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Research Article

Some Properties of Solutions to a Class of Dirichlet Boundary Value Problems

Tingting Wang and Gejun Bao

Department of Mathematics, Harbin Institute of Technology, Harbin 150001, China

Correspondence should be addressed to Gejun Bao, baogj@hit.edu.cn

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This paper deals with the following Dirichlet problem: $d^*A(x, g + du) = d^*h$ in $\Omega, u_T = 0$ on $\partial\Omega$. Based on its solvability, we derive some properties of its solutions. In this paper, we mainly get three results. Firstly, we establish an integral estimate for the solutions of the above Dirichlet boundary value problem. Secondly, a stability result of solutions for varying differential forms g and h is obtained. Lastly, we present a weak reverse Hölder inequality for solutions.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. Throughout this paper we assume that $1 < p$, $q < \infty$, is a Hölder conjugate pair, $p + q = pq$. First, we consider the nonhomogeneous A-harmonic equation:

$$d^*A(x, g + du) = d^*h, \tag{1.1}$$

where $(g, h) \in L^p(\Omega, \wedge^l) \times W^{1,q}(\Omega, \wedge^l)$ and $A : \Omega \times \wedge^l(\mathbb{R}^n) \to \wedge^l(\mathbb{R}^n)$ is a Carathéodory mapping satisfying the following assumptions for fixed $0 < \alpha \le \beta < \infty$:

(1) Lipschitz continuity

$$|A(x, \xi) - A(x, \zeta)| \le \beta|\xi - \zeta|(|\xi| + |\zeta|)^{p-2}; \tag{1.2}$$

(2) uniform monotonicity

$$\langle A(x,\xi) - A(x,\zeta), \xi - \zeta \rangle \geq \alpha|\xi - \zeta|^2 (|\xi| + |\zeta|)^{p-2}; \tag{1.3}$$

(3) homogeneity

$$A(x, \lambda\xi) = \lambda|\lambda|^{p-2} A(x,\xi), \tag{1.4}$$

for almost every $x \in \Omega$ and all $\xi, \zeta \in \wedge^l(\mathbb{R}^n)$, $\lambda \in \mathbb{R}$.

In particular, for $A(x,\xi) = |\xi|^{p-2}\xi$, then (1.1) is simplified to the nonhomogeneous p-harmonic equation:

$$d^*\left(\left|g + du\right|^{p-2}(g + du)\right) = d^*h. \tag{1.5}$$

By Browder-Minty theory, see [1], the existence and uniqueness of a solution to the Dirichlet problem

$$\begin{aligned} d^*A(x, g + du) &= d^*h \quad \text{in } \Omega, \\ u_T &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.6}$$

in $W^{1,p}(\Omega, \wedge^{l-1})$ has been obtained by Iwaniec et al., see [2]. For the solution $u \in W^{1,p}(\Omega, \wedge^{l-1})$ with vanishing tangential component on $\partial\Omega$, we write it as $u \in W_T^{1,p}(\Omega, \wedge^{l-1})$.

Definition 1.1. Given that $(g, h) \in L^p(\Omega, \wedge^l) \times W^{1,q}(\Omega, \wedge^l)$, a differential form u is called a solution to the Dirichlet problem (1.6) if $u \in W_T^{1,p}(\Omega, \wedge^{l-1})$ and it holds

$$\int_\Omega \langle A(x, g + du), d\phi \rangle \mathrm{d}x = \int_\Omega \langle h, d\phi \rangle \mathrm{d}x, \tag{1.7}$$

for all $\phi \in W_T^{1,p}(\Omega, \wedge^{l-1})$.

A-harmonic equations for differential forms have been a very active field in recent years because they are an invaluable tool to describe various systems of partial differential equations and to express different geometrical structures on manifolds. Moreover, they can be used in many fields, such as physics, nonlinear elasticity theory, and the theory of quasiconformal mappings, see [3–10]. The purpose of this paper is to study properties of solutions of the Dirichlet boundary value problem (1.6) based on the existence of its solutions.

2. Notation and Preliminary Results

This section is devoted to the notation of the exterior calculus and a few necessary preliminaries. For more details the reader can refer to [2, 3].

