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Research Article

Hyers-Ulam Stability of Jensen Functional Inequality in p-Banach Spaces

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We prove the Hyers-Ulam stability of the following Jensen functional inequality $\|f((x-y)/n+z)+f((y-z)/n+x)+f((z-x)/n+y)\| \le \|f((x+y+z)\|$ in p-Banach spaces for any fixed nonzero integer n.

1. Introduction

The stability problem of equations originated from a question of Ulam [1] concerning the stability of group homomorphisms.

We are given a group G_1 and a metric group G_2 with metric $\rho(\cdot,\cdot)$. Given $\epsilon > 0$, does there exist a number $\delta > 0$ such that if $f : G_1 \to G_2$ satisfies $\rho(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G_1$, then a homomorphism $h : G_1 \to G_2$ exists with $\rho(f(x), h(x)) < \epsilon$ for all $x \in G_1$?

In 1941, Hyers [2] considered the case of approximately additive mappings between Banach spaces and proved the following result.

Suppose that E_1 and E_2 are Banach spaces and $f : E_1 \to E_2$ satisfies the following condition: if there is a number $\epsilon \ge 0$ such that

$$\|f(x+y) - f(x) - f(y)\| \le \epsilon \tag{1.1}$$

for all $x, y \in E_1$, then the limit $h(x) = \lim_{n\to\infty} f(2^n x)/2^n$ exists for all $x \in E_1$ and there exists a unique additive mapping $h : E_1 \to E_2$ such that

$$\|f(x) - h(x)\| \le \epsilon. \tag{1.2}$$

Moreover, if $f(tx)$ is continuous in $t \in \mathbb{R}$ for each $x \in E_1$, then the mapping h is $\mathbb{R}$-linear.

The method which was provided by Hyers, and which produces the additive mapping h, is called a direct method. This method is the most important and most powerful tool for studying the stability of various functional equations. Hyers' theorem was generalized by Aoki [3] and Bourgin [4] for additive mappings by considering an unbounded Cauchy difference. In 1978, Rassias [5] also provided a generalization of Hyers' theorem for linear mappings which allows the Cauchy difference to be unbounded. Let E_1 and E_2 be two Banach spaces and let $f : E_1 \to E_2$ be a mapping such that $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed x. Assume that there exist $\epsilon > 0$ and $0 \leq p < 1$ such that

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon\left(\|x\|^p + \|y\|^p\right), \quad \forall x, y \in E_1. \tag{1.3}$$

Then, there exists a unique $\mathbb{R}$-linear mapping $T : E_1 \to E_2$ such that

$$\|f(x) - T(x)\| \leq \frac{2\epsilon}{2 - 2^p}\|x\|^p \tag{1.4}$$

for all $x \in E_1$. A generalized result of Rassias' theorem was obtained by Găvruţa in [6] and Jung in [7]. In 1990, Rassias [8] during the 27th International Symposium on Functional Equations asked the question whether such a theorem can also be proved for $p \geq 1$. In 1991, Gajda [9], following the same approach as in [5], gave an affirmative solution to this question for $p > 1$. It was shown by Gajda [9], as well as by Rassias and Šemrl [10], that one cannot prove a Rassias' type theorem when $p = 1$. The counterexamples of Gajda [9], as well as of Rassias and Šemrl [10], have stimulated several mathematicians to invent new approximately additive or approximately linear mappings.

We recall some basic facts concerning quasinormed spaces and some preliminary results. Let X be a real linear space. A quasinorm is a real-valued function on X satisfying the following:

(1) $\|x\| \geq 0$ for all $x \in X$ and $\|x\| = 0$ if and only if $x = 0$.

(2) $\|\lambda x\| = |\lambda|\|x\|$ for all $\lambda \in \mathbb{R}$ and all $x \in X$.

(3) There is a constant $M \geq 1$ such that $\|x + y\| \leq M(\|x\| + \|y\|)$ for all $x, y \in X$.

The pair $(X, \|\cdot\|)$ is called a quasinormed space if $\|\cdot\|$ is a quasinorm on X [11, 12]. The smallest possible M is called the modulus of concavity of $\|\cdot\|$. A quasi-Banach space is a complete quasinormed space.

A quasinorm $\|\cdot\|$ is called a p-norm $(0 < p \leq 1)$ if

$$\|x + y\|^p \leq \|x\|^p + \|y\|^p \tag{1.5}$$

for all $x, y \in X$. In this case, a quasi-Banach space is called a p-Banach space.

Given a p-norm, the formula $d(x, y) := \|x - y\|^p$ gives us a translation invariant metric on X. By the Aoki-Rolewicz theorem [12], each quasinorm is equivalent to some p-norm (see also [11]). Since it is much easier to work with p-norms, henceforth, we restrict our attention mainly to p-norms. We observe that if $x_1, x_2, \ldots, x_n$ are nonnegative real numbers, then

$$\left(\sum_{i=1}^{n} x_i\right)^p \leq \sum_{i=1}^{n} x_i^p, \tag{1.6}$$

where $0 < p \leq 1$.

