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Research Article

Generalized k-Uniformly Close-to-Convex Functions Associated with Conic Regions

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We define and study some subclasses of analytic functions by using a certain multiplier transformation. These functions map the open unit disc onto the domains formed by parabolic and hyperbolic regions and extend the concept of uniformly close-to-convexity. Some interesting properties of these classes, which include inclusion results, coefficient problems, and invariance under certain integral operators, are discussed. The results are shown to be the best possible.

1. Introduction

Let A denote the class of analytic functions f defined in the unit disc $E = \{z : |z| < 1\}$ and satisfying the condition $f(0) = 0, f'(0) = 1$. Let S, $S^*(\gamma)$, $C(\gamma)$ and $K(\gamma)$ be the subclasses of A consisting of functions which are univalent, starlike of order γ, convex of order γ, and close-to-convex of order γ, respectively, $0 \leq \gamma < 1$. Let $S^*(0) = S^*$, $C(0) = C$ and $K(0) = K$.

For analytic functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$, by $f * g$ we denote the convolution (Hadamard product) of f and g, defined by

$$(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n. \tag{1.1}$$

We say that a function $f \in A$ is subordinate to a function $F \in A$ and write $f(z) \prec F(z)$ if and only if there exists an analytic function $w(z)$, $w(0) = 0$, $|w(z)| < 1$ for $z \in E$ such that $f(z) = F(w(z))$, $z \in E$.

If F is univalent in E, then

$$f(z) \prec F(z) \Longleftrightarrow f(0) = F(0), \qquad f(E) \subset F(E). \tag{1.2}$$

For $k \in [0,1]$, define the domain Ω_k as follows, see [1]:

$$\Omega_k = \left\{ u + iv : u > k\sqrt{(u-1)^2 + v^2} \right\}. \tag{1.3}$$

For fixed k, Ω_k represents the conic region bounded, successively, by the imaginary axis ($k = 0$), the right branch of hyperbola ($0 < k < 1$), a parabola ($k = 1$).

Related with Ω_k, the domain $\Omega_{k,\gamma}$ is defined in [2] as follows:

$$\Omega_{k,\gamma} = (1-\gamma)\Omega_k + \gamma, \qquad (0 \le \gamma < 1). \tag{1.4}$$

The functions which play the role of extremal functions for the conic regions $\Omega_{k,\gamma}$ are denoted by $p_{k,\gamma}(z)$ with $p_{k,\gamma}(0) = 1$, and $p'_{k,\gamma}(0) > 0$ are univalent, map E onto $\Omega_{k,\gamma}$, and are given as

$$p_{k,\gamma}(z) = \begin{cases} \dfrac{1 + (1-2\gamma)z}{(1-z)}, & k = 0, \\ 1 + \dfrac{2(1-\gamma)}{\pi^2}\left(\log \dfrac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2, & k = 1, \\ 1 + \dfrac{2(1-\gamma)}{1-k^2}\sinh^2\left[\left(\dfrac{2}{\pi}\text{arc}\cos k\right)\text{arc}\tanh\sqrt{z}\right], & (0 < k < 1). \end{cases} \tag{1.5}$$

It has been shown [3, 4] that $p_{k,\gamma}(z)$ is continuous as regards to k and has real coefficients for all $k \in [0,1]$.

Let $P(p_{k,\gamma})$ be the class of functions $p(z)$ which are analytic in E with $p(0) = 1$ such that $p(z) \prec p_{k,\gamma}(z)$ for $z \in E$. It can easily be seen that $P(p_{k,\gamma}) \subset P$, where P is the class of Caratheodory functions of positive real part.

The class $P_m(p_{k,\gamma})$ is defined in [5] as follows.

Let $p(z)$ be analytic in E with $p(0) = 1$. Then $p \in P_m(p_{k,\gamma})$ if and only if, for $m \ge 2$, $0 \le \gamma < 1$, $k \in [0,1]$, $z \in E$,

$$p(z) = \left(\frac{m}{4} + \frac{1}{2}\right)p_1(z) - \left(\frac{m}{4} - \frac{1}{2}\right)p_2(z), \qquad p_1, p_2 \in P(p_{k,\gamma}). \tag{1.6}$$

For $k = 0$, $\gamma = 0$, the class $P_m(p_{0,0})$ coincides with the class P_m introduced by Pinchuk in [6]. Also $P_2 = P$.

The generalized Harwitz-Lerch Zeta function [7] $\phi(z,\lambda,\mu)$ is given as

$$\phi(z,\lambda,\mu) = \sum_{n=0}^{\infty} \frac{z^n}{(\mu+n)^{\lambda}}, \qquad (\lambda \in \mathbb{C},\ \mu \in \mathbb{C} \setminus Z^- = \{-1, -2, \ldots\}). \tag{1.7}$$

Using (1.7), the following family of linear operators, see [7–9], is defined in terms of the Hadamard product as

$$J_{\lambda,\mu}f(z) = H_{\lambda,\mu}(z) * f(z), \tag{1.8}$$

where $f \in A$,

$$H_{\lambda,\mu}(z) = (1+\mu)^{\lambda}\left[\phi(z,\lambda,\mu) - \mu^{-\lambda}\right], \qquad (z \in E), \tag{1.9}$$

and $\phi(z,\lambda,\mu)$ is given by (1.7).

From (1.7) and (1.8), we can write

$$J_{\lambda,\mu}f(z) = z + \sum_{n=2}^{\infty}\left(\frac{1+\mu}{n+\mu}\right)^{\lambda} a_n z^n. \tag{1.10}$$

For the different permissible values of parameters λ and μ, the operator $J_{\lambda,\mu}$ has been studied in [3, 4, 7, 10–12].

