Research Article

Characterization of Eigenvalues in Spectral Gap for Singular Differential Operators

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The spectral properties for \( n \)th-order differential operators are considered. When given a spectral gap \((a, b)\) of the minimal operator \( T_0 \) with deficiency index \( r \), arbitrary \( m \) points \( \beta_i \) (\( i = 1, 2, \ldots, m \)) in \((a, b)\), and a positive integer function \( p \) such that \( \sum_{i=1}^{m} p(\beta_i) \leq r \), \( T_0 \) has a self-adjoint extension \( \tilde{T} \) such that each \( \beta_i \) (\( i = 1, 2, \ldots, m \)) is an eigenvalue of \( \tilde{T} \) with multiplicity at least \( p(\beta_i) \).

1. Introduction

In this paper, we consider the following \( n \)th-order formal symmetric differential expression:

\[
\tau y = w^{-1} \left\{ \sum_{j=0}^{[n/2]} (-1)^j \left(y^{(j)}(t)\right)^{(j)} + \sum_{j=0}^{[n-1/2]} (-1)^j \left( q_j y^{(j)}(t) - y^{(j+1)}(t) \right)^{(j)} \right\},
\]

on \((\alpha, \beta)\), where \(-\infty \leq \alpha < \beta \leq +\infty\), \( y = y(t) \) is a complex-valued \( m \)-vector function, \( p_j(t), q_j(t) \) and \( w(t) \) are measurable and locally integrable \( m \times m \) matrices, \( p_j(t), w(t) \) are Hermitian, and \( w(t) > 0 \) a.e.

The spectral properties of \( n \)th-order differential expression, particularly the distribution of eigenvalues, have been widely researched in these years (see [1–8] and references cited therein).

Let us recall some known results due to Neumann [1], Stone [2], Friedrichs [3] and Krein [4]. For convenience of the reader in addition to these original sources we will also give text book references [7].
An open interval \((a, b)\) with \(-\infty < a < b < \infty\) is called a spectral gap of a symmetric operator \(A\) if

\[
\left\| \left( A - \frac{a + b}{2} \right) f \right\|_w \geq \frac{b - a}{2} \left\| f \right\|_w \quad \forall f \in D(A).
\]  

\(\| \cdot \|_w\) and \(\langle \cdot, \cdot \rangle_w\) are defined in Section 2. It is easy to find that \((a + b)/2\) is a real regular point of \(A\). If \(\langle Af, f \rangle_w \geq b\|f\|_w^2\), we shall also say that \((-\infty, b)\) is a spectral gap of \(A\) (the latter definition is not generally used but convenient for our purpose; and it is justified since each interval \((a, b)\) with \(a < b\) is a spectral gap of \(A\)). Let \((a, b)\) be a spectral gap of \(A\), then there exists a self-adjoint extension \(\tilde{A}\) of \(A\) (for instance the famous Friedrichs extension) such that \((a, b) \subset \Gamma(\tilde{A})\) (here \(\Gamma(\tilde{A})\) is the regular-form domain of \(\tilde{A}\) which will be defined in Definition 2.4), this is the reason why we call \((a, b)\) a spectral gap of \(A\).

Suppose in addition that the deficiency index of a symmetric operator \(A\) is equal to \(n\) \((n \in \mathbb{N})\). Let \(\tilde{A}\) be a self-adjoint extension of \(A\). The sum of the multiplicities of the eigenvalues of \(\tilde{A}\) within the interval \((a, b)\) is at most \(n\), and no point of the continuous spectrum of \(\tilde{A}\) lies in the interval \((a, b)\) (cf. [5, Theorem 8.19 and Corollary 2 in Section 8.3]). Conversely, given any finite subset \(\mathcal{E}\) of \((a, b)\) and positive integers \(p(\lambda)\), \(\lambda \in \mathcal{E}\), such that \(\sum_{\lambda \in \mathcal{E}} p(\lambda) \leq n\) there exists a self-adjoint extension \(\tilde{A}\) of \(A\) such that \(\sigma(\tilde{A}) \cap (a, b) = \mathcal{E}\) and the multiplicities of \(\lambda\) as an eigenvalue of \(\tilde{A}\) equal \(p(\lambda)\) for each \(\lambda \in \mathcal{E}\) (cf. [4]). If one only requires that \((a, b)\) is some interval within the set of real regular points of \(A\), then the corresponding statement is false. For instance, one can give a symmetric operator \(\tilde{A}\), such that each \(\lambda \in \mathbb{R}\) is a regular point of \(\tilde{A}\), the deficiency index of \(\tilde{A}\) is equal one and each self-adjoint extension \(\tilde{A}\) of \(\tilde{A}\) has a periodic point spectrum with period \(p\) independent of \(\tilde{A}\) (cf. [4]).