In 2009, Moslehian and Najati [13] introduced the Hyers-Ulam stability of the additive functional inequality:

$$\left\| f\left(\frac{x-y}{2}+z\right)+f\left(\frac{y-z}{2}+x\right)+f\left(\frac{z-x}{2}+y\right)\right\| \leq \|f(x+y+z)\|, \tag{1.7}$$

and then have investigated the general solution and the Hyers-Ulam stability problem for the functional inequality. The stability problems of several functional equations in quasi-normed spaces and several functional inequalities have been investigated by a number of authors and there are many interesting results concerning the stability of various functional inequalities [14–17].

In this paper, we consider a modified and general Jensen functional inequality:

$$\left\| f\left(\frac{x-y}{n}+z\right)+f\left(\frac{y-z}{n}+x\right)+f\left(\frac{z-x}{n}+y\right)\right\| \leq \|f(x+y+z)\| \tag{1.8}$$

for any fixed nonzero integer n. First of all, it is easy to see that a function f satisfies the inequality (1.8) if and only if $f(x)$ is additive. Thus the inequality (1.8) may be called the Jensen functional inequality and the general solution of inequality (1.8) may be called the Jensen function. In the sequel, we investigate the generalized Hyers-Ulam stability of (1.8) in p-Banach spaces for any fixed nonzero integer n by using the techniques of [14, 15].

2. Generalized Hyers-Ulam Stability

First, we present the general solution of the inequality (1.8).

Lemma 2.1. *Let both X and Y be real vector spaces. A function $f : X \to Y$ satisfies* (1.8) *for all $x, y, z \in X$ if and only if f is additive.*

Proof. Letting $x = y = z = 0$ in (1.8), we have $f(0) = 0$. Putting $y = -(n+1)x/2$ and $z = (n-1)x/2$ in (1.8), we get

$$\left\| f\left(\frac{(n^2+3)x}{2n}\right)+f\left(\frac{-(n^2+3)x}{2n}\right)\right\| \leq \|f(0)\| \tag{2.1}$$

for all $x \in X$. Hence $f(-x) = -f(x)$ for all $x \in X$. Replacing z by $-x-y$ in (1.8), we obtain

$$\left\| f\left(\frac{(1-n)x-(n+1)y}{n}\right)+f\left(\frac{(n+1)x+2y}{n}\right)+f\left(\frac{-2x+(n-1)y}{n}\right)\right\| \leq \|f(0)\|, \tag{2.2}$$

that is,

$$f((1-n)x-(n+1)y)+f((n+1)x+2y)+f(-2x+(n-1)y) = 0 \tag{2.3}$$

for all $x, y \in X$. Putting $u = (n+1)x + 2y$ and $v = -2x + (n-1)y$ in (2.3), we get by oddness of f,

$$f(u+v) = f(u) + f(v) \tag{2.4}$$

for all $u, v \in X$. So f is additive.

The proof of the converse is trivial. □

From now on, assume that X is a quasinormed space with quasinorm $\|\cdot\|$ and that Y is a p-Banach space with p-norm $\|\cdot\|$. Let M be the modulus of concavity of $\|\cdot\|$ in Y.

Before taking up the main subject, given a mapping $f : X \to Y$, we define the difference operator $Df : X^3 \to Y$ by

$$Df(x,y,z) := \left\| f\left(\frac{x-y}{n} + z\right) + f\left(\frac{y-z}{n} + x\right) + f\left(\frac{z-x}{n} + y\right) \right\| - \|f(x+y+z)\| \tag{2.5}$$

for all $x, y, z \in X$ and for any fixed nonzero integer n.

Theorem 2.2. *Suppose that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the functional inequality*

$$Df(x,y,z) \leq \varphi(x,y,z) \tag{2.6}$$

for all $x, y, z \in X$ and the perturbing function $\varphi : X^3 \to \mathbb{R}^+$ satisfies

$$\Phi(x,y,z) := \sum_{i=0}^{\infty} \frac{1}{2^{ip}} \varphi\left(2^i x, 2^i y, 2^i z\right)^p < \infty \tag{2.7}$$

for all $x, y, z \in X$. Then, there exists a unique additive mapping $h : X \to Y$ defined by $h(x) = \lim_{k\to\infty}(1/2^k) f(2^k x)$ such that

$$\begin{aligned} \|f(x) - h(x)\| \leq \frac{M}{2} \Bigg[&\Phi\left(\frac{n(n-3)x}{n^2+3}, \frac{n(n+3)x}{n^2+3}, \frac{-2n^2 x}{n^2+3}\right) \\ &+ \Phi\left(\frac{-2n(n+1)x}{n^2+3}, \frac{2n(n-1)x}{n^2+3}, \frac{4nx}{n^2+3}\right)\Bigg]^{1/p} \end{aligned} \tag{2.8}$$

for all $x \in X$.