We denote by $\wedge^l = \wedge^l(\mathbb{R}^n)$ the space of l-covectors in $\mathbb{R}^n$ and the direct sum

$$\wedge(\mathbb{R}^n) = \bigoplus_{l=0}^{n} \wedge^l(\mathbb{R}^n) \tag{2.1}$$

is a graded algebra with respect to the wedge product $\wedge$. We will make use of the exterior derivative:

$$d : C^\infty\left(\Omega, \wedge^l\right) \longrightarrow C^\infty\left(\Omega, \wedge^{l+1}\right) \tag{2.2}$$

and its formal adjoint operator

$$d^* = (-1)^{nl+1} * d* : C^\infty\left(\Omega, \wedge^{l+1}\right) \longrightarrow C^\infty\left(\Omega, \wedge^l\right), \tag{2.3}$$

known as the Hodge codifferential, where the symbol $*$ denotes the Hodge star duality operator. Note that each of the operators d and d^* applied twice gives zero.

Let $C^\infty(\overline{\Omega}, \wedge^l)$ be the class of infinitely differentiable l-forms on $\overline{\Omega} \subset \mathbb{R}^n$. Since Ω is a smooth domain, near each boundary point one can introduce a local coordinate system $(x_1, x_2, \ldots, x_n)$ such that $x_n = 0$ on $\partial\Omega$ and such that the x_n-curve is orthogonal to $\partial\Omega$. Near this boundary point, every differential form $\omega \in C^\infty(\overline{\Omega}, \wedge^l)$ can be decomposed as $\omega(x) = \omega_T(x) + \omega_N(x)$, where

$$\begin{aligned} \omega_T(x) &= \sum_{1 \le i_1 < \cdots < i_l < n} \omega_{i_1,\ldots,i_l}(x) dx_{i_1} \wedge \cdots \wedge dx_{i_l}, \\ \omega_N(x) &= \sum_{1 \le i_1 < \cdots < i_l = n} \omega_{i_1,\ldots,i_l}(x) dx_{i_1} \wedge \cdots \wedge dx_{i_l} \end{aligned} \tag{2.4}$$

are called the tangential and the normal parts of ω, respectively. Now, the duality between d and d^* is expressed by the integration by parts formula

$$\int_\Omega \langle du, v\rangle = \int_\Omega \langle u, d^* v\rangle, \tag{2.5}$$

for all $u \in C^\infty(\overline{\Omega}, \wedge^l)$ and $v \in C^\infty(\overline{\Omega}, \wedge^{l+1})$, provided $u_T = 0$ or $v_N = 0$. The symbol $\langle\cdot,\cdot\rangle$ denotes the inner product, that is, let $\alpha = \sum_I \alpha_I(x) \mathrm{d}x_I$ and $\beta = \sum_I \beta_I(x) \mathrm{d}x_I$, then $\langle\alpha, \beta\rangle = \sum_I \alpha_I(x)\beta_I(x)$.

Due to (2.5), extended definitions for d and d^* can be introduced as the introduction of weak derivatives.

Definition 2.1 (see [2]). Suppose that $\omega \in L^1_{\mathrm{loc}}(\Omega, \wedge^l)$ and $v \in L^1_{\mathrm{loc}}(\Omega, \wedge^{l+1})$. If

$$\int_\Omega \langle \omega, d^*\eta \rangle = \int_\Omega \langle v, \eta \rangle \tag{2.6}$$

for every test form $\eta \in C_0^\infty(\Omega, \wedge^{l+1})$, one says that ω has generalized exterior derivative v and write $v = \tilde{d}\omega$.

The notion of the generalized exterior coderivative $\tilde{d}^*$ can be defined analogously.

Definition 2.2 (see [2]). Suppose that $\omega \in L^1_{\mathrm{loc}}(\Omega, \wedge^l)$ and $v \in L^1_{\mathrm{loc}}(\Omega, \wedge^{l-1})$. If

$$\int_\Omega \langle \omega, d\eta \rangle = \int_\Omega \langle v, \eta \rangle \tag{2.7}$$

for every test form $\eta \in C_0^\infty(\Omega, \wedge^{l-1})$, one says that ω has generalized exterior coderivative v and write $v = \tilde{d}^*\omega$.