This paper consists of three sections including the introduction. In Section 2, we present some preliminary materials that include definitions and theorems needed for the rest of the paper. In Section 3, we give three main results in the study of self-adjoint extensions of a minimal operator generated by differential expression \(\tau\). Firstly, we present a partial self-adjointness of the minimal operator \(T_0\). Secondly, if \((a, b)\) is a spectral gap of \(T_0\), for all \(\beta_1, \beta_2, \ldots, \beta_m \in (a, b)\) \((m \leq r, r\) is the deficiency index of \(T_0)\) and positive integer function \(p\) satisfying \(\sum_{i=1}^{m} p(\beta_i) = r\), \(T_0\) has a self-adjoint extension \(\tilde{T}\) with the following properties:

(i) \(\sigma_{p}(\tilde{T}) \cap (a, b) = \{\beta_1, \beta_2, \ldots, \beta_m\}\);

(ii) each \(\beta_i\) \((i = 1, 2, \ldots, m)\) is an eigenvalue of \(\tilde{T}\) with multiplicity equal to \(p(\beta_i)\);

(iii) \(\tilde{T}\) has pure point spectrum within \((a, b)\).

Finally, given a symmetric operator \(T_0\) with the deficiency index being equal to \(r\), \((a, b) \subset \mathbb{R}\), \(\beta_1, \beta_2, \ldots, \beta_m \in (a, b)\) \((m \leq r)\) being real regular points of \(T_0\), the interval \((a, b)\) need not be a spectral gap of \(T_0\) and positive integer function \(p\) satisfying \(\sum_{i=1}^{m} p(\beta_i) \leq r\), \(T_0\) has a self-adjoint extension \(\tilde{T}\) with the properties that each \(\beta_i\) \((i = 1, 2, \ldots, m)\) is an eigenvalue of \(\tilde{T}\) with multiplicity at least \(\beta_i\).
Abstract and Applied Analysis

2. Preliminaries

In this section, we introduce notations, definitions, and some theorems that are needed in this paper.

First, we define the following space:

\[
L_2(\alpha, \beta; w) := \left\{ f : \int_\alpha^\beta f(t)w(t)f(t) < +\infty \right\},
\]

with the inner product

\[
\langle f, g \rangle_w = \int_\alpha^\beta g(t)f(t)w(t)dt,
\]

where the weight function \( w(t) \) is the same as that in (1.1). Denote \( \|f\|_w = ((f, f)_w)^{1/2} \) for \( f \in L_2(\alpha, \beta; w) \). If \( f \in L_2(\alpha, \beta; w) \), then \( f \) is called square integrable. Here, we note that if \( w \) is singular, \( L_2(\alpha, \beta; w) \) is a quotient space in the sense that \( y = z \) if and only if \( \|y - z\|_w = 0 \). In this case, \( L_2(\alpha, \beta; w) \) is a Hilbert space.

Now we introduce the maximal operator \( T \) and minimal operator \( T_0 \) generated by the expression \( \tau \).