Proof. Replacing z by $-x-y$ in (2.6), we obtain

$$\begin{aligned} &\left\| f\left(\frac{(1-n)x - (n+1)y}{n}\right) + f\left(\frac{(n+1)x + 2y}{n}\right) + f\left(\frac{-2x + (n-1)y}{n}\right) \right\| \\ &\quad \leq \varphi(x, y, -x-y) \end{aligned} \tag{2.9}$$

for all $x, y \in X$. Letting $x = (n-3)x/(n^2+3)$ and $y = (n+3)x/(n^2+3)$ in (2.9), we get

$$\left\| f\left(-\frac{2x}{n}\right) + 2f\left(\frac{x}{n}\right) \right\| \le \varphi\left(\frac{(n-3)x}{n^2+3}, \frac{(n+3)x}{n^2+3}, \frac{-2nx}{n^2+3}\right) \tag{2.10}$$

for all $x \in X$. Putting $x = -(n+1)z/2$ and $y = (n-1)z/2$ in (2.6), we have

$$\left\| f\left(\frac{-(n^2+3)z}{2n}\right) + f\left(\frac{(n^2+3)z}{2n}\right) \right\| \le \varphi\left(\frac{-(n+1)z}{2}, \frac{(n-1)z}{2}, z\right) \tag{2.11}$$

for all $z \in X$. Replacing z by $4x/(n^2+3)$ in (2.11), we obtain

$$\left\| f\left(-\frac{2x}{n}\right) + f\left(\frac{2x}{n}\right) \right\| \le \varphi\left(\frac{-2(n+1)x}{n^2+3}, \frac{2(n-1)x}{n^2+3}, \frac{4x}{n^2+3}\right) \tag{2.12}$$

for all $x \in X$. It follows from (2.10) and (2.12) that

$$\begin{aligned} \left\| f\left(\frac{2x}{n}\right) - 2f\left(\frac{2x}{n}\right) \right\| &\le M\left[\left\| f\left(-\frac{2x}{n}\right) + 2f\left(\frac{x}{n}\right) \right\| + \left\| f\left(-\frac{2x}{n}\right) + f\left(\frac{2x}{n}\right) \right\|\right] \\ &\le M\left[\varphi\left(\frac{(n-3)x}{n^2+3}, \frac{(n+3)x}{n^2+3}, \frac{-2nx}{n^2+3}\right)\right. \\ &\qquad \left. +\varphi\left(\frac{-2(n+1)x}{n^2+3}, \frac{2(n-1)x}{n^2+3}, \frac{4x}{n^2+3}\right)\right] \end{aligned} \tag{2.13}$$

for all $x \in X$. If we replace x by nx in (2.13), then we get that

$$\begin{aligned} \|f(2x) - 2f(x)\| \le M&\left[\varphi\left(\frac{n(n-3)x}{n^2+3}, \frac{n(n+3)x}{n^2+3}, \frac{-2n^2x}{n^2+3}\right)\right. \\ &\left. +\varphi\left(\frac{-2n(n+1)x}{n^2+3}, \frac{2n(n-1)x}{n^2+3}, \frac{4nx}{n^2+3}\right)\right]. \end{aligned} \tag{2.14}$$

It follows from (2.14) that

$$\begin{aligned}\left\|\frac{f(2^l x)}{2^l}-\frac{f(2^m x)}{2^m}\right\|^p &\le \sum_{i=l}^{m-1}\left\|\frac{1}{2^i}f\left(2^i x\right)-\frac{1}{2^{i+1}}f\left(2^{i+1}x\right)\right\|^p \\ &= \sum_{i=l}^{m-1}\frac{1}{2^{ip}}\left\|f\left(2^i x\right)-\frac{1}{2}f\left(2^{i+1}x\right)\right\|^p \\ &\le \frac{M^p}{2^p}\sum_{i=l}^{m-1}\frac{1}{2^{ip}}\left[\varphi\left(\frac{n(n-3)2^i x}{n^2+3},\frac{n(n+3)2^i x}{n^2+3},\frac{(-2n^2)2^i x}{n^2+3}\right)^p\right. \\ &\qquad \left.+\varphi\left(\frac{-2n(n+1)2^i x}{n^2+3},\frac{2n(n-1)2^i x}{n^2+3},\frac{(4n)2^i x}{n^2+3}\right)^p\right]\end{aligned} \tag{2.15}$$

for all nonnegative integers m and l with $m > l \ge 0$ and $x \in X$. Since the right-hand side of (2.15) tends to zero as $l \to \infty$, by the convergence of the series (2.7), we obtain that the sequence $\{f(2^m x)/2^m\}$ is Cauchy for all $x \in X$. Because of the fact that Y is complete, it follows that the sequence $\{f(2^m x)/2^m\}$ converges in Y. Therefore, we can define a mapping $h : X \to Y$ as

$$h(x) = \lim_{m\to\infty}\frac{f(2^m x)}{2^m}, \quad x \in X. \tag{2.16}$$

Moreover, letting $l = 0$ and taking $m \to \infty$ in (2.15), we get

$$\begin{aligned}\|f(x)-h(x)\| \le \frac{M}{2}&\left[\Phi\left(\frac{n(n-3)x}{n^2+3},\frac{n(n+3)x}{n^2+3},\frac{-2n^2 x}{n^2+3}\right)\right. \\ &\left.+\Phi\left(\frac{-2n(n+1)x}{n^2+3},\frac{2n(n-1)x}{n^2+3},\frac{4nx}{n^2+3}\right)\right]^{1/p}\end{aligned} \tag{2.17}$$

for all $x \in X$.