Remark 2.3. (i) Observe that generalized exterior derivatives have many properties similar to those of weak derivatives. For example, (i_1) if it exists, it is unique; (i_2) if ω is differentiable in the conventional sense, then its generalized exterior derivative $\tilde{d}\omega$ is identical to its the classical exterior differential $d\omega$. Analogous results hold for generalized exterior coderivative.

(ii) If the generalized exterior derivative of ω, $\tilde{d}\omega$, exists, then $\tilde{d}\omega$ also has its generalized exterior derivative $\tilde{d}(\tilde{d}\omega)$. Moreover, $\tilde{d}(\tilde{d}\omega) = 0$.

In fact, according to Definition 2.1, it holds

$$\int_\Omega \langle \omega, d^*\phi \rangle = \int_\Omega \left\langle \tilde{d}\omega, \phi \right\rangle \tag{2.8}$$

for every test form $\phi \in C_0^\infty(\Omega, \wedge^{l+1})$. Thus, for every $\eta \in C_0^\infty(\Omega, \wedge^{l+2})$, we have $d^*\eta \in C_0^\infty(\Omega, \wedge^{l+1})$ and by taking $\phi = d^*\eta$ in the above integral equality implies

$$\begin{aligned} \int_\Omega \left\langle \tilde{d}\omega, d^*\eta \right\rangle &= \int_\Omega \langle \omega, d^*d^*\phi \rangle \\ &= \int_\Omega \langle \omega, 0 \rangle \\ &= 0. \end{aligned} \tag{2.9}$$

Therefore, Definition 2.1 yields that $\tilde{d}(\tilde{d}\omega)$ exists and $\tilde{d}(\tilde{d}\omega) = 0$. Similarly, we have $\tilde{d}^*(\tilde{d}^*\omega) = 0$.

(iii) Together with the expression of differential forms, the definition of weak derivative and its uniqueness, we can prove that d and $\widetilde{d}$ have analogous expressions, that is, for $\omega(x) = \sum_I \omega_I(x)\mathrm{d}x_I$, we have

$$\begin{aligned} d\omega(x) &= \sum_I \sum_{k=1}^{n} \frac{\partial \omega_I}{\partial x_k}\mathrm{d}x_k \wedge \mathrm{d}x_I, \\ \widetilde{d}\omega(x) &= \sum_I \sum_{k=1}^{n} \frac{\widetilde{\partial} \omega_I}{\partial x_k}\mathrm{d}x_k \wedge \mathrm{d}x_I, \end{aligned} \tag{2.10}$$

where ∂ denotes the ordinary derivative and $\widetilde{\partial}$ the weak derivative. So in the next we use d to represent the action instead of $\widetilde{d}$, similar for d^* and $\widetilde{d}^*$.

(iv) Lastly, we refer to

$$\ker d = \left\{\omega \in L^1_{\mathrm{loc}}\left(\Omega, \wedge^l\right) : d\omega = 0\right\} \tag{2.11}$$

as the closed l-forms and to

$$\ker d^* = \left\{\omega \in L^1_{\mathrm{loc}}\left(\Omega, \wedge^l\right) : d^*\omega = 0\right\} \tag{2.12}$$

as the coclosed l-forms.

Definition 2.4 (see [2]). A l-form ω is said to have vanishing tangential component at $\partial\Omega$ in a generalized sense, if both ω and $d\omega$ belong to $L^1(\Omega, \wedge^l)$ and

$$\int_\Omega \langle \omega, d^*\eta\rangle = \int_\Omega \langle d\omega, \eta\rangle \tag{2.13}$$

holds for any $\eta \in C^\infty(\Omega, \wedge^{l+1})$. One writes $\omega_T = 0$.

The notion of vanishing normal part $\omega_N = 0$ can be defined analogously. Now, the following extension of the identity (2.5) can be introduced, by an approximation argument proved by Iwaniec and Lutoborski [3].

