-dependent PDE.

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