It follows from (2.6) and (2.7) that

$$\begin{aligned}&\left\|h\left(\frac{x-y}{n}+z\right)+h\left(\frac{y-z}{n}+x\right)+h\left(\frac{z-x}{n}+y\right)\right\|^p \\ &\quad = \lim_{m\to\infty}\left\|\frac{1}{2^m}\left\{f\left(2^m\left(\frac{x-y}{n}+z\right)\right)+f\left(2^m\left(\frac{y-z}{n}+x\right)\right)+f\left(2^m\left(\frac{z-x}{n}+y\right)\right)\right\}\right\|^p \\ &\quad \le \lim_{m\to\infty}\left\{\left\|\frac{1}{2^m}f(2^m(x+y+z))\right\|^p+\frac{1}{2^{mp}}\varphi(2^m x,2^m y,2^m z)^p\right\} \\ &\quad = \|h(x+y+z)\|^p\end{aligned} \tag{2.18}$$

for all $x, y, z \in X$. So the mapping h is additive.

Next, let $h' : X \to Y$ be another additive mapping satisfying (2.8). Then, we have

$$
\begin{aligned}
\|h(x) - h'(x)\|^p
&= \left\| \frac{1}{2^k} h\left(2^k x\right) - \frac{1}{2^k} h'\left(2^k x\right) \right\|^p \\
&\le \frac{1}{2^{kp}} \left(\left\| h\left(2^k x\right) - f\left(2^k x\right) \right\|^p + \left\| f\left(2^k x\right) - h'\left(2^k x\right) \right\|^p \right) \\
&\le \sum_{i=0}^{\infty} \frac{2M^p}{2^{(i+k+1)p}} \left[\varphi\left(\frac{n(n-3)2^{i+k}x}{n^2+3}, \frac{n(n+3)2^{i+k}x}{n^2+3}, \frac{(-2n^2)2^{i+k}x}{n^2+3} \right)^p \right. \\
&\qquad \left. + \varphi\left(\frac{-2n(n+1)2^{i+k}x}{n^2+3}, \frac{2n(n-1)2^{i+k}x}{n^2+3}, \frac{(4n)2^{i+k}x}{n^2+3} \right)^p \right] \\
&= \sum_{i=k}^{\infty} \frac{2M^p}{2^{(i+1)p}} \left[\varphi\left(\frac{n(n-3)2^{i}x}{n^2+3}, \frac{n(n+3)2^{i}x}{n^2+3}, \frac{(-2n^2)2^{i}x}{n^2+3} \right)^p \right. \\
&\qquad \left. + \varphi\left(\frac{-2n(n+1)2^{i}x}{n^2+3}, \frac{2n(n-1)2^{i}x}{n^2+3}, \frac{(4n)2^{i}x}{n^2+3} \right)^p \right]
\end{aligned}
\tag{2.19}
$$

for all $k \in \mathbb{N}$ and all $x \in X$. Taking the limit as $k \to \infty$, we conclude that

$$h(x) = h'(x) \tag{2.20}$$

for all $x \in X$. This completes the proof. □

If we put $\varphi(x, y, z) := \theta(\|x\|^{r_1}\|y\|^{r_2}\|z\|^{r_3})$ and $\varphi(x, y, z) := \theta_1\|x\|^{r_1} + \theta_2\|y\|^{r_2} + \theta_3\|z\|^{r_3}$ in the following corollaries, respectively, then we lead to the desired results.

Corollary 2.3. *Let $r_i > 0$ for $i = 1, 2, 3$ with $\sum_{i=1}^{3} r_i < 1$ and $\theta \ge 0$. If a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the following functional inequality*

$$Df(x, y, z) \le \theta\left(\|x\|^{r_1}\|y\|^{r_2}\|z\|^{r_3}\right) \tag{2.21}$$

for all $x, y, z \in X$, then there exists a unique additive mapping $h : X \to Y$ such that

$$
\begin{aligned}
\|f(x) - h(x)\| \le \frac{M\theta\|x\|^r}{\sqrt[p]{2^p - 2^{rp}}} &\left(\left| \frac{n(n-3)}{n^2+3} \right|^{r_1 p} \left| \frac{n(n+3)}{n^2+3} \right|^{r_2 p} \left| \frac{2n^2}{n^2+3} \right|^{r_3 p} \right. \\
&\left. + \left| \frac{2n(n+1)}{n^2+1} \right|^{r_1 p} \left| \frac{2n(n-3)}{n^2+3} \right|^{r_2 p} \left| \frac{4n}{n^2+3} \right|^{r_3 p} \right)^{1/p}
\end{aligned}
\tag{2.22}
$$

for all $x \in X$, where $r = \sum_{i=1}^{3} r_i$.

Corollary 2.4. *Let $0 < r_i < 1$ and $\theta_i \geq 0$ for $i = 1,2,3$. If a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the following functional inequality*

$$Df(x,y,z) \leq \theta_1 \|x\|^{r_1} + \theta_2 \|y\|^{r_2} + \theta_3 \|z\|^{r_3} \tag{2.23}$$

for all $x, y, z \in X$, then there exists a unique additive mapping $h : X \to Y$ such that

$$\begin{aligned} \|f(x) - h(x)\| \leq M \Bigg[&\left(\left| \frac{n(n-3)}{n^2+3} \right|^{r_1 p} + \left| \frac{2n(n+1)}{n^2+3} \right|^{r_1 p} \right) \frac{\theta_1^p \|x\|^{r_1 p}}{2^p - 2^{r_1 p}} \\ &+ \left(\left| \frac{n(n+3)}{n^2+3} \right|^{r_2 p} + \left| \frac{2n(n-1)}{n^2+3} \right|^{r_2 p} \right) \frac{\theta_2^p \|x\|^{r_2 p}}{2^p - 2^{r_2 p}} \\ &+ \left(\left| \frac{2n^2}{n^2+3} \right|^{r_3 p} + \left| \frac{4n}{n^2+3} \right|^{r_3 p} \right) \frac{\theta_3^p \|x\|^{r_3 p}}{2^p - 2^{r_3 p}} \Bigg]^{1/p} \end{aligned} \tag{2.24}$$

for all $x \in X$.