We observe some special cases of the operator (1.10) as given below

(i) $J_{0,\mu}f(z) = f(z)$,

(ii) $J_{1,0}f(z) = z + \sum_{n=2}^{\infty}(a_n/n)z^n = \int_0^z (f(t)/t)dt$,

(iii) $J_{1,\mu}f(z) = z + \sum_{n=2}^{\infty}((1+\mu)/(n+\mu))a_n z^n = ((1+\mu)/z^{\mu})\int_0^z t^{\mu-1}f(t)dt$, $(\mu > -1)$.

We remark that $J_{1,1}f(z)$ is the well-known Libera operator and $J_{1,\mu}f(z)$ is the generalized Bernardi operator, see [13, 14]. Also $J_{\lambda,1}f(z) = L_{\lambda}f(z)$ represents the operator closely related to the multiplier transformation studied by Flett [3].

We define the operator $I_{\lambda,\mu} : A \longrightarrow A$ as

$$I_{\lambda,\mu}f(z) * J_{\lambda,\mu}f(z) = \frac{z}{(1-z)}, \qquad (\lambda \text{ real}, \ \mu > -1), \tag{1.11}$$

see [15]. This gives us

$$I_{\lambda,\mu}f(z) = z + \sum_{n=2}^{\infty}\left(\frac{n+\mu}{1+\mu}\right)^{\lambda} a_n z^n, \qquad (\lambda \text{ real}, \ \mu > -1). \tag{1.12}$$

From (1.12), the following identity can easily be verified

$$z\left(I_{\lambda,\mu}f(z)\right)' = (\mu+1)I_{\lambda+1,\mu}f(z) - \mu I_{\lambda,\mu}f(z). \tag{1.13}$$

Remark 1.1.

(i) For $k \in (0,1)$, we note that the domain Ω_k given by (1.3) represents the following hyperbolic region:

$$\left(u + \frac{k^2}{1-k^2}\right)^2 - \frac{k^2}{1-k^2}v^2 > \left(\frac{k}{1-k^2}\right)^2, \qquad u > \frac{k}{k+1}. \tag{1.14}$$

The extremal function $p_{k,\gamma}(z)$, for $0 < k < 1$, can be written as

$$p_{k,\gamma}(z), = (1-\gamma)p_k(z) + \gamma, \tag{1.15}$$

where $p_k(z)$, in a simplified form, is given below

$$\begin{aligned} p_k(z) &= 1 + \frac{1}{2\sin^2\sigma}\left\{\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^{2\sigma/\pi} + \left(\frac{1-\sqrt{z}}{1+\sqrt{z}}\right)^{2\sigma/\pi} - 2\right\}, \\ &= 1 + \frac{1}{2\sin^2\sigma}\sum_{n=1}^{\infty}\left(\sum_{j=0}^{2n}(-1)^j\binom{2\sigma/\pi}{j}\binom{-2\sigma/\pi}{2n-j}\right)z^n, \\ &= 1 + 8\left(\frac{\sigma}{\pi\sin\sigma}\right)^2 z + \cdots, \qquad (z \in E,\ \sigma = \text{arc}\cos k), \end{aligned} \tag{1.16}$$

and the branch of $\sqrt{z}$ is chosen such that $\operatorname{Im}\sqrt{z} \geq 0$.

It is easy to see that, for $h \in P(p_{k,\gamma})$, $\operatorname{Re} h(z) > (k+\gamma)/(k+1)$, $k \in (0,1)$. That is

$$P(p_{k,\gamma}) \subset P\left(\frac{k+\gamma}{1+k}\right), \tag{1.17}$$

and the order $(k+\gamma)/(1+k)$ is sharp with the extremal function $p(z) = (1-\gamma)p_k(z)+\gamma$, where $p_k(z)$ is given by (1.16).

(ii) For $k = 1$, the extremal function

$$\begin{aligned} p_1(z) &= 1 + \frac{2}{\pi^2}\left(\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2, \\ &= 1 + \frac{8}{\pi^2}\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{j=0}^{n-1}\frac{1}{2j+1}\right)z^n, \\ &= 1 + \frac{8}{\pi^2}\left(z + \frac{2}{3}z^2 + \frac{23}{45}z^3 + \frac{44}{105}z^4 + \cdots\right), \qquad (z \in E), \end{aligned} \tag{1.18}$$

maps E conformally onto the parabolic region $\Omega_1 = \{u + iv : u > (v^2+1)/2\}$.

It can easily be verified that $\operatorname{Re} p_1(z) > 1/2$ and, in this case, the order $1/2$ is sharp.

We now define the following.

Definition 1.2. Let $f \in A$ and let the operator $I_{\lambda,\mu} f$ be defined by (1.12). Then $f \in k - \cup R_m^{\gamma}(\lambda,\mu)$ for $m \geq 2$, $k \in [0,1]$ and $\gamma \in [0,1)$ if and only if

$$\left\{ \frac{z(I_{\lambda,\mu} f(z))'}{I_{\lambda,\mu} g(z)} \right\} \in P_m(p_{k,\gamma}), \qquad z \in E. \tag{1.19}$$

We note the following.

(i) For $m = 2$, $k = 0$, and $\lambda = 0$, the class $k - \cup R_m^{\gamma}(\lambda,\mu)$ reduces to $S^*(\gamma)$, and $\lambda = 0$ gives us the class $k - \cup ST$ of uniformly starlike functions, see [2, 16].

(ii) $0 - \cup R_m^{0}(0,\mu) = R_m$ is the class of functions of bounded radius rotation, see [13, 14].

(iii) We denote $k - \cup R_m^{\gamma}(0,\mu)$ as $k - \cup R_m^{\gamma}$, see [5].