Definition 2.1. The maximal operator \( T \) generated by \( \tau \) is defined by

\[
D(T) = \left\{ f \in L_2(\alpha, \beta; w) : f^{(0)}, f^{(1)}, \ldots, f^{(n-1)} \in AC_{loc}(\alpha, \beta), \; \tau f \in L_2(\alpha, \beta; w) \right\},
\]

\[
Tf = \tau f \quad \text{for } f \in D(T),
\]

where \( AC_{loc}(\alpha, \beta) \) denotes the collection of functions on \( (\alpha, \beta) \) which are absolutely continuous locally. Roughly speaking the \( j \)th quasi-derivative \( u^{(j)} \) will be collection of terms that, if differentiated \( (n - j) \) times, is “part” of the differential expression \( \tau \tau u \) (see [7] for details). We know that \( T \) is densely defined and closed.

Definition 2.2. The preminimal operator \( T_{00} \) generated by \( \tau \) is defined by

\[
D(T_{00}) = \left\{ f \in D(T) : f \text{ has compact support in } (\alpha, \beta) \right\},
\]

\[
T_{00}f = \tau f = Tf \quad \text{for } f \in D(T_{00}).
\]

Obviously, \( T_{00} \subset T \) and \( T_{00} \) is Hermitian. It is easy to know that \( D(T_{00}) \) is dense, so \( T_{00} \) is symmetric, and \( T_{00} \) is not closed [8]. The closure of \( T_{00} \) is called the (closed) minimal operator denoted by \( T_0 \).

Definition 2.3. Given a linear operator \( A \) with domain and range in a Hilbert space \( \mathbb{H} \), the resolvent set of \( A \) is

\[
\rho(A) = \left\{ \lambda \in \mathbb{C} \mid (\lambda - A)^{-1} \in B(\mathbb{H}) \right\},
\]
the spectrum of \( A \) is the set \( \sigma(A) = \mathbb{C} \setminus \rho(A) \). Denote

\[
\sigma_p = \{ \lambda : (\lambda - A)^{-1} \text{ does not exist} \},
\]

\[
\sigma_c = \{ \lambda : (\lambda - A)^{-1} \text{ exists and } \overline{R(\lambda - A)} = \mathbb{H}, \text{but } (\lambda - A)^{-1} \text{ is not continuous} \},
\]

\[
\sigma_r = \{ \lambda : (\lambda - A)^{-1} \text{ exists and } \overline{R(\lambda - A)} \neq \mathbb{H} \}.
\]

Obviously, \( \sigma(A) = \sigma_p \cup \sigma_c \cup \sigma_r \) (see [8]).

**Definition 2.4.** Let \( A \) be a linear operator in a Hilbert space \( \mathbb{H} \). The set of regular points of \( A \), is called the regular-form domain of \( A \), denoted by \( \gamma(A) \):

\[
\Gamma(A) = \{ z \in \mathbb{C} : A - z \text{ is continuously invertible} \}
\]

\[
= \{ z \in \mathbb{C} : \text{there exists } a \, v = v(z) > 0 \text{ such that } ||(A - z)f|| \geq v||f||, \forall f \in D(A) \}
\]

(2.6)

it is an open subset of \( \mathbb{C} \) containing \( \mathbb{C} \setminus \mathbb{R} \).

**Definition 2.5.** For a closed operator \( A \) in a Hilbert space \( \mathbb{H} \), the essential spectrum of \( A \) is defined as

\[
\sigma_c(A) = \{ \lambda \in \sigma(A) : R(\lambda - A) \neq \overline{R(\lambda - A)} \}.
\]

(2.7)

We assume that \( T_0 \) has real regular points, that is, \( \Gamma(T_0) \cap \mathbb{R} \neq \emptyset \). In this case the deficiency index

\[
n(T_0, z) = n(T_0) := \dim \ker(T - z) = \dim R(T_0 - z)^\perp
\]

(2.8)

does not depend on the special choice of the regular point of \( T_0 \) and consequently \( T_0 \) has self-adjoint extensions.

The kernel \( N(T - z) \) is the vector space of those solutions of the differential equation \((\tau - z)u = 0\) which are elements of \( L_2(a, \beta; u) \). Since the space of all solutions of \((\tau - z)u = 0\) (which in general is not contained in \( L_2(a, \beta; u) \)) has dimension \( p := n \times m \), this implies

\[
0 \leq n(T_0, z) \leq p.
\]

(2.9)

**Definition 2.6.** We say that an operator \( A \) has pure point spectrum within \((a, b)\) if \( \sigma(A) \cap (a, b) = \overline{\sigma_p(A)} \cap (a, b) \).