Proposition 2.5 (see [3]). *For (p, q) a Hölder conjugate pair and $u \in W^{1,p}(\Omega, \wedge^l)$, $v \in W^{1,q}(\Omega, \wedge^{l+1})$ one has*

$$\int_\Omega \langle du, v\rangle = \int_\Omega \langle u, d^*v\rangle \tag{2.14}$$

provided $u_T = 0$ or $v_N = 0$.

Finally, we present briefly some spaces of differential forms:

$L^p(\Omega, \wedge^l)$—the space of l-forms ω with coefficients in $L^p(\Omega)$;

$L_1^p(\Omega, \wedge^l)$—the space of l-forms ω such that $\nabla\omega$ is a regular distribution of $L^p(\Omega, \wedge^l)$;

$W^{1,p}(\Omega, \wedge^l)$—the Sobolev space of l-forms ω defined by $L^p(\Omega, \wedge^l) \cap L_1^p(\Omega, \wedge^l)$;

$W_T^{1,p}(\Omega, \wedge^l)$—the space of l-forms ω in $W^{1,p}(\Omega, \wedge^l)$ with vanishing tangential component on $\partial\Omega$.

which are used throughout this paper.

3. Main Results

We study the properties of solutions of the Dirichlet boundary value problem (1.6) whose existence can be deduced by the following.

Lemma 3.1 (see [2]). *For each data $(g, h) \in L^p(\Omega, \wedge^l) \times W^{1,q}(\Omega, \wedge^l)$, there exists a solution $u \in W_T^{1,p}(\Omega, \wedge^{l-1})$ to the Dirichlet problem* (1.6).

3.1. An Integral Estimate

We start with a proposition which gives an important estimate for solutions of (1.6).

Proposition 3.2. *Given $(g, h) \in L^p(\Omega, \wedge^l) \times W^{1,q}(\Omega, \wedge^l)$, suppose that u is a solution of* (1.6) *in $W_T^{1,p}(\Omega, \wedge^{l-1})$. Then one has*

$$\int_\Omega |du|^p \mathrm{d}x \le c(p, \alpha, \beta)\left(\int_\Omega |g|^p \mathrm{d}x + \int_\Omega |h|^q \mathrm{d}x\right). \tag{3.1}$$

Proof. Taking the solution u as the test function in (1.7) yields

$$\int_\Omega \langle A(x, g + du), du\rangle = \int_\Omega \langle h, du\rangle, \tag{3.2}$$

thus we have

$$\int_\Omega \langle A(x, g + du), g + du\rangle = \int_\Omega \langle A(x, g + du), g\rangle + \int_\Omega \langle h, du\rangle. \tag{3.3}$$

Next we apply Young's inequality as follows:

$$ab \le \varepsilon a^r + \varepsilon^{-1/(r-1)} b^{r/(r-1)}, \quad \varepsilon > 0,\ r > 1. \tag{3.4}$$

It follows from the structural assumptions (1) and (2) that

$$\begin{aligned}
\alpha\int_{\Omega}|g+du|^{p} &\le \int_{\Omega}\langle A(x,g+du),g+du\rangle \\
&= \int_{\Omega}\langle h,du\rangle+\int_{\Omega}\langle A(x,g+du),g\rangle \\
&\le \int_{\Omega}|h||du|+\beta\int_{\Omega}|g+du|^{p-1}|g| \\
&\le \int_{\Omega}|h||du|+\beta\int_{\Omega}\left(\varepsilon|g+du|^{p}+c(\varepsilon,p)|g|^{p}\right) \\
&= \varepsilon\beta\int_{\Omega}|g+du|^{p}+\int_{\Omega}|h||du|+c(\varepsilon,p)\beta\int_{\Omega}|g|^{p}.
\end{aligned} \tag{3.5}$$