(iv) Let $m = 2$. Then $f \in k - \cup R_2^{\gamma}(\lambda,\mu)$ implies that

$$\operatorname{Re}\left\{ \frac{z(I_{\lambda,\mu} f(z))'}{I_{\lambda,\mu} f(z)} \right\} > k\left| \frac{z(I_{\lambda,\mu} f(z))'}{I_{\lambda,\mu} f(z)} - 1 \right| + \gamma, \tag{1.20}$$

and we note that, for $0 \leq k_2 < k_1$, $k_1 - \cup R_2^{\gamma}(\lambda,\mu) \subset k_2 - \cup R_2^{\gamma}(\lambda,\mu)$.

Definition 1.3. Let $f \in A$. Then $f \in k - \cup T_m^{\gamma}(\lambda,\mu)$ if and only if there exists $g \in k - \cup R_2^{\gamma}(\lambda,\mu)$ such that

$$\left\{ \frac{z(I_{\lambda,\mu} f(z))'}{I_{\lambda,\mu} g(z)} \right\} \in P_m(p_{k,\gamma}) \quad \text{in } E. \tag{1.21}$$

Special Cases.

(i) $0 - \cup T_2^{\gamma}(0,\mu) = K(\gamma)$.

(ii) For $k = \gamma = \lambda = 0$, we obtain the class T_m introduced and discussed in [17].

(iii) When we take $m = 2$ and $\lambda = 0$, then $k - \cup T_2^{\gamma}(0,\mu) = k - \cup K_{\gamma}$, the class of uniformly close-to-convex functions, see [2].

2. Preliminary Results

We need the following results in our investigation.

Lemma 2.1 (see [18]). *Let $q(z)$ be convex in E and $j : E \to \mathbb{C}$ with $\operatorname{Re}[j(z)] > 0$, $z \in E$. If $p(z)$, analytic in E with $p(0) = 1$, satisfies*

$$(p(z) + j(z) z p'(z)) \prec q(z), \tag{2.1}$$

then

$$p(z) \prec q(z). \tag{2.2}$$

In the following, one gives an easy extension of a result proved in [1].

Lemma 2.2 (see [5]). *Let $k \geq 0$ and let $\beta,\ \delta$ be any complex numbers with $\beta \neq 0$ and $\mathrm{Re}((\beta k/(k+1)) + \delta) > \gamma$. If $h(z)$ is analytic in E, $h(0) = 1$ and satisfies*

$$\left(h(z) + \frac{zh'(z)}{\beta h(z) + \delta}\right) \prec p_{k,\gamma}(z), \tag{2.3}$$

and $q_{k,\gamma}(z)$ is an analytic solution of

$$q_{k,\gamma}(z) + \frac{zq'_{k,\gamma}(z)}{\beta q_{k,\gamma}(z) + \delta} = p_{k,\gamma}(z), \tag{2.4}$$

then $q_{k,\gamma}(z)$ is univalent,

$$h(z) \prec q_{k,\gamma}(z) \prec p_{k,\gamma}(z), \tag{2.5}$$

and $q_{k,\gamma}(z)$ is the best dominant of (2.3).

Lemma 2.3 (see [19]). *If $f \in \mathbb{C}$, $g \in S^*$, then for each h analytic in E with $h(0) = 1$,*

$$\frac{(f * hg)(E)}{(f * g)(E)} \subset \overline{Co}h(E), \tag{2.6}$$

where $\overline{Co}h(E)$ denotes the convex hull of $h(E)$.

Lemma 2.4 (see [18]). *Let $u = u_1 + i\ u_2$, $v = v_1 + iv_2$ and let $\psi(u,\ v)$ be a complex-valued function satisfying the conditions:*

(i) *$\psi(u,\ v)$ is continuous in a domain $D \subset \mathbb{C}^2$,*

(ii) *$(1,0) \in D$ and $\mathrm{Re}\,\psi(1,0) > 0$,*

(iii) *$\mathrm{Re}\,\psi(iu_2,\ v_1) \leq 0$, whenever $(iu_2,\ v_1) \in D$ and $v_1 \leq -(1/2)(1+u_2^2)$.*

If $h(z) = 1 + c_1 z + c_2 z^2 + \cdots$ is a function analytic in E such that $(h(z), zh'(z)) \in D$ and $\mathrm{Re}\,\psi(h(z), zh'(z)) > 0$ for $z \in E$, then $\mathrm{Re}\,h(z) > 0$ in E.

Lemma 2.5 (see [20]). *Let $h \in P_m(\rho)$, $0 \leq \rho < 1$. Then, with $= re^{i\theta}$, $z \in E$, one has*

(i) *$(1/2\pi)\int_0^{2\pi} |h(re^{i\theta})|^2 d\theta \leq (1 - [m^2(1-\rho)^2 - 1]r^2)/(1-r^2)$,*

(ii) *$(1/2\pi)\int_0^{2\pi} |h'(re^{i\theta})| d\theta \leq m(1-\rho)/(1-r^2)$.*

Lemma 2.6 (see [5]). *Let $f \in k - \cup R_m^\gamma$. Then there exist $s_1, s_2 \in k - \cup R_2^\gamma$ such that*

$$f(z) = \frac{(s_1(z))^{(m+2)/4}}{(s_2(z))^{(m-2)/4}}, \qquad k \geq 0,\ m \geq 2,\ z \in E. \tag{2.7}$$

Lemma 2.7. *Let $p \in P_m(p_{k,\gamma})$ and $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$. Then*

$$|c_n| \leq \frac{m}{2}\left|\delta_{k,\gamma}\right|, \qquad n \geq 1, \tag{2.8}$$

where

$$\delta_{k,\gamma} = \begin{cases} \dfrac{8(1-\gamma)\left(\cos^{-1}k\right)^2}{\pi^2(1-k^2)}, & 0 \leq k < 1, \\ \dfrac{8(1-\gamma)}{\pi^2}, & k = 1. \end{cases} \tag{2.9}$$

Proof. Let $p(z) = (m/4 + 1/2)p_1(z) - (m/4 - 1/2)p_2(z)$. Then,

$$p_i(z) \prec p_{k,\gamma}(z) = 1 + \delta_{k,\gamma} z + \cdots, \qquad i = 1, 2. \tag{2.10}$$

Now the proof follows immediately by using the well-known Rogosinski's result, see [21]. □

3. Main Results

We shall assume throughout, unless stated otherwise, that $k \in [0,1]$, $m \geq 0$, $0 \leq \gamma < 1$, $\lambda \in \mathbb{C}$, $\mu > -1$ and $z \in E$.