**Proposition 2.7.** Let \( A \) be a self-adjoint operator in a Hilbert space \( \mathbb{H} \). One has \( \rho(A) = \Gamma(A) \).

**Proof.** From Definitions 2.3–2.5, we have \( \rho(A) \subseteq \Gamma(A) \) and \( \sigma_p, \sigma_c \) do not belong to \( \Gamma(A) \), so \( \rho(A) \cup (\sigma_r \cap \Gamma(A)) = \Gamma(A) \). \( A \) is densely defined since it is a self-adjoint operator, so \( \sigma_r = \emptyset \), then we get \( \rho(A) = \Gamma(A) \). \( \square \)

The following results give the relation between deficiency index and self-adjoint extension.
Proposition 2.8. If $T_{00}$, $T_0$, $T$ are the preminimal operator, minimal operator and maximal operator generated by $\tau$, and $\tilde{T}$ is one of the self-adjoint extension of $T_0$, then One has:

$$T_{00} \subset \overline{T_{00}} = T_0 \subset \tilde{T} = (\tilde{T})^* \subset T = T_0^*.$$  \hfill (2.10)

Proposition 2.9. If $A$ is Hermitian, then $C \setminus \mathbb{R} \subset \Gamma(A)$ (cf. [5, Proposition 2, page 229]).

Proposition 2.10. The deficiency index is constant on each connected subset of $\Gamma(A)$. If $A$ is Hermitian, then the deficiency index is constant in the upper and lower half-planes (cf. [5, Theorem 8.1]).

Proposition 2.11. A closed symmetric operator is self-adjoint if and only if its deficiency index is equal to 0.

Proposition 2.12. Let $A$ be a symmetric operator,

(a) $A$ has self-adjoint extension if and only if its deficiency indices are equal,

(b) $\Gamma(A) \cap \mathbb{R} \neq \emptyset$, then $A$ has self-adjoint extensions,

(c) if $A$ is semibounded, then $A$ has self-adjoint extensions,

We can find the proof from Theorem 8.8 in [7].

Proposition 2.13 (see [8]). Let $A$ be a closed symmetric operator in $L_2(\alpha, \beta; \omega)$ and $A_1 = A^*$. Set $N_0(\lambda - A_1) = N(\lambda - A_1) \cap D(A)$. If there exists a $\lambda \in \mathbb{R}$ such that $\lambda \notin \sigma_e(A)$, then $A$ has a self-adjoint extension and

$$\dim N(\lambda - A_1) = n(A) + \dim N_0(\lambda - A_1).$$  \hfill (2.11)

Furthermore, there exists a self-adjoint extension $\tilde{A}$ of $A$ such that

$$[N(\lambda - A_1) \oplus N_0(\lambda - A_1)] \cap D(\tilde{A}) = \{0\}. $$  \hfill (2.12)

3. Main Results

Let $T_0$, $T$ be the minimal operator and maximal operator generated by $\tau$ in (1.1) and let the deficiency index of $T_0$ be equal to $r$ ($0 < r \leq p$). In this section we assume that $T_0$ has real regular points. That is, $\Gamma(T_0) \cap \mathbb{R} \neq \emptyset$.

Definition 3.1. A closed subspace $\mathcal{M}$ of $L_2(\alpha, \beta; \omega)$ is called a reducing subspace of $T_0$ if $Pf \in D(T_0)$, and $T_0Pf = PT_0f$ for all $f \in D(T_0)$, where $P$ denotes the orthogonal projection in $D(T_0)$ onto $\mathcal{M}$.