By choosing $\varepsilon=\alpha/(2\beta)$ and using again Young's inequality, we have

$$\begin{aligned}
\int_{\Omega}|g+du|^{p} &\le \frac{2}{\alpha}\int_{\Omega}|h|\ |du|+c(p,\alpha,\beta)\int_{\Omega}|g|^{p} \\
&\le \frac{2}{\alpha}\int_{\Omega}\left(\varepsilon|du|^{p}+c(\varepsilon,p)|h|^{q}\right)+c(p,\alpha,\beta)\int_{\Omega}|g|^{p}.
\end{aligned} \tag{3.6}$$

Therefore, we get that

$$\begin{aligned}
\int_{\Omega}|du|^{p} &\le \int_{\Omega}\left(|g+du|+|g|\right)^{p} \\
&\le 2^{p}\int_{\Omega}|g+du|^{p}+2^{p}\int_{\Omega}|g|^{p} \\
&\le \frac{2^{p+1}}{\alpha}\varepsilon\int_{\Omega}|du|^{p}+\frac{2^{p+1}}{\alpha}c(\varepsilon,p)\int_{\Omega}|h|^{q}+c(p,\alpha,\beta)\int_{\Omega}|g|^{p}.
\end{aligned} \tag{3.7}$$

Finally, we obtain by choosing $\varepsilon=\alpha/2^{p+2}$ that

$$\int_{\Omega}|du|^{p}\mathrm{d}x \le c(p,\alpha,\beta)\left(\int_{\Omega}|g|^{p}\mathrm{d}x+\int_{\Omega}|h|^{q}\mathrm{d}x\right). \tag{3.8}$$

The theorem follows. □

3.2. Stability of Solutions

In this section, we establish the weak convergence of solutions of (1.6) with varying differential forms g and h. Particularly, given a sequence $(g_j, h_j) \in L^p(\Omega, \wedge^l) \times W^{1,q}(\Omega, \wedge^l)$, Lemma 3.1 implies that there exists solution $u_j \in W_T^{1,p}(\Omega, \wedge^{l-1})$ to

$$d^* A(x, g_j + du_j) = d^* h_j. \tag{3.9}$$

Suppose that $(g_j, h_j) \to (g, h)$ in $L^p(\Omega, \wedge^l) \times W^{1,q}(\Omega, \wedge^{l+1})$, and $u_j \rightharpoonup u$ in $W^{1,p}(\Omega, \wedge^{l-1})$, we will show that $u \in W_T^{1,p}(\Omega, \wedge^{l-1})$ is a solution of (1.6).

Theorem 3.3. *Under the hypotheses above, $u \in W_T^{1,p}(\Omega, \wedge^{l-1})$ is a solution of* (1.6).

To prove Theorem 3.3, we need firstly to give the definition of the weak convergence for sequences in spaces of differential forms.

Definition 3.4. One says that φ_j is weakly convergent to φ in $L^p(\Omega, \wedge^l)$ if

$$\int_\Omega \langle \varphi_j, h \rangle \longrightarrow \int_\Omega \langle \varphi, h \rangle, \tag{3.10}$$

whenever $h \in L^q(\Omega, \wedge^l)$ and write $\varphi_j \rightharpoonup \varphi$ in $L^p(\Omega, \wedge^l)$.

It is easy to verify that it has the following equivalent definition.

Definition 3.5. One says that φ_j weakly convergent to φ in $L^p(\Omega, \wedge^l)$ if

$$\int_\Omega \varphi_j \wedge h \longrightarrow \int_\Omega \varphi \wedge h \tag{3.11}$$

whenever $h \in L^q(\Omega, \wedge^{n-l})$ and write $\varphi_j \rightharpoonup \varphi$ in $L^p(\Omega, \wedge^l)$.