Theorem 3.1. *Let $f \in k - \cup R_m^\gamma(\lambda, \mu)$. Let, for $\alpha,\ \beta > 0$,*

$$F(z) = \left[(1+\beta) z^{-\beta} \int_0^z t^{\beta-1'} f^{\alpha}(t) dt\right]^{1/\alpha}. \tag{3.1}$$

Then, $F \in k - \cup R_m^\gamma(\lambda, \mu)$ in E.

Proof. Set

$$\frac{z\left(I_{\lambda,\mu}F(z)\right)'}{I_{\lambda,\mu}F(z)} = H(z) = \left(\frac{m}{4} + \frac{1}{2}\right)H_1(z) - \left(\frac{m}{4} - \frac{1}{2}\right)H_2(z). \tag{3.2}$$

We note $H(z)$ is analytic in E with $H(0) = 1$.

From (3.1), we have

$$\left\{z^{\beta}\left(I_{\lambda,\mu}F(z)\right)^{\alpha}\right\}' = z^{\beta-1}\left(I_{\lambda,\mu}f(z)\right)^{\alpha}. \tag{3.3}$$

That is

$$\left(I_{\lambda,\mu}F(z)\right)^{\alpha}\left[\beta + \alpha H(z)\right] = \left(I_{\lambda,\mu}f(z)\right)^{\alpha}. \tag{3.4}$$

Logarithmic differentiation of (3.4) and simple computations give us

$$H(z) + \frac{zH'(z)}{\alpha H(z) + \beta} = \frac{z\left(I_{\lambda,\mu}f(z)\right)'}{I_{\lambda,\mu}f(z)} \in P_m(p_{k,\gamma}). \tag{3.5}$$

Define

$$\phi_{a,b}(z) = \frac{1}{1+b}\frac{z}{(1-z)^{a+1}} + \frac{b}{b+1}\frac{z}{(1-z)^{a+2}}, \qquad (a > 0, b \geq 0), \tag{3.6}$$

then, with $a = 1/\alpha$, $b = \beta/\alpha$, we have

$$H(z) * \left(\frac{\phi_{a,b}(z)}{z}\right) = \left\{H(z) + \frac{zH'(z)}{\alpha H(z) + \beta}\right\}. \tag{3.7}$$

From (3.2), (3.5), and (3.7), it follows that

$$\left\{H_i(z) + \frac{zH_i'(z)}{\alpha H_i(z) + \beta}\right\} \in P(p_{k,\gamma}), \qquad z \in E,\ i = 1, 2. \tag{3.8}$$

On applying Lemma 2.2, we obtain

$$H_i(z) \prec q_{k,\gamma}(z) \prec p_{k,\gamma}(z) \quad \text{in } E, \tag{3.9}$$

where $q_{k,\gamma}(z)$ is the best dominant and is given as

$$q_{k,\gamma}(z) = \left[\alpha \int_0^1 \left(t^{\beta+\alpha-1} \exp \int_z^{tz} \frac{p_{k,\gamma}(u) - 1}{u} du\right)^{\alpha} dt\right]^{-1} - \frac{\beta}{\alpha}. \tag{3.10}$$

Consequently it follows, from (3.2), that $H \in P_m(p_{k,\gamma})$ and $F \in k - \cup T_m^{\gamma}(\lambda, \mu)$ in E. □

For $k = 0$, $\gamma = 0$, we have the following special case.

Corollary 3.2. *Let $f \in 0 - \cup R_m^0(\lambda,\mu) = R_m(\lambda,\mu)$ and let $F(z)$ be defined by* (3.1). *Then, $F \in R_m^{\gamma_1}(\lambda,\mu)$, where*

$$\gamma_1 = \frac{2}{\left\{(1+2\beta)+\sqrt{(1+2\beta)^2+8\alpha}\right\}}. \tag{3.11}$$

Proof. We write

$$\begin{aligned}\frac{z(I_{\lambda,\mu}F(z))'}{I_{\lambda,\mu}F(z)} &= (1-\gamma_1)H(z)+\gamma_1\\ &= \left(\frac{m}{4}+\frac{1}{2}\right)\{(1-\gamma_1)H_1(z)+\gamma_1\} - \left(\frac{m}{4}-\frac{1}{2}\right)\{(1-\gamma_1)H_2(z)+\gamma_1\},\end{aligned} \tag{3.12}$$

and proceeding as in Theorem 3.1, we obtain

$$\begin{aligned}\frac{z(I_{\lambda,\mu}f(z))'}{I_{\lambda,\mu}f(z)} &= \left(\frac{m}{4}+\frac{1}{2}\right)\left[(1-\gamma_1)\left\{H_1(z)+\frac{\alpha z H_1'(z)}{\alpha H_1(z)+(\alpha\gamma_1+\beta)/(1-\gamma_1)}\right\}+\gamma_1\right]\\ &\quad - \left(\frac{m}{4}-\frac{1}{2}\right)\left[(1-\gamma_1)\left\{H_2(z)+\frac{\alpha z H_2'(z)}{\alpha H_2(z)+(\alpha\gamma_1+\beta)/(1-\gamma_1)}\right\}+\gamma_1\right].\end{aligned} \tag{3.13}$$