Theorem 2.5. *Suppose that a mapping $f : X \to Y$ satisfies the functional inequality*

$$Df(x,y,z) \leq \varphi(x,y,z) \tag{2.25}$$

for all $x, y, z \in X$, and the perturbing function $\varphi : X^3 \to \mathbb{R}^+$ satisfies

$$\Phi(x,y,z) := \sum_{i=0}^{\infty} 2^{ip} \varphi\left(\frac{x}{2^{i+1}}, \frac{y}{2^{i+1}y}, \frac{z}{2^{i+1}} \right)^p < \infty \tag{2.26}$$

for all $x, y, z \in X$. Then, there exists a unique additive mapping $h : X \to Y$ defined by $h(x) = \lim_{k\to\infty} 2^k f(x/2^k)$ such that

$$\begin{aligned} \|f(x) - h(x)\| \leq M \Bigg[&\Phi\left(\frac{n(n-3)x}{n^2+3}, \frac{n(n+3)x}{n^2+3}, \frac{-2n^2 x}{n^2+3} \right) \\ &+\Phi\left(\frac{-2n(n+1)x}{n^2+3}, \frac{2n(n-1)x}{n^2+3}, \frac{4nx}{n^2+3} \right) \Bigg]^{1/p} \end{aligned} \tag{2.27}$$

for all $x \in X$.

Proof. We note that $f(0) = 0$ since $\varphi(0,0,0) = 0$ by the convergence of (2.26). Now, if we replace x by $x/2$ in (2.14),

$$\left\|f(x) - 2f\left(\frac{x}{2}\right)\right\| \leq M\left[\varphi\left(\frac{n(n-3)x}{2(n^2+3)}, \frac{n(n+3)x}{2(n^2+3)}, \frac{-n^2x}{(n^2+3)}\right)\right.$$
$$\left. +\varphi\left(\frac{-n(n+1)x}{n^2+3}, \frac{n(n-1)x}{n^2+3}, \frac{2nx}{n^2+3}\right)\right] \tag{2.28}$$

for all $x \in X$. Then, it follows from the last inequality that

$$\left\|f(x) - 2^m f\left(\frac{x}{2^m}\right)\right\|^p \leq M^p \sum_{i=0}^{m-1} 2^{ip}\left[\varphi\left(\frac{n(n-3)x}{2^{i+1}(n^2+3)}, \frac{n(n+3)x}{2^{i+1}(n^2+3)}, \frac{-2n^2x}{2^{i+1}(n^2+3)}\right)^p\right.$$
$$\left. +\varphi\left(\frac{-2n(n+1)x}{2^{i+i}(n^2+3)}, \frac{2n(n-1)x}{2^{i+1}(n^2+3)}, \frac{4nx}{2^{i+1}(n^2+3)}\right)^p\right] \tag{2.29}$$

for all nonnegative integer m and all $x \in X$. The remaining proof is similar to the corresponding part of Theorem 2.2. This completes the proof. □

If we put $\varphi(x,y,z) := \theta(\|x\|^{r_1}\|y\|^{r_2}\|z\|^{r_3})$ and $\varphi(x,y,z) := \theta_1\|x\|^{r_1} + \theta_2\|y\|^{r_2} + \theta_3\|z\|^{r_3}$ in the following corollaries, respectively, then we lead to the desired results.

Corollary 2.6. *Let $r_i > 0$ for $i = 1,2,3$ with $\sum_{i=1}^{3} r_i > 1$ and $\theta \geq 0$. If a mapping $f : X \to Y$ satisfies the following functional inequality*

$$Df(x,y,z) \leq \theta\left(\|x\|^{r_1}\|y\|^{r_2}\|z\|^{r_3}\right) \tag{2.30}$$

for all $x,y,z \in X$, then there exists a unique additive mapping $h : X \to Y$ such that

$$\|f(x) - h(x)\| \leq \frac{M\theta\|x\|^r}{\sqrt[p]{2^{rp} - 2^p}}\left(\left|\frac{n(n-3)}{n^2+3}\right|^{r_1p}\left|\frac{n(n+3)}{n^2+3}\right|^{r_2p}\left|\frac{2n^2}{n^2+3}\right|^{r_3p}\right.$$
$$\left. +\left|\frac{2n(n+1)}{n^2+1}\right|^{r_1p}\left|\frac{2n(n-3)}{n^2+3}\right|^{r_2p}\left|\frac{4n}{n^2+3}\right|^{r_3p}\right)^{1/p} \tag{2.31}$$

for all $x \in X$, where $r = \sum_{i=1}^{3} r_i$.