Obviously along with $\mathcal{M}$ the orthogonal complement $\mathcal{M}^\perp$ of $\mathcal{M}$ is also a reducing subspace of $T_0$. It is easy to see that the closed span of an orthogonal system of eigenvectors of $T_0$ is a reducing subspace of $T_0$. If $\mathcal{M}$ is a reducing subspace of $T_0$, then the part of $T_0$ in $\mathcal{M}$, that is, the restriction of $T_0$ to $D(T_0) \cap \mathcal{M}$ may be (and in the following will be) regarded as an operator in the Hilbert space $\mathcal{M}$. 
Since $\Gamma(T_0) \cap \mathbb{R} \neq \emptyset$, there exists a real point $\lambda$ of $T_0$, and the positive and negative deficiency indices of $T_0$ are equal. We choose a one-dimensional subspace $\mathcal{L}_\Lambda$ of $\ker(T_0 - \lambda)$ ($= \ker(T - \lambda)$), set

$$D(T_0) := D(T_0) + \mathcal{L}_\Lambda,$$

$$T_0'(f + g) := T_0f + \lambda g \quad \forall f \in D(T_0) \text{ and } \forall g \in \mathcal{L}_\Lambda.$$  \hfill (3.1)

Obviously $T_0'$ is a symmetric extension of $T_0$, and $n(T_0', z) = n(T_0, z) - 1$ for each regular point $z$ of $T_0$ and $\mathcal{L}_\Lambda$ the corresponding eigenspace. Since the graph of $T_0'$ is a one-dimensional extension of the closed graph of $T_0$, therefore the operator $T_0'$ is closed [6].

**Theorem 3.2.** Assume that the deficiency index of $T_0$ is equal to $r$, let $(a, b)$ be a spectral gap of $T_0$, and let $T_0'$ be defined as above.

1. If $T_0'$ has an eigenvalue $\mu \in (a, b)$, then $(a, b)$ is a spectral gap of the restriction of $T_0'$ to the Hilbert space $\ker (T_0' - \mu)^\perp$.

2. The deficiency index of the restriction of $T_0'$ to the Hilbert space $\ker(T_0' - \mu)^\perp$ is equal to $r - 1$.

**Proof.** (1) First we consider the case that $-\infty < a < b < +\infty$. Without loss of generality, we may assume that $(a, b) = (-v, v)$. We have to show that

$$\|T_0'f\| \geq v\|f\| \quad \forall f \in D(T_0) \cap \ker (T_0' - \mu)^\perp.$$  \hfill (3.2)

Assume that $\|T_0'f\| < v\|f\|$ for some $f \in D(T_0') \cap \ker(T_0' - \mu)^\perp$. Let $g \in \ker(T_0' - \mu)$ and $g \neq 0$. Since $\ker(T_0' - \mu)$ reduces $T_0'$,

$$\|T_0'\left(c_1f + c_2g\right)\|^2 = |c_1|^2\|T_0'f\|^2 + |c_2|^2\|T_0'g\|^2$$

$$= |c_1|^2\|T_0'f\|^2 + |c_2|^2\|g\|^2$$

$$< |c_1|^2v^2\|f\|^2 + |c_2|^2v^2\|g\|^2$$

$$= v^2\|c_1f + c_2g\|^2,$$  \hfill (3.3)

for all $(c_1, c_2) \in \mathbb{C}^2 \setminus (0, 0)$. Since $f$ and $g$ span a two-dimensional subspace of $D(T_0)$ and $\dim D(T_0') \setminus D(T_0) = 1$, it follows that $\|T_0'h\| < v\|h\|$ for some $h \in D(T_0)$, this is a contradiction to the hypothesis that $(-v, v)$ is a spectral gap of $T_0$. This completes the proof in the case that $-\infty < a < b < +\infty$. \hfill \Box

The proof in the semibounded case is similar. One only has to replace expression of the form $\|T_0'f\|$ by those of the form $\langle T_0'f, f \rangle$, here we omit the details.

(2) Since $n(T_0', z) = n(T_0, z) - 1$, that is, $n(T_0', z) = r - 1$, and the restriction of $T_0'$ to the Hilbert space $\ker(T_0' - \mu)$ is self-adjoint, so the deficiency index of the restriction of $T_0'$ onto Hilbert space $\ker(T_0' - \mu)^\perp$ is equal to $r - 1$. 