We construct the functional $\psi(u,\ v)$ by taking $u = H_i(z)$,$v = zH_i{}'(z)$, as

$$\psi(u,\ v) = u + \frac{v}{\alpha u + (\alpha\gamma_1+\beta)/(1-\gamma_1)} + \frac{\gamma_1}{1-\gamma_1}. \tag{3.14}$$

The first two conditions of Lemma 2.4 can easily be verified. For condition (iii), we proceed as follows:

$$\begin{aligned}\operatorname{Re}\psi(iu_2,\ v_1) &= \frac{\gamma_1}{1-\gamma_1} + \operatorname{Re}\frac{(1-\gamma_1)v_1}{\alpha\gamma_1+\beta+i\alpha(1-\gamma_1)u_2},\\ &= \frac{1}{1-\gamma_1}\left[\gamma_1 + \frac{(1-\gamma_1)(\alpha\gamma_1+\beta)v_1}{(\alpha\gamma_1+\beta)^2+\alpha^2(1-\gamma_1)^2u_2^2}\right],\\ &\le \frac{1}{1-\gamma_1}\left[\gamma_1 - \frac{(1-\gamma_1)(\alpha\gamma_1+\beta)(1+u_2^2)}{2\left\{(\alpha\gamma_1+\beta)^2+\alpha^2(1-\gamma_1)^2u_2^2\right\}}\right], \qquad \left(v_1 \le \frac{-(1+u_2^2)}{2}\right),\\ &= \frac{1}{1-\gamma_1}\left[\frac{A+Bu_2^2}{2C}\right],\end{aligned} \tag{3.15}$$

where $A = 2\gamma_1(\alpha\gamma_1+\beta)^2-(1-\gamma_1)(\alpha\gamma_1+\beta)$, $B = 2\gamma_1\alpha^2(1-\gamma_1)^2-(1-\gamma_1)(\alpha\gamma_1+\beta)$, $C = \{(\alpha\gamma_1+\beta)^2+\alpha^2(1-\gamma_1)^2u_2^2\} > 0$.

The right-hand side of (3.15) is less than equal to zero when $A \le 0$ and $B \le 0$. From $A \le 0$, we obtain γ_1 as given by (3.11), and $B \le 0$ ensures that $\gamma_1 \in [0,1)$.

This shows that all the conditions of Lemma 2.4 are satisfied and therefore $\operatorname{Re} H_i(z) > 0$. This implies $H \in P_m$ and consequently $I_{\lambda,\mu}F \in R_m^{\gamma_1}$. That is $F \in R_m^{\gamma_1}(\lambda,\mu)$ as required. □

By taking $\alpha = 1$, $\beta = 0$, $\lambda = 0$, and $m = 2$, we obtain a well-known result that every convex function is starlike of order $1/2$. Also, for $\beta = 1$, $\lambda = 0$, $\alpha = 1$, and $m = 2$, we obtain from (3.1) the Libera operator and in this case we obtain a known result with $\gamma_1 = 2/(3+\sqrt{17})$ for starlike functions, see [18].

Assigning permissible values to different parameters, we obtain several new and known results from Theorem 3.1 and Corollary 3.2.

Theorem 3.3. *Let $f \in k-\cup T_m^{\gamma}(\lambda,\mu)$ and let $F(z)$ be defined by* (3.1). *Then $F \in k-\cup T_m^{\gamma}(\lambda,\mu)$.*

Proof. We can write (3.1) as

$$\begin{aligned} I_{\lambda,\mu}F(z) &= \left[(1+\beta)z^{-\beta}\int_0^z t^{\beta-1'}\left(I_{\lambda,\mu}f(t)\right)^{\alpha}dt\right]^{1/\alpha}, \qquad f \in k-\cup T_m^{\gamma}(\lambda,\mu), \\ &= \left[\left(\frac{I_{\lambda,\mu}f(z)}{z}\right)^{\alpha} * \frac{h_{\alpha,\beta}(z)}{z}\right]^{1/\alpha}, \end{aligned} \tag{3.16}$$

where

$$h_{\alpha,\beta}(z) = \sum_{n=1}^{\infty}\frac{z^n}{n+\alpha+\beta}, \tag{3.17}$$

is convex in E.

Let $f \in k-\cup T_m^{\gamma}(\lambda,\mu)$. Then there exists some $g \in k-\cup R_2^{\gamma}(\lambda,\mu)$ such that

$$\left\{\frac{z(I_{\lambda,\mu}f(z))'}{I_{\lambda,\mu}g(z)}\right\} \in P_m(p_{k,\gamma}). \tag{3.18}$$

$$G(z) = \left[(\beta+1)z^{-\beta}\int_0^z t^{\beta-1}g^{\alpha}(t)dt\right]^{1/\alpha}. \tag{3.19}$$

From Theorem 3.1, it follows that $G \in k-\cup R_2^{\gamma}(\lambda,\mu)$. We can write (3.19) as

$$I_{\lambda,\mu}G(z) = z\left[\frac{I_{\lambda,\mu}g(z)}{z} * \frac{h_{\alpha,\beta}(z)}{z}\right]^{1/\alpha}, \tag{3.20}$$

where $h_{\alpha,\beta}(z)$, given by (3.17), is convex in E.

Since $g \in k - \cup R_2^{\gamma}(\lambda, \mu)$, so $I_{\lambda,\mu} g \in S^*((k+\gamma)/(k+1)) \subset S^*$. It can easily be shown that $z(I_{\lambda,\mu} g/z)^{\alpha}$ and $z(I_{\lambda,\mu} G/z)^{\alpha}$ are in the class S^*.