Corollary 2.7. *Let $r_i > 1$ and $\theta_i \geq 0$ for $i = 1,2,3$. If a mapping $f : X \to Y$ satisfies the following functional inequality*

$$Df(x,y,z) \leq \theta_1 \|x\|^{r_1} + \theta_2 \|y\|^{r_2} + \theta_3 \|z\|^{r_3} \tag{2.32}$$

for all $x,y,z \in X$, then there exists a unique additive mapping $h : X \to Y$ such that

$$\begin{aligned}\|f(x) - h(x)\| \leq M\Bigg[&\left(\left|\frac{n(n-3)}{n^2+3}\right|^{r_1 p} + \left|\frac{2n(n+1)}{n^2+3}\right|^{r_1 p}\right)\frac{\theta_1^p \|x\|^{r_1 p}}{2^{r_1 p - 2^p}} \\ &+ \left(\left|\frac{n(n+3)}{n^2+3}\right|^{r_2 p} + \left|\frac{2n(n-1)}{n^2+3}\right|^{r_2 p}\right)\frac{\theta_2^p \|x\|^{r_2 p}}{2^{r_2 p} - 2^p} \\ &+ \left(\left|\frac{2n^2}{n^2+3}\right|^{r_3 p} + \left|\frac{4n}{n^2+3}\right|^{r_3 p}\right)\frac{\theta_3^p \|x\|^{r_3 p}}{2^{r_3 p} - 2^p}\Bigg]^{1/p}\end{aligned} \tag{2.33}$$

for all $x \in X$.

The following is a simple example that the additive functional inequality $Df(x,y,z) \leq \theta(\|x\| + \|y\| + \|z\|)$ is not stable for the singular case $r_1, r_2, r_3 = 1$ in Corollaries 2.4 and 2.7.

Example 2.8. Fix $\theta \geq 0$ and put $\mu := \theta/8$. Let $\phi : \mathbb{R} \to \mathbb{R}$ be defined by

$$\phi(x) = \begin{cases} \mu & \text{for } x \in [1,\infty), \\ \mu x & \text{for } x \in (-1,1), \\ -\mu & \text{for } x \in (-\infty,-1], \end{cases} \tag{2.34}$$

and define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \sum_{i=0}^{\infty} \frac{\phi(2^i x)}{2^i}, \quad \forall x \in \mathbb{R}, \tag{2.35}$$

which can be found in [9]. It follows from the same argument as in the example of [9] that f satisfies the functional inequality

$$\begin{aligned}&\left|\left|f\left(\frac{x-y}{n} + z\right) + f\left(\frac{y-z}{n} + x\right) + f\left(\frac{z-x}{n} + y\right)\right| - |f(x+y+z)|\right| \\ &\quad \leq 8\mu(|x| + |y| + |z|)\end{aligned} \tag{2.36}$$

for all $x, y, z \in \mathbb{R}$. In fact, if $x = y = z = 0$, then (2.36) is trivially fulfilled. Next, if $0 < |x| + |y| + |z| < 1$, then there exists an $N \in \mathbb{N}$ such that

$$\frac{1}{2^N} \leq |x| + |y| + |z| < \frac{1}{2^{N-1}}, \tag{2.37}$$

which implies that

$$2^i\left(\frac{x-y}{n}+z\right), 2^i\left(\frac{y-z}{n}+x\right), 2^i\left(\frac{z-x}{n}+y\right), 2^i(x+y+z) \in (-1,1), \quad \forall i \in \{0,\ldots,N-1\}. \tag{2.38}$$

Thus, we see that

$$\phi\left(2^i\left(\frac{x-y}{n}+z\right)\right) + \phi\left(2^i\left(\frac{y-z}{n}+x\right)\right) + \phi\left(2^i\left(\frac{z-x}{n}+y\right)\right) - \phi\left(2^i(x+y+z)\right) = 0 \tag{2.39}$$

for all $i \in \{0,\ldots,N-1\}$. As a result, we infer that

$$\begin{aligned}
&\frac{\left|f(((x-y)/n)+z) + f(((y-z)/n)+x) + f(((z-x)/n)+y) - f(x+y+z)\right|}{|x|+|y|+|z|} \\
&\leq \sum_{i=N}^{\infty} \frac{\left|\phi(2^i(((x-y)/n)+z)) + \phi(2^i(((y-z)/n)+x)) + \phi(2^i(((z-x)/n)+y)) - \phi(2^i(x+y+z))\right|}{2^i(|x|+|y|+|z|)} \\
&\leq 8\mu
\end{aligned} \tag{2.40}$$

for all $x, y, z \in \mathbb{R}$. Finally, if $|x| + |y| + |z| \geq 1$, then one has by use of boundedness of f

$$\frac{\left|f(((x-y)/n)+z) + f(((y-z)/n)+x) + f(((z-x)/n)+y) - f(x+y+z)\right|}{|x|+|y|+|z|} \leq 8\mu \tag{2.41}$$

for all $x, y, z \in \mathbb{R}$. Therefore, f satisfies the functional inequality (2.36) and so

$$Df(x,y,z) \leq 8\mu(|x| + |y| + |z|) \tag{2.42}$$

for all $x, y, z \in \mathbb{R}$. However, there do not exist an additive function $T : \mathbb{R} \to \mathbb{R}$ and a constant $c > 0$ such that

$$\left|f(x) - T(x)\right| \leq c|x| \quad \forall x \in \mathbb{R}. \tag{2.43}$$

Remark 2.9. The stability problem on the singular case $r = 1$ in Corollaries 2.3 and 2.6 is not easy and it remains with us unsolved for providing a counterexample on the singular case $r = 1$.