Theorem 3.3. Suppose the deficiency index of $T_0$ is equal to $r$ and $\lambda$ is a real regular point of $T_0$. Let $f_1, f_2, \ldots, f_m \in L_2(\alpha, \beta; \omega)$ be linearly independent solutions of $(\tau - \lambda)f = 0$. One has the following:

1. $T_0$ has a $j$ dimensional symmetric extension $T^j_0$ ($j = 1, 2, \ldots, m$) with the following properties:

\[
\begin{align*}
D(T^j_0) &:= D(T_0) + \text{span}\{f_1\} + \cdots + \text{span}\{f_m\}, \\
T^j_0(f + g_1 + g_2 + \cdots + g_m) &= T_0f + \lambda(g_1 + g_2 + \cdots + g_m), \\
\forall f &\in D(T_0) \text{ and } \forall g_j \in \text{span}\{f_j\} (j = 1, 2, \ldots, m).
\end{align*}
\] (3.4)

2. $m \leq r$.

3. If $m = r$, then $T_0$ has a self-adjoint extension.

Proof. (1) Since $\text{span}\{f_1\}$ is a one-dimensional subspace of $\ker(T - \lambda)$, we define

\[
\begin{align*}
D(T^1_0) &:= D(T_0) + \text{span}\{f_1\}, \\
T^1_0(f + g) &= Tf + \lambda g \quad \forall f \in D(T_0) \text{ and } \forall g \in \text{span}\{f_1\}.
\end{align*}
\] (3.5)

Obviously, $T^1_0$ is a symmetric extension of $T_0$, $n(T^1_0, z) = n(T_0, z) - 1 = r - 1$, and $\lambda$ is an eigenvalue of $T^1_0$, and $\text{span}\{f_1\}$ is the corresponding eigenspace, and $T^1_0$ is also closed. By induction, we get

\[
n(T^j_0, z) = n(T_0, z) - j = r - j,
\] (3.6)

and $T^j_0$ ($j = 1, 2, \ldots, m$) is a closed symmetric operator.

(2) When $m > r$, then $n(T^m_0, z) = n(T_0, z) - m = r - m < 0$, that is a contradiction, so we have $m \leq r$.

(3) When $m = r$, then $n(T^m_0, z) = n(T_0, z) - m = r - r = 0$, so from Proposition 2.11, we have that $T^m_0$ is a self-adjoint extension of $T_0$. \hfill \square

Theorem 3.4. Assume that the deficiency index of $T_0$ is equal to $r$. If $(a, b)$ is a spectral gap of $T_0$, then for all $\beta_1, \beta_2, \ldots, \beta_m \in (a, b)$ (where $m \leq r$) and positive integer function $p$ satisfying $\sum_{i=1}^m p(\beta_i) = r$, the minimal operator $T_0$ has a self-adjoint extension $\overline{T}$ with the following properties:

(i) $\sigma_p(\overline{T}) \cap (a, b) = \{\beta_1, \beta_2, \ldots, \beta_m\}$,

(ii) each $\beta_j$ ($j = 1, 2, \ldots, m$) is an eigenvalue of $\overline{T}$ with multiplicity $p(\beta_j)$,

(iii) $\overline{T}$ has pure point spectrum within $(a, b)$.

Proof. We choose $\lambda_1, \lambda_2, \ldots, \lambda_r$ in $(a, b)$, such that $\#\{l \leq r : \lambda_l = \beta_j\} = p(\beta_j)$ ($j = 1, \ldots, m$) ($\#M$ denote the cardinality of the set $M$), in other words each $\beta_j \in (a, b)$ occurs exactly $p(\beta_j)$ times in the sequence $\{\lambda_1, \lambda_2, \ldots, \lambda_r\}$. 

We choose a one-dimensional space \( L_{\lambda_1} \) of ker\((T - \lambda_1)\) and define

\[
D(T'_1) := D(T_0) + L_{\lambda_1},
\]

\[
T'_1(f + g) := T_0f + \lambda g \quad \forall f \in D(T_0) \text{ and } \forall g \in L_{\lambda_1}.
\] (3.7)

Then \( T'_1 \) is a closed symmetric extension of \( T_0 \) with deficiency index being equal to \( r - 1 \) and \( \lambda_1 \) is an eigenvalue of \( T'_1 \) with eigensubspace being \( L_{\lambda_1} \).

