Research Article

Approximate \( n \)-Lie Homomorphisms and Jordan \( n \)-Lie Homomorphisms on \( n \)-Lie Algebras

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Using fixed point methods, we establish the stability of \( n \)-Lie homomorphisms and Jordan \( n \)-Lie homomorphisms on \( n \)-Lie algebras associated to the following generalized Jensen functional equation

\[
\mu f\left(\sum_{i=1}^{n} x_i/n\right) + \mu \sum_{j=2}^{n} f\left(\sum_{i=1, i \neq j}^{n} x_i - (n-1)x_j/n\right) = f(\mu x_1)(n \geq 2).
\]

1. Introduction

Let \( n \) be a natural number greater or equal to 3. The notion of an \( n \)-Lie algebra was introduced by Filippov in 1985 [1]. The Lie product is taken between \( n \) elements of the algebra instead of two. This new bracket is \( n \)-linear, antisymmetric and satisfies a generalization of the Jacobi identity. For \( n = 3 \) this product is a special case of the Nambu bracket, well known in physics, which was introduced by Nambu [2] in 1973, as a generalization of the Poisson bracket in Hamiltonian mechanics.

An \( n \)-Lie algebra is a natural generalization of a Lie algebra. Namely, a vector space \( V \) together with a multilinear, antisymmetric \( n \)-ary operation \( [\cdot]:\wedge^n V \to V \) is called an \( n \)-Lie algebra, \( n \geq 3 \), if the \( n \)-ary bracket is a derivation with respect to itself, that is,

\[
[[x_1, \ldots, x_n], x_{n+1}, \ldots, x_{2n-1}] = \sum_{i=1}^{n} [x_1, \ldots, x_{i-1} [x_i, x_{n+1}, \ldots, x_{2n-1}], \ldots, x_n], \quad (1.1)
\]

where \( x_1, x_2, \ldots, x_{2n-1} \in V \). Equation (1.1) is called the generalized Jacobi identity. The meaning of this identity is similar to that of the usual Jacobi identity for a Lie algebra (which is a 2-Lie algebra).
In [1] and several subsequent papers, [3–5] a structure theory of finite-dimensional $n$-Lie algebras over a field $\mathbb{F}$ of characteristic 0 was developed.

$n$-ary algebras have been considered in physics in the context of Nambu mechanics [2, 6] and, recently (for $n = 3$), in the search for the effective action of coincident M2-branes in $M$-theory initiated by the Bagger-Lambert-Gustavsson (BLG) model [7, 8] (further references on the physical applications of $n$-ary algebras are given in [9]).

From now on, we only consider $n$-Lie algebras over the field of complex numbers. An $n$-Lie algebra $A$ is a normed $n$-Lie algebra if there exists a norm $\| \|$ on $A$ such that $\| [x_1, x_2, \ldots, x_n] \| \leq \| x_1 \| \cdots \| x_n \|$ for all $x_1, x_2, \ldots, x_n \in A$. A normed $n$-Lie algebra $A$ is called a Banach $n$-Lie algebra, if $(A, \| \|)$ is a Banach space.

Let $(A, [\cdot]_A)$ and $(B, [\cdot]_B)$ be two Banach $n$-Lie algebras. A $\mathbb{C}$-linear mapping $H : (A, [\cdot]_A) \rightarrow (B, [\cdot]_B)$ is called an $n$-Lie homomorphism if

$$H([x_1 x_2 \cdots x_n]_A) = [H(x_1) H(x_2) \cdots H(x_n)]_B$$

(1.2)

for all $x_1, x_2, \ldots, x_n \in A$. A $\mathbb{C}$-linear mapping $H : (A, [\cdot]_A) \rightarrow (B, [\cdot]_B)$ is called a Jordan $n$-Lie homomorphism if

$$H([x x \cdots x]_A) = [H(x) H(x) \cdots H(x)]_B$$

(1.3)

for all $x \in A$.