Now, from (3.1), we have

$$\begin{aligned}
\frac{z(I_{\lambda,\mu}F(z))'(I_{\lambda,\mu}F(z))^{\alpha-1}}{(I_{\lambda,\mu}G(z))^{\alpha}} &= \frac{h_{\alpha,\beta}(z) * z(I_{\lambda,\mu}g(z)/z)^{\alpha}\left(z(I_{\lambda,\mu}f(z))'(I_{\lambda,\mu}f(z))^{\alpha-1}/(I_{\lambda,\mu}g(z))^{\alpha}\right)}{h_{\alpha,\beta}(z) * z(I_{\lambda,\mu}g(z)/z)^{\alpha}} \\
&= \left(\frac{m}{4}+\frac{1}{2}\right)\frac{\left[h_{\alpha,\beta}(z) * z(I_{\lambda,\mu}g(z)/z)^{\alpha}h_1(z)\right]}{h_{\alpha,\beta}(z) * z(I_{\lambda,\mu}g(z)/z)^{\alpha}} \\
&\quad - \left(\frac{m}{4}-\frac{1}{2}\right)\frac{\left[h_{\alpha,\beta}(z) * z(I_{\lambda,\mu}g(z)/z)^{\alpha}h_2(z)\right]}{h_{\alpha,\beta}(z) * z(I_{\lambda,\mu}g(z)/z)^{\alpha}}.
\end{aligned} \tag{3.21}$$

We use Lemma 2.3 with $h_i \prec p_{k,\gamma}, \ i = 1, 2$, to have

$$\left\{\frac{h_{\alpha,\beta}(z) * z(I_{\lambda,\mu}g(z)/z)^{\alpha}h_i(z)}{h_{\alpha,\beta}(z) * z(I_{\lambda,\mu}g(z)/z)^{\alpha}}\right\} \prec p_{k,\gamma}(z) \quad \text{in } E. \tag{3.22}$$

Thus from (3.19), (3.21), and (3.22) we obtain the required result that $F \in k - \cup T_m^{\gamma}(\lambda, \mu)$. This completes the proof. □

As a special case we note that, for $\lambda = 0 = k$, the subclass $T_m^{\gamma} \subset T_m$ is invariant under the integral operator defined by (3.1).

Theorem 3.4. *One has*

$$k - \cup R_m^{\gamma}(\lambda+1, \mu) \subset k - \cup R_m^{\gamma}(\lambda, \mu). \tag{3.23}$$

Proof. Let $f \in k - \cup R_m^{\gamma}(\lambda+1, \mu)$ and let

$$\frac{z(I_{\lambda,\mu}f(z))'}{I_{\lambda,\mu}f(z)} = H(z), \tag{3.24}$$

where $H(z)$ is analytic in E and is defined by (3.2).

Then, from (1.13), we have

$$\frac{z(I_{\lambda+1,\mu}f(z))'}{I_{\lambda+1,\mu}f(z)} = \left\{H(z) + \frac{zH'(z)}{H(z)+\mu}\right\} \in P_m(p_{k,\gamma}). \tag{3.25}$$

Applying similar technique used before, we have from (3.2) and (3.7) for $i = 1, 2$

$$\left\{H_i(z) + \frac{zH_i'(z)}{H_i(z)+\mu}\right\} \prec p_{k,\gamma}. \tag{3.26}$$

Thus, using Lemma 2.2, it follows that $H_i \prec p_{k,\gamma}$, $i = 1,2$ and $z \in E$, consequently $H \in P_m(p_{k,\gamma})$ in E and this completes the proof. □

As special cases, we have the following.

(i) Let $m = 2$, $\lambda \geq 0$. Then, from Theorem 3.4, it easily follows that

$$k - \cup R_2^{\gamma}(\lambda, \mu) \subset k - \cup R_2^{\gamma}(0, \mu) \subset S^*\left(\frac{k+\gamma}{1+k}\right) \subset S^*. \tag{3.27}$$

(ii) Let $k = 0$ and $\lambda \geq 0$. Then $f \in 0 - \cup R_m^{\gamma}(\lambda, \mu)$ implies $f \in R_m^{\gamma} \subset R_m$, that is, $f(z)$ is a function of bounded radius rotation in E.

Theorem 3.5. *One has*

$$k - \cup T_m^{\gamma}(\lambda + 1, \mu) \subset k - \cup T_m^{\gamma}(\lambda, \mu), \qquad \mu, \lambda \geq 0. \tag{3.28}$$

Proof. Let $f \in k - \cup T_m^{\gamma}(\lambda + 1, \mu)$. Then, for $z \in E$,

$$\frac{z(I_{\lambda+1,\mu} f(z))'}{I_{\lambda+1,\mu} g(z)} \in P_m(p_{k,\gamma}), \tag{3.29}$$

for some $g \in k - \cup R_2^{\gamma}(\lambda + 1, \mu)$.

We define an analytic function $h(z)$ in E such that

$$\frac{z(I_{\lambda,\mu} f(z))'}{I_{\lambda,\mu} g(z)} = h(z) = \left(\frac{m}{4} + \frac{1}{2}\right)h_1(z) - \left(\frac{m}{4} - \frac{1}{2}\right)h_2(z), \tag{3.30}$$

where $h(0) = 1$. We shall show that $h \in P_m(p_{k,\gamma})$ in E.

Since $g \in k - \cup R_2^{\gamma}(\lambda + 1, \mu)$ and $k - \cup R_2^{\gamma}(\lambda + 1, \mu) \subset k - \cup R_2^{\gamma}(\lambda, \mu)$, we have

$$\frac{z(I_{\lambda,\mu} g(z))'}{I_{\lambda,\mu} g(z)} = h_0(z), \qquad h_0 \in P(p_{k,\gamma}),\ z \in E. \tag{3.31}$$

Now, on using (1.13), we have

$$\begin{aligned} \frac{z(I_{\lambda+1,\mu} f(z))'}{I_{\lambda+1,\mu} g(z)} &= \frac{(1/(\mu+1))z\left[I_{\lambda,\mu}(zf'(z))'\right] + (\mu/(\mu+1))\left[I_{\lambda,\mu} zf'(z)\right]}{(1/(\mu+1))z(I_{\lambda,\mu} g(z))' + (\mu/(\mu+1))I_{\lambda,\mu} g(z)}, \\ &= \frac{\left[\left(z\left[I_{\lambda,\mu}(zf'(z))'\right]/I_{\lambda,\mu} g(z)\right) + \mu h(z)\right]}{h_0(z) + \mu}. \end{aligned} \tag{3.32}$$