Set \( \mathcal{H}_1 = L_{\lambda_1}^\perp \) and denote by \( M_1 \) and \( T_1 \) the restriction of \( T_1' \) to \( \mathcal{H}_1 \) and \( \mathcal{H}_1 \), respectively. Obviously \( M_1 \) is a self-adjoint operator in the one-dimensional Hilbert space \( L_{\lambda_1} \) with \( \sigma(M_1) = \{ \lambda_1 \} \). \( T_1 \) is a closed symmetric operator in Hilbert space \( \mathcal{H}_1 \), with deficiency index being equal to \( r - 1 \), and \( T'_1 = M_1 + T_1 \). Moreover, by Lemma 3.1, \((a, b)\) is a spectral gap of \( T_1 \).

Thus we can replace \( L_2(\alpha, \beta; W) \) by \( \mathcal{H}_1 \), \( T_0 \) by \( T_1 \), and \( \lambda_1 \) by \( \lambda_2 \) in the above conclusions.

Proceeding further in this way by induction, we obtain sequences \( \{ \mathcal{H}_1, \ldots, \mathcal{H}_r \} \) and \( \{ \mathcal{H}_1, \ldots, \mathcal{H}_r \} \) of Hilbert spaces and sequence \( \{ M_1, \ldots, M_r \} \) and \( \{ T'_1, \ldots, T'_r \} \) of the operators with the following properties:

(i) \( T'_l \) are closed symmetric extension of \( T_0 \) which satisfy: \( T'_l \subset T'_2 \subset \cdots \subset T'_r \), \( T'_l \) with deficiency indices \( r - l \) \((l = 1, 2, \ldots, r)\),

(ii) \( M_l \) is a self-adjoint operator on a one-dimensional subspace \( L_{\lambda_l} \) of \( \mathcal{H}_{l-1} \) \((\mathcal{H}_0 := L_2(\alpha, \beta; W), \mathcal{H}_1 = \mathcal{H}_{l-1} \oplus L_{\lambda_l}, \text{ with } \sigma(M_l) = \{ \lambda_l \} \) for each \( \lambda_l \) \((l = 1, 2, \ldots, r)\),

(iii) \( T_l \) is a closed symmetric operator in the Hilbert space \( \mathcal{H}_l \) with spectral gap \((a, b)\) for all \( l = 1, 2, \ldots, r \).

(iv) \( T'_l = \{ \oplus_{i=1}^{l} M_j \} \oplus T_i \) \((l = 1, \ldots, r)\).

Then we have that \( T'_r \) is a closed symmetric extension of \( T_0 \) with deficiency index 0. So \( \overline{T} = T'_r \) is a self-adjoint extension of \( T_0 \) with the required properties.

**Theorem 3.5.** Assume the deficiency index of \( T_0 \) is equal to \( r \). Let \( \beta_1, \beta_2, \ldots, \beta_m \in (a, b) \) \( (\text{where } m \leq r, \ (a, b) \ \text{need not be a spectral gap of } T_0) \) be real regular points of \( T_0 \) and let positive integer function \( p \) satisfy \( \sum_{j=1}^{m} p(\beta_j) \leq r \). Moreover, one assumes that \( (\tau - \lambda_1) f = 0 \) has at least \( p(\beta_1) \) square integrable solutions. Then \( T_0 \) has a self-adjoint extension \( \overline{T} \) such that each \( \beta_j \) \((j = 1, 2, \ldots, m)\) is an eigenvalue of \( \overline{T} \) with multiplicity at least \( p(\beta_j) \).