The study of stability problems had been formulated by Ulam [10] during a talk in 1940. Under what condition does there exist a homomorphism near an approximate homomorphism? In the following year, Hyers [11] answered affirmatively the question of Ulam for Banach spaces, which states that if $\varepsilon > 0$ and $f : X \rightarrow Y$ is a map with $X$ a normed space, $Y$ a Banach spaces such that

$$\| f(x + y) - f(x) - f(y) \| \leq \varepsilon$$

(1.4)

for all $x, y \in X$, then there exists a unique additive map $T : X \rightarrow Y$ such that

$$\| f(x) - T(x) \| \leq \varepsilon$$

(1.5)

for all $x \in X$. A generalized version of the theorem of Hyers for approximately linear mappings was presented by Rassias [12] in 1978 by considering the case when inequality (1.4) is unbounded. Due to that fact, the additive functional equation $f(x + y) = f(x) + f(y)$ is said to have the generalized Hyers-Ulam-Rassias stability property. A large list of references concerning the stability of functional equations can be found in [13–32].

In 1982–1994, Rassias (see [26–28]) solved the Ulam problem for different mappings and for many Euler-Lagrange type quadratic mappings, by involving a product of different powers of norms. In addition, Rassias considered the mixed product sum of powers of norms control function. For more details see [33–57].

In 2003 Cădariu and Radu applied the fixed-point method to the investigation of the Jensen functional equation [58]. They could present a short and a simple proof (different of the “direct method”, initiated by Hyers in 1941) for the generalized Hyers-Ulam stability of Jensen functional equation [58] and for quadratic functional equation.
Park and Rassias [59] proved the stability of homomorphisms in C*-algebras and Lie C*-algebras and also of derivations on C*-algebras and Lie C*-algebras for the Jensen-type functional equation

\[ \mu f \left( \frac{x + y}{2} \right) + \mu f \left( \frac{x - y}{2} \right) - f(\mu x) = 0 \] (1.6)

for all \( \mu \in \mathbb{T}^1 := \{\lambda \in \mathbb{C}; |\lambda| = 1\} \).

In this paper, by using the fixed-point methods, we establish the stability of homomorphisms and Jordan \( n \)-Lie homomorphisms on \( n \)-Lie Banach algebras associated to the following generalized Jensen type functional equation:

\[ \mu f \left( \frac{\sum_{i=1}^{n} x_i}{n} \right) + \mu \sum_{j=2}^{n} f \left( \frac{\sum_{i=1, i \neq j}^{n} x_i - (n - 1)x_j}{n} \right) - f(\mu x_1) = 0 \] (1.7)

for all \( \mu \in (\mathbb{T}^1_{1/n_o} := \{e^{i\theta}; 0 \leq \theta \leq 2\pi/n_o\} \cup \{1\}) \), where \( n \geq 2 \).

Throughout this paper, assume that \((A, \{\cdot\}_A), (B, \{\cdot\}_B)\) are two \( n \)-Lie Banach algebras.

2. Main Results

Before proceeding to the main results, we recall a fundamental result in fixed point theory.

**Theorem 2.1** (see [60]). Let \((\Omega, d)\) be a complete generalized metric space, and let \(T : \Omega \to \Omega\) be a strictly contractive function with Lipschitz constant \( L \). Then for each given \( x \in \Omega\), either

\[ d(T^m x, T^{m+1} x) = \infty \quad \forall m \geq 0, \] (2.1)

or other exists a natural number \( m_0 \) such that

(i) \( d(T^m x, T^{m+1} x) < \infty \) for all \( m \geq m_0 \);

(ii) the sequence \( \{T^m x\} \) is convergent to a fixed point \( y^* \) of \( T \);

(iii) \( y^* \) is the unique fixed point of \( T \) in \( \Lambda = \{y \in \Omega : d(T^{m_0} x, y) < \infty\} \);

(iv) \( d(y, y^*) \leq (1/(1 - L))d(y, Ty) \) for all \( y \in \Lambda \).

We start our work with the main theorem of the our paper.