Differentiation of (3.30) gives us

$$\frac{z\left(z\left(I_{\lambda,\mu}f(z)\right)'\right)'}{I_{\lambda,\mu}g(z)} = zh'(z) + (h(z))(h_0(z)), \tag{3.33}$$

and using (3.33) in (3.32), we obtain

$$\begin{aligned}\frac{z\left(I_{\lambda+1,\mu}f(z)\right)'}{I_{\lambda+1,\mu}g(z)} &= h(z) + \frac{zh'(z)}{h_0(z)+\mu} \\ &= \left(\frac{m}{4}+\frac{1}{2}\right)\left[h_1(z) + \frac{zh_1'(z)}{h_0(z)+\mu}\right] - \left(\frac{m}{4}-\frac{1}{2}\right)\left[h_2(z) + \frac{zh_2'(z)}{h_0(z)+\mu}\right].\end{aligned} \tag{3.34}$$

Since $f \in k - \cup T_m^{\gamma}(\lambda+1,\mu)$, we have with $1/H_0(z) = \{h_0(z)+\mu\} \in P$,

$$\left\{h_i(z) + H_0(z)\left[zh_i'(z)\right]\right\} \prec p_{k,\gamma}(z) \quad \text{in } E, \tag{3.35}$$

and thus, applying Lemma 2.1, we have $h_i(z) \prec p_{k,\gamma}(z)$ in E. This shows $h \in P_m(p_{k,\gamma})$ in E and consequently $f \in k - \cup T_m^{\gamma}(\lambda,\mu)$. □

As a special case, we note that $f \in k - \cup T_2^{\gamma}(\lambda,\mu)$ is a close-to-convex function for $z \in E$.

Theorem 3.6. *Let $f \in k - \cup R_2^{\gamma}(\lambda,\mu)$ and let $\phi(z)$ be convex in E. Then $(f * \phi) \in k - \cup R_2^{\gamma}(\lambda,\mu)$ for $z \in E$.*

Proof. We have

$$\begin{aligned}\frac{z\left(I_{\lambda,\mu}(\phi * f)\right)'}{I_{\lambda,\mu}(\phi * f)} &= \frac{\phi * z\left[I_{\lambda,\mu}f\right]'}{\phi * I_{\lambda,\mu}f}, \\ &= \frac{\phi * \left[z\left(I_{\lambda,\mu}f\right)'/I_{\lambda,\mu}f\right]I_{\lambda,\mu}f}{\phi * I_{\lambda,\mu}f}.\end{aligned} \tag{3.36}$$

Now $k - \cup R_2^{\gamma}(\lambda,\mu) \subset S^*((k+\gamma)/(1+k)) \subset S^*$, and ϕ is a convex in E, we use Lemma 2.3 to (3.36) and conclude that $(\phi * f) \in k - \cup R_2^{\gamma}(\lambda,\mu)$ for $z \in E$. This completes the proof. □

Remark 3.7. Following the similar technique, we can easily extend Theorem 3.6 to the class $k - \cup T_m^{\gamma}(\lambda,\mu)$, that is, $k - \cup T_m^{\gamma}(\lambda,\mu)$, is invariant under convolution with convex function.

3.1. Applications of Theorem 3.6

The classes $k - \cup R_2^{\gamma}(\lambda,\mu)$ and $k - \cup T_m^{\gamma}(\lambda,\mu)$ are preserved under the following integral operators:

(1) $f_1(z) = \int_0^z (f(t)/t)dt = (\phi_1 * f)(z)$, where $\phi_1(z) = -\log(1-z)$,

(2) $f_2(z) = (2/z)\int_0^z f(t)dt = (\phi_2 * f)(z)$, where $\phi_2(z) = -2[z - \log(1-z)]/z$,

(3) $f_3(z) = \int_0^z ((f(t) - f(tx))/(t - tx))dt = (\phi_3 * f)(z)$, $|x| \le 1$, $x \ne 1$, where $\phi_3(z) = (1/(1-x))\log((1-xz)/(1-z))$, $|x| \le 1$, $x \ne 1$,

(4) $f_4(z) = ((1+c)/z^c)\int_0^z t^{c-1}f(t)dt = (\phi_4 * f)(z)$, $\operatorname{Re} c > 0$, where $\phi_4(z) = \sum_{n=1}^{\infty}((1+c)/(n+c))z^n$, $\operatorname{Re} c > 0$,

The proof is immediate since $\phi_i(z)$ is convex in E for $i = 1, 2, 3, 4$.

With essentially the same method together with Lemma 2.7, we can easily prove the following sharp coefficient results.

Theorem 3.8. *Let $f \in k - \cup R_m^{\gamma}(\lambda, m)$ and let it be given by*

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{3.37}$$

Then

$$|a_n| \leq \frac{m}{2[(n-1)!]}\left[\left(\frac{1+\mu}{n+\mu}\right)^{\lambda}(|\delta_{k,\gamma}|)_{n-1}\right], \qquad (n \geq 2), \tag{3.38}$$

where $(\rho)_n$ is Pochhamer symbol defined, in terms of Gamma function Γ, by

$$(\rho)_n = \frac{\Gamma(n+\rho)}{\Gamma(\rho)} = \begin{cases} 1, & n = 0, \\ \rho(\rho+1)(\rho+2)\cdots(\rho+n-1), & n \in N, \end{cases} \tag{3.39}$$

and $\delta_{k,\gamma}$ is as given by (2.9).