**Proof.** Choose a sequence \( \lambda_1, \ldots, \lambda_r \) in \((a, b)\) such that each \( \beta_i \) occurs at least \( p(\beta_i) \) times in this sequence \( (\text{the times } k_i \text{ of the } \lambda_i \text{ appears in the sequence satisfies } p(\beta_i) \leq k \leq k_i, \text{ } k_i \text{ is the number of square integrable solutions of } (\tau - \lambda_i) f = 0; \text{ the } \lambda_i \text{ which is not in } \{ \beta_1, \ldots, \beta_m \} \text{ will be selected as a regular point of } T \text{ satisfying } \ker(T - \lambda_i) \neq \emptyset) \). Since the deficiency index of \( T_0 \) is equal to \( r \), the negative and positive deficiency indices are equal to \( r \), so we select a normalized orthogonal basis \( f_1^+ \cdots f_r^+ \in R(T_0 + i)^{\perp} \) and \( f_1^- \cdots f_r^- \in \ker(T - i)^{\perp} \). Now, we define by induction, orthonormal sequences \( g_1, \ldots, g_r \in D(T) \) as follows:

for \( k = 1 \), we choose normalized element \( g_1 \in \ker(T - \lambda_1) \) with

\[
\langle f_1^+, g_1 \rangle = 0 = \langle f_1^-, g_1 \rangle.
\] (3.8)
Suppose for given \( k < r-1 \), the \( g_1, \ldots, g_k \) (\( g_j \in \ker(T - \lambda_j) \)) are well defined with the properties

\[
\langle f_i^+, g_j \rangle = 0 = \langle f_i^-, g_j \rangle \quad \forall 1 \leq i, j \leq k,
\]

(those \( g_i \) are eigenfunctions of \( T \) relating to \( \lambda_i \), so they are orthogonal if \( \lambda_i \neq \lambda_j \), and if \( \lambda_i = \lambda_j \), we select different orthogonal eigenfunctions of \( T \) relating to \( \lambda_j \)), we choose a normalized element \( g_{k+1} \in \ker(T - \lambda_{k+1}) \) with

\[
\langle g_{k+1}, f_j^+ \rangle = 0 = \langle g_{k+1}, f_j^- \rangle \quad \forall 1 \leq j \leq k + 1.
\]

When \( k = r-1 \), we stop the induction process. So \( \{g_1, \ldots, g_r\} \) is an orthonormal sequence with the following properties:

(i) \( g_k \in \ker(T - \lambda_k) \) for all \( k = 1, \ldots, r \),
(ii) \( f_k^+ \in R(T_0 + i) \) and \( f_k^- \in R(T_0 - i) \) for all \( k = 1, \ldots, r \),
(iii) \( \langle g_j, f_k^+ \rangle = \langle g_j, f_k^- \rangle = 0 \) for all \( j, k \in \{1, \ldots, r\} \).

Set

\[
D\left( \tilde{T} \right) := D(T_0) + \text{span}\{g_1, \ldots, g_r\},
\]

\[
\tilde{T} h := Th \quad \forall h \in D\left( \tilde{T} \right).
\]

Obviously \( \tilde{T} \) is a symmetric extension of \( T_0 \) and \( \tilde{T}g_j = Tg_j = \lambda_j g_j \) for \( j = 1, \ldots, m \), so each \( \beta_j \) is an eigenvalue of \( \tilde{T} \) with multiplicity at least equal to \( p(\beta_j) \). For every \( k \in \{1, \ldots, r\} \), we have

\[
\langle f_k^+, (\tilde{T} + i)f \rangle = \langle f_k^+, (T_0 + i)f \rangle = 0 \quad \forall f \in D(T_0),
\]

\[
\langle f_k^+, (\tilde{T} + i)g_j \rangle = \langle f_k^+, (\lambda_j + i)g_j \rangle = 0 \quad \forall j \in \{1, \ldots, r\}.
\]

It follows that \( f_k^+ \in R(\tilde{T} + i) \) for all \( k = 1, \ldots, r \). Thus we have

\[
\dim R(\tilde{T} + i)^\perp = \dim R(T_0 + i)^\perp - r = 0.
\]

Analogously, it can be shown that

\[
\dim R(\tilde{T} - i)^\perp = \dim R(T_0 - i)^\perp - r = 0.
\]

Thus \( \dim R(\tilde{T} + i)^\perp = \dim R(\tilde{T} - i)^\perp = 0 \) and consequently \( \tilde{T} \) is a self-adjoint extensions which has the properties mentioned above.
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References