**Theorem 2.2.** Let \( n_0 \in \mathbb{N} \) be a fixed positive integer number. Let \( f : A \to B \) be a function for which there exists a function \( \phi : A^n \to [0, \infty) \) such that

\[ \left\| \mu f \left( \frac{\sum_{i=1}^{n} x_i}{n} \right) + \mu \sum_{j=2}^{n} f \left( \frac{\sum_{i=1, i \neq j}^{n} x_i - (n - 1)x_j}{n} \right) - f(\mu x_1) \right\|_B \leq \phi(x_1, x_2, \ldots, x_n) \] (2.2)
for all \( \mu \in (T^n_{1/n_0} := \{e^{i\theta}; 0 \leq \theta \leq 2\pi/n_0 \} \cup \{1\}) \) and all \( x_1, \ldots, x_n \in A \), and that
\[
\|f([x_1 x_2 \cdots x_n]_A) - [f(x_1) f(x_2) \cdots f(x_n)]_B \|_B \leq \phi(x_1, x_2, \ldots, x_n)
\] (2.3)
for all \( x_1, \ldots, x_n \in A \). If there exists an \( L < 1 \) such that
\[
\phi(x_1, x_2, \ldots, x_n) \leq nL \phi\left(\frac{x_1}{n}, \frac{x_2}{n}, \ldots, \frac{x_n}{n}\right)
\] (2.4)
for all \( x_1, \ldots, x_n \in A \), then there exists a unique \( n \)-Lie homomorphism \( H : A \to B \) such that
\[
\|f(x) - H(x)\| \leq \frac{L}{1-L} \phi(x, 0, \ldots, 0)
\] (2.5)
for all \( x \in A \).

**Proof.** Let \( \Omega \) be the set of all functions from \( A \) into \( B \) and let
\[
d(g, h) := \inf \{ C \in \mathbb{R}^+ : \|g(x) - h(x)\|_B \leq C \phi(x, 0, \ldots, 0), \forall x \in A \}.
\] (2.6)
It is easy to show that \((\Omega, d)\) is a generalized complete metric space [61].

Now we define the mapping \( J : \Omega \to \Omega \) by \( J(h)(x) = (1/n)h(nx) \) for all \( x \in A \). Note that for all \( g, h \in \Omega \),
\[
d(g, h) < C \implies \|g(x) - h(x)\| \leq C \phi(x, 0, \ldots, 0), \quad \forall x \in A,
\]
\[
\implies \left\|\frac{1}{n}g(nx) - \frac{1}{n}h(nx)\right\| \leq \frac{1}{|n|} C \phi(nx, 0, \ldots, 0), \quad \forall x \in A,
\]
\[
\implies \left\|\frac{1}{n}g(nx) - \frac{1}{n}h(nx)\right\| \leq L C \phi(x, 0, \ldots, 0), \quad \forall x \in A,
\]
\[
\implies d(J(g), J(h)) \leq L C.
\] (2.7)
Hence we see that
\[
d(J(g), J(h)) \leq L d(g, h)
\] (2.8)
for all \( g, h \in \Omega \). It follows from (2.4) that
\[
\lim_{m \to \infty} \frac{1}{n^m} \phi(n^mx_1, n^mx_2, \ldots, n^mx_n) = 0
\] (2.9)
for all \( x_1, \ldots, x_n \in A \). Putting \( \mu = 1, \ x_1 = x, \) and \( x_j = 0 \ (j = 2, \ldots, n) \) in (2.2), we obtain
\[
\left\|nf\left(\frac{x}{n}\right) - f(x)\right\|_B \leq \phi(x, 0, \ldots, 0)
\] (2.10)
for all $x \in A$. Thus by using (2.4), we obtain that

$$\left\| \frac{1}{n} f(nx) - f(x) \right\|_B \leq \frac{1}{n} \phi(nx, 0, \ldots, 0) \leq L \phi(x, 0, \ldots, 0)$$  \hspace{1cm} (2.11)

for all $x \in A$, that is,

$$d(f, J(f)) \leq L < \infty.$$  \hspace{1cm} (2.12)

By Theorem 2.1, $J$ has a unique fixed point in the set $X_1 := \{ h \in \Omega : d(f, h) < \infty \}$. Let $H$ be the fixed point of $J$. $H$ is the unique mapping with

$$H(nx) = nH(x)$$  \hspace{1cm} (2.13)

for all $x \in A$, such that there exists $C \in (0, \infty)$ satisfying

$$\left\| f(x) - H(x) \right\|_B \leq C \phi(x, 0, \ldots, 0)$$  \hspace{1cm} (2.14)

for all $x \in A$. On the other hand we have $\lim_{m \to \infty} d(J^m(f), H) = 0$, so

$$\lim_{m \to \infty} \frac{1}{n^m} f(n^m x) = H(x)$$  \hspace{1cm} (2.15)

for all $x \in A$. Also by Theorem 2.1, we have

$$d(f, H) \leq \frac{1}{1 - L} d(f, J(f)).$$  \hspace{1cm} (2.16)