According to the well-known results in Sobolev space in terms of functions, and together with the expression of differential forms and the diagonal rule we can easily obtain that

Proposition 3.6. *For $1 < p < \infty$, $L^p(\Omega, \wedge^l)$ is reflexive.*

Proposition 3.7. *$\varphi_j \rightharpoonup \varphi$ in $W^{1,p}(\Omega, \wedge^l)$ if and only if $\varphi_j \rightharpoonup \varphi$ in $L^p(\Omega, \wedge^l)$ and $\nabla\varphi_j \rightharpoonup \nabla\varphi$ in $L^p(\Omega, \wedge^l)$, where $\nabla\varphi = ((\partial\varphi/\partial x_1), \ldots, (\partial\varphi/\partial x_n))$ and the partial differentiation is applied to the coefficients of φ.*

Lemma 3.8. *Suppose that a sequence of differential forms $\{\varphi_j\}$ converges to φ weakly in $L^p(\Omega, \wedge^l)$ while the generalized exterior derivatives of φ_j, $d\varphi_j$, exist and stay bounded in $L^p(\Omega, \wedge^{l+1})$. Then the generalized exterior derivative of φ, $d\varphi$, exists and $d\varphi_j \rightharpoonup d\varphi$ in $L^p(\Omega, \wedge^{l+1})$.*

Proof. On the one hand, since φ_j has generalized exterior derivative $d\varphi_j$, then according to Definition 2.1 we have

$$\int_\Omega \langle d\varphi_j, \eta\rangle = \int_\Omega \langle \varphi_j, d^*\eta\rangle, \tag{3.12}$$

for every $\eta \in C_0^\infty(\Omega, \wedge^{l+1})$. Notice that $\varphi_j \rightharpoonup \varphi$ in $L^p(\Omega, \wedge^l)$ and $d^*\eta \in C_0^\infty(\Omega, \wedge^l) \subset L^q(\Omega, \wedge^l)$, it follows from Definition 3.4 that

$$\int_\Omega \langle \varphi_j, d^*\eta\rangle \longrightarrow \int_\Omega \langle \varphi, d^*\eta\rangle. \tag{3.13}$$

Thus, we obtain

$$\int_\Omega \langle d\varphi_j, \eta\rangle \longrightarrow \int_\Omega \langle \varphi, d^*\eta\rangle, \tag{3.14}$$

for every $\eta \in C_0^\infty(\Omega, \wedge^{l+1})$.

On the other hand, since $d\varphi_j$ stays bounded in $L^p(\Omega, \wedge^{l+1})$, then it follows from Proposition 3.6 that there exists weakly convergence subsequence of $d\varphi_j$, we may assume that

$$d\varphi_j \rightharpoonup g \tag{3.15}$$

in $L^p(\Omega, \wedge^{l+1})$. Hence, for $\eta \in C_0^\infty(\Omega, \wedge^{l+1}) \subset L^q(\Omega, \wedge^{l+1})$ we have

$$\int_\Omega \langle d\varphi_j, \eta\rangle \longrightarrow \int_\Omega \langle g, \eta\rangle. \tag{3.16}$$

Combining (3.14) and (3.16), the uniqueness of the limit yields

$$\int_\Omega \langle \varphi, d^*\eta\rangle = \int_\Omega \langle g, \eta\rangle, \tag{3.17}$$

for every $\eta \in C_0^\infty(\Omega, \wedge^{l+1})$. Thus, Definition 2.1 implies that the generalized exterior derivative of φ exists and $g = d\varphi$. Furthermore, we have from (3.15) that $d\varphi_j \rightharpoonup d\varphi$ in $L^p(\Omega, \wedge^{l+1})$. The lemma follows. □

Proof of Theorem 3.3. Taking $(u_j - u_k)$ as test differential form in (1.7) for both

$$\begin{aligned} d^*A(x, g_j + du_j) &= d^*h_j, \\ d^*A(x, g_k + du_k) &= d^*h_k, \end{aligned} \tag{3.18}$$

then we obtain

$$\int_{\Omega}\langle A(x,g_j+du_j)-A(x,g_k+du_k),du_j-du_k\rangle=\int_{\Omega}\langle h_j-h_k,du_j-du_k\rangle, \tag{3.19}$$

or, equivalently, we have

$$\begin{aligned}&\int_{\Omega}\langle A(x,g_j+du_j)-A(x,g_k+du_k),(g_j+du_j)-(g_k+du_k)\rangle\\&\quad=\int_{\Omega}\langle A(x,g_j+du_j)-A(x,g_k+du_k),g_j-g_k\rangle+\int_{\Omega}\langle h_j-h_k,du_j-du_k\rangle.\end{aligned} \tag{3.20}$$