As special case, one notes that

(i) *$\lambda = 0$, $m = 2$, then one has*

$$|a_n| \leq \frac{(|\delta_{k,\gamma}|)_{n-1}}{(n-1)!}, \tag{3.40}$$

see [2].

(ii) *Let $\lambda = 0$, $n = 2$. Then,*

$$|a_n| \leq \frac{m}{2}|\delta_{k,\gamma}|. \tag{3.41}$$

This coefficient bound is well known for $m = 2$, see [2].

Using Theorem 3.8 with $m = 2$, the following result can easily be proved.

Theorem 3.9. *Let $f : f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in k - \cup T_m^{\gamma}(\lambda, \mu)$. Then, for $n \geq 2$*

$$|a_n| \leq \left(\frac{1+\mu}{n+\mu}\right)^{\lambda}\left[\frac{(|\delta_{k,\gamma}|)_{n-1}}{n!} + \frac{m|\delta_{k,\gamma}|}{2n}\sum_{j=1}^{n-1}\frac{(|\delta_{k,\gamma}|)_{j-1}}{(j-1)!}\right], \tag{3.42}$$

where $\delta_{k,\gamma}$ is as given by (2.9).

By assigning different permissible values to the parameters, we obtain several known results, see [2, 22].

We now prove the following.

Theorem 3.10. *Let* $f : f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in k - \cup R_m^{\gamma}(\lambda, \mu)$ *with* $(m+2)(1-\gamma)/2(1+k) > 1$. *Then, for* $n \geq 1$,

$$||a_{n+1}| - |a_n|| \leq c_1(m, \gamma, k, \lambda, \mu) n^{[\{((m/2)+1)((1-\gamma)/(1+k))\}-(2+\lambda)]}, \tag{3.43}$$

where $c_1(m, \gamma, k, \lambda, \mu)$ *is a constant.*

Proof. Let $F(z) = I_{\lambda,\mu} f(z) = z + \sum_{n=2}^{\infty} A_n z^n$, $A_n = ((n+\mu)/(1+\mu))^{\lambda} a_n$. Since $f \in k - \cup R_m^{\gamma}(\lambda, \mu)$, we can write

$$zF'(z) = F(z)h(z), \qquad h \in P_m(p_{k,\gamma}). \tag{3.44}$$

That is

$$z(zF'(z))' = F(z)\left[h^2(z) + zh'(z)\right]. \tag{3.45}$$

Now, $F \in k - \cup R_m^{\gamma} \subset R_m((k+\gamma)/(1+k))$, and it follows from a result proved in [5] that there exist $s_1, s_2 \in S^*((k+\gamma)/(k+1))$ such that

$$\begin{aligned} F(z) &= \frac{(s_1(z))^{(m+2)/4}}{(s_2(z))^{(m-2)/4}}, \qquad m \geq 2, \\ &= \frac{\left((g_1(z))^{(1-(k+\gamma)/(k+1))}\right)^{(m+2)/4}}{\left((g_2(z))^{(1-(k+\gamma)/(k+1))}\right)^{(m-2)/4}}, \qquad g_1, g_2 \in S^*, \\ &= \frac{(g_1(z))^{((1-\gamma)/(k+1))((m+2)/4)}}{(g_2(z))^{((1-\gamma)/(k+1))((m-2)/4)}}. \end{aligned} \tag{3.46}$$

Thus, for $\xi \in E$ and $n \geq 1$,

$$\begin{aligned} \left|(n+1)^2 \xi A_{n+1} - n^2 A_n\right| &\leq \frac{1}{2\pi r^{n+2}} \int_0^{2\pi} |z-\xi| \left|z(zF'(z))'\right| d\theta \\ &= \frac{1}{2\pi r^{n+2}} \int_0^{2\pi} |z-\xi||F(z)| \left|h^2(z) + zh'(z)\right| d\theta \\ &= \frac{1}{2\pi r^{n+2}} \int_0^{2\pi} |z-\xi| \left|\frac{(g_1(z))^{(m/4+1/2)}}{(g_2(z))^{(m/4-1/2)}}\right|^{((1-\gamma)/(k+1))} \left|h^2(z) + zh'(z)\right| d\theta, \end{aligned} \tag{3.47}$$

where $g_1, g_2 \in S^*$ and $h \in P_m(p_{k,\gamma})$ in E.

Let $0 < r < 1$. Then, by a result [23], there exists a ξ with $|\xi| = r$ such that, for z, $|z| = r$

$$|z-\xi||g_1(z)| \leq \frac{2r^2}{1-r^2}. \tag{3.48}$$

We now use (3.48), distortion theorems for starlike functions g_1, g_2 for $((m/2)+1)((1-\gamma)/(1+k)) > 1$, and Lemma 2.5 with $\rho = (k+\rho)/(k+1), r = (1-1/n)$, $n \to \infty$ and obtain from (3.47),

$$\left|(n+1)^2\xi\left(\frac{n+\mu+1}{1+\mu}\right)^{\lambda} a_{n+1} - n^2\left(\frac{n+\mu}{1+\mu}\right)^{\lambda} a_n\right| \leq C(m,\gamma,k)n^{\{((m/2)+1)((1-\gamma)/(1+k))\}}. \tag{3.49}$$

From (3.48), we easily obtain the required result given by (3.43), $(n \to \infty)$.

This completes the proof. □

Using the similar technique, we can easily prove the following.

Theorem 3.11. *Let* $f : f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in k - \cup T_m^{\gamma}(\lambda,\mu)$. *Then*

$$a_n = O(1)n^{\{2((1-\gamma)/(1+k))-\lambda-1\}}, \qquad (n \longrightarrow \infty), \tag{3.50}$$

where $O(1)$ *is a constant depending on* γ, k, m, μ, *and* λ *only. The exponent* $\{2((1-\gamma)/(1+k)) - \lambda - 1\}$ *is best possible.*

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