It follows from (2.12) and (2.16) that

$$d(f, H) \leq \frac{L}{1 - L}.$$  \hspace{1cm} (2.17)

This implies the inequality (2.5). By (2.21), we have

$$\left\| H([x_1 x_2 \cdots x_n]_A) - [H(x_1) H(x_2) H(x_3) \cdots H(x_n)]_B \right\|_B \leq \lim_{m \to \infty} \frac{1}{n^m} \phi(n^m x_1, n^m x_2, \ldots, n^m x_n) = 0$$  \hspace{1cm} (2.18)
for all $x_1, \ldots, x_n \in A$. Hence

\[ H([x_1x_2 \cdots x_n]_A) = [H(x_1)H(x_2)H(x_3) \cdots H(x_n)]_B \]  

(2.19)

for all $x_1, \ldots, x_n \in A$.

On the other hand, it follows from (2.2), (2.9), and (2.15) that

\[
\begin{align*}
\left\| H \left( \frac{\sum_{i=1}^{n} x_i}{n} \right) + \sum_{j=2}^{n} H \left( \frac{\sum_{i=1, i \neq j}^{n} x_i - (n-1)x_j}{n} \right) - H(x_1) \right\|_B \\
= \lim_{m \to \infty} \frac{1}{h_m} \left\| f \left( n^{m-1} \sum_{i=1}^{n} x_i \right) + \sum_{j=2}^{n} \left( f \left( n^{m-1} \left( \sum_{i=1, i \neq j}^{n} x_i - (n-1)x_j \right) \right) \right) - f(n^m x_1) \right\|_B \\
\leq \lim_{m \to \infty} \frac{1}{h_m} \phi(n^m x_1, n^m x_2, \ldots, n^m x_n) = 0
\end{align*}
\]

(2.20)

for all $x_1, \ldots, x_n \in A$. Then

\[ H \left( \frac{\sum_{i=1}^{n} x_i}{n} \right) + \sum_{j=2}^{n} H \left( \frac{\sum_{i=1, i \neq j}^{n} x_i - (n-1)x_j}{n} \right) = H(x_1) \]  

(2.21)

for all $x_1, \ldots, x_n \in A$. Putting $s_1 = \sum_{i=1}^{n} x_i / n$ and $s_j = \sum_{i=1, i \neq j}^{n} x_i - (n-1)x_j / n$ ($j = 2, 3, \ldots, n$) in (2.21), we obtain

\[ H \left( \sum_{j=1}^{n} s_j \right) = \sum_{j=1}^{n} H(s_j) \]  

(2.22)

for all $s_1, \ldots, s_n \in A$. Setting $s_j = 0$ ($j = 3, 4, \ldots, n$) in (2.22) to get

\[ H(s_1 + s_2) = H(s_1) + H(s_2) \]  

(2.23)

hence $H$ is cauchy additive. Letting $x_i = x$ for all $i = 1, 2, \ldots, n$ in (2.2), we obtain

\[ \left\| \mu f(x) - f(\mu x) \right\|_B \leq \phi(x, x, \ldots, x) \]  

(2.24)

for all $x \in A$. It follows that

\[
\begin{align*}
\left\| H(\mu x) - \mu H(x) \right\| &= \lim_{m \to \infty} \frac{1}{h_m} \left\| f(\mu^m x) - \mu f(n^m x) \right\|_B \\
&\leq \lim_{m \to \infty} \frac{1}{h_m} \phi(n^m x, n^m x, \ldots, n^m x) = 0
\end{align*}
\]  

(2.25)
for all $\mu \in T^1_{1/n_0}$, and all $x \in A$. One can show that the mapping $H : A \to B$ is $\mathbb{C}$-linear. Hence, $H : A \to B$ is an $n$-Lie homomorphism satisfying (2.5), as desired.