3. Alternative Generalized Hyers-Ulam Stability of (1.8)

From now on, we investigate the generalized Hyers-Ulam stability of the functional inequality (1.8) using the contractive property of perturbing term of the inequality (1.8).

Theorem 3.1. *Suppose that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the functional inequality*

$$Df(x,y,z) \leq \varphi(x,y,z) \tag{3.1}$$

for all $x,y,z \in X$ and there exists a constant L with $0 < L < 1$ for which the perturbing function $\varphi : X^3 \to \mathbb{R}^+$ satisfies

$$\varphi(2x,2y,2z) \leq 2L\varphi(x,y,z) \tag{3.2}$$

for all $x,y,z \in X$. Then, there exists a unique additive mapping $h : X \to Y$ given by $h(x) = \lim_{k\to\infty}(1/2^k)f(2^k x)$ such that

$$\begin{aligned}\|f(x)-h(x)\| \leq \frac{M}{2\sqrt[p]{1-L^p}}&\left[\varphi\left(\frac{n(n-3)x}{n^2+3},\frac{n(n+3)x}{n^2+3},\frac{-2n^2x}{n^2+3}\right)^p\right.\\ &\left.+\varphi\left(\frac{-2n(n+1)x}{n^2+3},\frac{2n(n-1)x}{n^2+3},\frac{4nx}{n^2+3}\right)^p\right]^{1/p}\end{aligned} \tag{3.3}$$

for all $x \in X$.

Proof. It follows from (2.15) and (3.2) that

$$\begin{aligned}\left\|\frac{f(2^l x)}{2^l}-\frac{f(2^m x)}{2^m}\right\|^p &\leq \frac{M^p}{2^p}\sum_{i=l}^{m-1}\frac{1}{2^{ip}}\left[\varphi\left(\frac{n(n-3)2^i x}{n^2+3},\frac{n(n+3)2^i x}{n^2+3},\frac{(-2n^2)2^i x}{n^2+3}\right)^p\right.\\ &\qquad\left.+\varphi\left(\frac{-2n(n+1)2^i x}{n^2+3},\frac{2n(n-1)2^i x}{n^2+3},\frac{(4n)2^i x}{n^2+3}\right)^p\right]\\ &\leq \frac{M^p}{2^p}\sum_{i=l}^{m-1}L^{ip}\left[\varphi\left(\frac{n(n-3)x}{n^2+3},\frac{n(n+3)x}{n^2+3},\frac{(-2n^2)x}{n^2+3}\right)^p\right.\\ &\qquad\left.+\varphi\left(\frac{-2n(n+1)x}{n^2+3},\frac{2n(n-1)x}{n^2+3},\frac{(4n)x}{n^2+3}\right)^p\right]\end{aligned} \tag{3.4}$$

for all nonnegative integers m and l with $m > l \geq 0$ and $x \in X$. Since the sequence $\{f(2^m x)/2^m\}$ is Cauchy for all $x \in X$, we can define a mapping $h : X \to Y$ by

$$h(x) = \lim_{m\to\infty} \frac{f(2^m x)}{2^m}, \quad x \in X. \tag{3.5}$$

Moreover, letting $l = 0$ and $m \to \infty$ in the last inequality yields the approximation (3.3).

The remaining proof is similar to the corresponding part of Theorem 2.2. This completes the proof. □

Corollary 3.2. *Let $\xi : [0,\infty) \to [0,\infty)$ be a nontrivial function satisfying*

$$\xi(2t) \leq \xi(2)\xi(t), \quad (t \geq 0),\ 0 < \xi(2) < 2. \tag{3.6}$$

If $f : X \to Y$ with $f(0) = 0$ is a mapping satisfying the following functional inequality

$$Df(x,y,z) \leq \theta\{\xi(\|x\|) + \xi(\|y\|) + \xi(\|z\|)\} \tag{3.7}$$

for all $x, y, z \in X$ and for some $\theta \geq 0$, then there exists a unique additive mapping $h : X \to Y$ such that

$$\begin{aligned}
&\|f(x) - h(x)\| \\
&\quad\leq \frac{M\theta}{\sqrt[p]{2^p - \xi(2)^p}}\left[\xi\left(\left|\frac{n(n-3)}{n^2+3}\right|\|x\|\right)^p + \xi\left(\left|\frac{n(n+3)}{n^2+3}\right|\|x\|\right)^p + \xi\left(\left|\frac{2n^2}{n^2+3}\right|\|x\|\right)^p\right. \\
&\qquad\left. + \xi\left(\left|\frac{2n(n+1)}{n^2+3}\right|\|x\|\right)^p + \xi\left(\left|\frac{2n(n-1)}{n^2+3}\right|\|x\|\right)^p + \xi\left(\left|\frac{4n}{n^2+3}\right|\|x\|\right)^p\right]^{1/p}
\end{aligned} \tag{3.8}$$

for all $x \in X$.