Write $\xi=g_j+du_j$ and $\zeta=g_k+du_k$, then the above identity can be simplified to

$$\begin{aligned}&\int_{\Omega}\langle A(x,\xi)-A(x,\zeta),\xi-\zeta\rangle\\&\quad=\int_{\Omega}\langle A(x,\xi)-A(x,\zeta),g_j-g_k\rangle+\int_{\Omega}\langle h_j-h_k,du_j-du_k\rangle.\end{aligned} \tag{3.21}$$

Then it follows from the Lipschitz condition (1), the monotonicity condition (2), and Hölder inequality that:

$$\begin{aligned}\alpha\int_{\Omega}|\xi-\zeta|^p&\le\beta\int_{\Omega}(|\xi|+|\zeta|)^{p-1}|g_j-g_k|\\&\quad+\int_{\Omega}|h_j-h_k||du_j-du_k|\\&\le\beta\left(\int_{\Omega}(|\xi|+|\zeta|)^p\right)^{1/q}\left(\int_{\Omega}|g_j-g_k|^p\right)^{1/p}\\&\quad+\left(\int_{\Omega}|du_j-du_k|^p\right)^{1/p}\left(\int_{\Omega}|h_j-h_k|^q\right)^{1/q}\\&\le c(p,\beta)\left(\|\xi\|_p^{p-1}+\|\zeta\|_p^{p-1}\right)\|g_j-g_k\|_p+\|du_j-du_k\|_p\|h_j-h_k\|_q.\end{aligned} \tag{3.22}$$

Therefore, we obtain

$$\begin{aligned}\|\xi-\zeta\|_p&\le c(p,\alpha,\beta)\left(\|\xi\|_p^{1/q}+\|\zeta\|_p^{1/q}\right)\|g_j-g_k\|_p^{1/p}\\&\quad+c(p,\alpha)\|du_j-du_k\|_p^{1/p}\|h_j-h_k\|_q^{1/p}.\end{aligned} \tag{3.23}$$

Since g_j and h_j are Cauchy sequences in $L^p(\Omega, \wedge^l)$ and $L^q(\Omega, \wedge^l)$, respectively, we have $\|g_j\|_p < \infty$ and $\|h_j\|_q < \infty$ for all $j \in \mathbb{N}$, and further we get by applying Proposition 3.2 that

$$\begin{aligned}
\|\xi\|_p^{1/q} + \|\zeta\|_p^{1/q} &\le c(p)\left(\|g_j\|_p^{1/q} + \|du_j\|_p^{1/q} + \|g_k\|_p^{1/q} + \|du_k\|_p^{1/q}\right) \\
&\le c(p)\Big(\|g_j\|_p^{1/q} + c(p,\alpha,\beta)\left(\|g_j\|_p^{1/q} + \|h_j\|_q^{1/p}\right) \\
&\qquad + \|g_k\|_p^{1/q} + c(p,\alpha,\beta)\left(\|g_k\|_p^{1/q} + \|h_k\|_q^{1/p}\right)\Big) \\
&\le c(p,\alpha,\beta)\left(\|g_j\|_p^{1/q} + \|g_k\|_p^{1/q} + \|h_j\|_q^{1/p} + \|h_k\|_q^{1/p}\right) \\
&< \infty, \\
\|du_j - du_k\|_p^{1/p} &\le c(p)\left(\|du_j\|_p^{1/p} + \|du_k\|_p^{1/p}\right) \\
&\le c(p,\alpha,\beta)\left(\|g_j\|_p^{1/p} + \|g_k\|_p^{1/p} + \|h_j\|_q^{(q-1)/p} + \|h_k\|_q^{(q-1)/p}\right) \\
&< \infty.
\end{aligned} \tag{3.24}$$