**Corollary 2.3.** Let $\theta$ and $p$ be nonnegative real numbers such that $p < 1$. Suppose that a function $f : A \to B$ satisfies

$$\|\mu f \left( \sum_{i=1}^{n} x_i \right) + \mu \sum_{j=2}^{n} f \left( \sum_{i=1, i \neq j}^{n} x_i - \frac{(n-1)x_j}{n} \right) - f(\mu x_1) \|_B \leq \theta \sum_{i=1}^{n} (\|x_i\|_A^p)$$

(2.26)

for all $\mu \in T^1$ and all $x_1, \ldots, x_n \in A$ and

$$\|f([x_1 x_2 \cdots x_n]_A) - [f(x_1) f(x_2) \cdots f(x_n)]_B\|_B \leq \theta \sum_{i=1}^{n} (\|x_i\|_A^p)$$

(2.27)

for all $x_1, \ldots, x_n \in A$. Then there exists a unique $n$-Lie homomorphism $H : A \to B$ such that

$$\|f(x) - H(x)\|_B \leq \frac{2^p}{\phi(2-2^p)} \theta \|x\|_A^p$$

(2.28)

for all $x \in A$.

**Proof.** Put $\phi(x_1, x_2, \ldots, x_n) := \theta \sum_{i=1}^{n} (\|x_i\|_A^p)$ for all $x_1, \ldots, x_n \in A$ in Theorem 2.2. Then (2.9) holds for $p < 1$, and (2.28) holds when $L = 2^{(p-1)}$.

**Theorem 2.4.** Let $n_0 \in \mathbb{N}$ be a fixed positive integer number. Let $f : A \to B$ be a function for which there exists a function $\phi : A^n \to [0, \infty)$ such that

$$\|\mu f \left( \sum_{i=1}^{n} x_i \right) + \mu \sum_{j=2}^{n} f \left( \sum_{i=1, i \neq j}^{n} x_i - \frac{(n-1)x_j}{n} \right) - f(\mu x_1) \|_B \leq \phi(x_1, x_2, \ldots, x_n)$$

(2.29)

for all $\mu \in \left( T^1_{1/n_0} := \{ e^{i\theta} \mid 0 \leq \theta \leq \frac{2\pi}{n_0} \} \cup \{ 1 \} \right)$ and all $x_1, \ldots, x_n \in A$, and that

$$\|f([x x \cdots x]_A) - [f(x) f(x) \cdots f(x)]_B\|_B \leq \phi(x, x, \ldots, x)$$

(2.30)

for all $x \in A$. If there exists an $L < 1$ such that

$$\phi(x_1, x_2, \ldots, x_n) \leq nL\phi \left( \frac{x_1}{n}, \frac{x_2}{n}, \ldots, \frac{x_n}{n} \right)$$

(2.31)

for all $x_1, \ldots, x_n \in A$, then there exists a unique Jordan $n$-Lie homomorphism $H : A \to B$ such that

$$\|f(x) - H(x)\|_B \leq \frac{L}{1-L} \phi(x, 0, 0, \ldots, 0)$$

(2.32)

for all $x \in A$.
Corollary 2.5. Let \( f \) be nonnegative real numbers such that \( p < 1 \). Suppose that a function \( f : A \to B \) satisfies

\[
\| \frac{\mu f \left( \frac{\sum_{i=1}^{n} x_i}{n} \right) + \mu \sum_{j=2}^{n} f \left( \frac{\sum_{i=1,i\neq j}^{n} x_i - (n - 1)x_j}{n} \right) - f(\mu x) \|_B \| < \theta \sum_{i=1}^{n} (\| x \|_A^p)
\]

for all \( \mu \in T^1 \) and all \( x_1, \ldots, x_n \in A \) and

\[
\| f([xx \cdots x]) - [f(x)f(x) \cdots f(x)] \|_B \leq n\theta(\| x \|_A^p)
\]

for all \( x \in A \). Then there exists a unique Jordan \( n \)-Lie homomorphism \( H : A \to B \) such that

\[
\| f(x) - H(x) \|_B \leq \frac{2p}{\ell (2-\ell)} \theta \| x \|_A^p
\]

for all \( x \in A \).

Proof. It follows by Theorem 2.4 by putting \( \phi(x_1, x_2, \ldots, x_n) \) := \( \theta \sum_{i=1}^{n} (\| x_i \|_A^p) \) for all \( x_1, \ldots, x_n \in A \) and \( L = 2^{(p-1)} \). \( \square \)

References

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