Proof. Letting $\varphi(x,y,z) = \theta\{\xi(\|x\|) + \xi(\|y\|) + \xi(\|z\|)\}$ and applying Theorem 3.1 with $L := \xi(2)/2$, we obtain the desired result. □

Theorem 3.3. *Suppose that a mapping $f : X \to Y$ satisfies the functional inequality*

$$Df(x,y,z) \leq \varphi(x,y,z) \tag{3.9}$$

for all $x, y, z \in X$ and there exists a constant L with $0 < L < 1$ for which the perturbing function $\varphi : X^3 \to \mathbb{R}^+$ satisfies

$$\varphi\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\right) \leq \frac{L}{2}\varphi(x,y,z) \tag{3.10}$$

for all $x, y, z \in X$. Then, there exists a unique additive mapping $h : X \to Y$ defined by $h(x) = \lim_{k\to\infty} 2^k f(x/2^k)$ such that

$$\|f(x) - h(x)\| \le \frac{ML}{2\sqrt[p]{1-L^p}} \left[\varphi\left(\frac{n(n-3)x}{n^2+3}, \frac{n(n+3)x}{n^2+3}, \frac{-2n^2x}{n^2+3} \right)^p + \varphi\left(\frac{-2n(n+1)x}{n^2+3}, \frac{2n(n-1)x}{n^2+3}, \frac{4nx}{n^2+3} \right)^p \right]^{1/p} \tag{3.11}$$

for all $x \in X$.

Proof. We observe that $f(0) = 0$ because $\varphi(0,0,0) = 0$, which follows from the condition $\varphi(0,0,0) \le L/2\ \varphi(0,0,0)$. It follows from (2.29) and (3.10) that

$$\begin{aligned} \left\| f(x) - 2^m f\left(\frac{x}{2^m}\right) \right\|^p &\le M^{\mathrm{p}} \sum_{i=0}^{m-1} 2^{ip} \left[\varphi\left(\frac{n(n-3)x}{2^{i+1}(n^2+3)}, \frac{n(n+3)x}{2^{i+1}(n^2+3)}, \frac{-2n^2x}{2^{i+1}(n^2+3)} \right)^p \right. \\ &\quad \left. + \varphi\left(\frac{-2n(n+1)x}{2^{i+i}(n^2+3)}, \frac{2n(n-1)x}{2^{i+1}(n^2+3)}, \frac{4nx}{2^{i+1}(n^2+3)} \right)^p \right] \\ &\le \frac{M^p}{2^p} \sum_{i=0}^{m-1} L^{(i+1)p} \left[\varphi\left(\frac{n(n-3)x}{(n^2+3)}, \frac{n(n+2)x}{(n^2+3)}, \frac{-2n^2x}{(n^2+3)} \right)^p \right. \\ &\quad \left. + \varphi\left(\frac{-2n(n+1)x}{(n^2+3)}, \frac{2n(n-1)x}{(n^2+3)}, \frac{4nx}{(n^2+3)} \right)^p \right] \end{aligned} \tag{3.12}$$

for all nonnegative integer m and all $x \in X$.

The remaining proof is similar to the corresponding part of Theorem 2.2. This completes the proof. □

Corollary 3.4. *Let $\xi : [0,\infty) \to [0,\infty)$ be a nontrivial function satisfying*

$$\xi\left(\frac{t}{2}\right) \le \xi\left(\frac{1}{2}\right)\xi(t), \quad (t \ge 0),\ 0 < \xi\left(\frac{1}{2}\right) < \frac{1}{2}. \tag{3.13}$$

If $f : X \to Y$ is a mapping satisfying the following functional inequality

$$Df(x,y,z) \le \theta\{\xi(\|x\|) + \xi(\|y\|) + \xi(\|z\|)\} \tag{3.14}$$

for all $x, y, z \in X$ and for some $\theta \geq 0$, then there exists a unique additive mapping $h : X \to Y$ such that

$$\begin{aligned}
&\|f(x) - h(x)\| \\
&\quad \leq \frac{M\theta\xi(1/2)}{\sqrt[p]{1 - 2^p\xi(1/2)^p}} \left[\xi\left(\left| \frac{n(n-3)}{n^2+3} \right| \|x\| \right)^p + \xi\left(\left| \frac{n(n+3)}{n^2+3} \right| \|x\| \right)^p + \xi\left(\left| \frac{2n^2}{n^2+3} \right| \|x\| \right)^p \right. \\
&\qquad \left. + \xi\left(\left| \frac{2n(n+1)}{n^2+3} \right| \|x\| \right)^p + \xi\left(\left| \frac{2n(n-1)}{n^2+3} \right| \|x\| \right)^p + \xi\left(\left| \frac{4n}{n^2+3} \right| \|x\| \right)^p \right]^{1/p}
\end{aligned} \tag{3.15}$$

for all $x \in X$.

Proof. Letting $\varphi(x, y, z) = \theta\{\xi(\|x\|) + \xi(\|y\|) + \xi(\|z\|)\}$ and applying Theorem 3.3 with $L := 2\xi(1/2)$, we lead to the approximation. □

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