Combining (3.23)-(3.24), we obtain

$$\|\xi - \zeta\|_p \longrightarrow 0, \tag{3.25}$$

as $j, k \to \infty$. Thus, the Minkowski inequality implies

$$\begin{aligned}
\|du_j - du_k\|_p &= \|(\xi - \zeta) - (g_j - g_k)\|_p \\
&\le \|\xi - \zeta\|_p + \|g_j - g_k\|_p \longrightarrow 0,
\end{aligned} \tag{3.26}$$

as $j, k \to \infty$, that is, $\{du_j\}$ is a Cauchy sequence in $L^p(\Omega, \wedge^l)$. Thus, we have du_j is bounded in $L^p(\Omega, \wedge^l)$. Then, it follows from Lemma 3.8 that du exists and $du_j \rightharpoonup du$ in $L^p(\Omega, \wedge^l)$. Note that $\{du_j\}$ is a Cauchy sequence in $L^p(\Omega, \wedge^l)$, then we have by the uniqueness of weak limit that

$$du_j \longrightarrow du \tag{3.27}$$

in $L^p(\Omega, \wedge^l)$. According to Definition 2.4, we know that

$$\int_\Omega \langle u_j, d^*\eta \rangle = \int_\Omega \langle du_j, \eta \rangle, \tag{3.28}$$

for any $\eta \in C^\infty(\Omega, \wedge^l)$. And it follows easily that

$$\int_\Omega \langle u, d^*\eta \rangle = \int_\Omega \langle du, \eta \rangle, \tag{3.29}$$

since $u_j \rightharpoonup u$ in $W^{1,p}(\Omega, \wedge^{l-1})$ and Ω is bounded. Hence, we have $u \in W_T^{1,p}(\Omega, \wedge^{l-1})$. The final step in our proof is to show that u satisfies (1.7). Recall that

$$\int_\Omega \langle A(x, g_j + du_j), d\phi\rangle = \int_\Omega \langle h_j, d\phi\rangle, \tag{3.30}$$

for all $\phi \in W_T^{1,p}(\Omega, \wedge^{l-1})$. It is rather easy to obtain the following:

$$\begin{aligned}
&\int_\Omega \langle A(x, g_j + du_j) - A(x, g + du), d\phi\rangle \\
&\quad \le \beta \int_\Omega \left|(g_j + du_j) - (g + du)\right|^{p-1} |d\phi| \\
&\quad \le \beta \left\|(g_j + du_j) - (g + du)\right\|_p^{p-1} \|d\phi\|_p \\
&\quad \le \beta \|d\phi\|_p \left(\|g_j - g\|_p^{p-1} + \|du_j - du\|_p^{p-1}\right) \longrightarrow 0,
\end{aligned} \tag{3.31}$$

as $j \to \infty$, that is, it holds

$$\int_\Omega \langle A(x, g_j + du_j), d\phi\rangle \longrightarrow \int_\Omega \langle A(x, g + du), d\phi\rangle, \tag{3.32}$$

as $j \to \infty$. On the other hand, an easy computation gives that

$$\int_\Omega \langle h_j, d\phi\rangle \longrightarrow \int_\Omega \langle h, d\phi\rangle, \tag{3.33}$$

as $j \to \infty$. Therefore, we obtain

$$\int_\Omega \langle A(x, g + du), d\phi\rangle = \int_\Omega \langle h, d\phi\rangle, \tag{3.34}$$

for all $\phi \in W_T^{1,p}(\Omega, \wedge^{l-1})$, which implies the desired result. □

3.3. Weak Reverse Hölder Inequality

In virtue of the fact that $d^*(d^*) = 0$, we can write (1.1) as the following:

$$A(x, g + du) = h + d^* v, \tag{3.35}$$

for some $v \in W^{1,q}(\Omega, \wedge^{l+1})$. For v in (3.35) we have

$$|d^* v| = \left|A(x, g + du) - h\right| \le \beta \left|g + du\right|^{p-1} + |h|, \tag{3.36}$$

thus, by using the Minkowski inequality and Proposition 3.2, we have the estimate

$$\|d^* v\|_q \leq c(p,\alpha,\beta)\|g\|_p^{p/q} + \|h\|_q. \tag{3.37}$$

For an arbitrary nonnegative test function $\eta \in C_0^\infty(\Omega)$, we multiply (3.35) for η^p and by the homogeneity assumption (3), and we obtain that

$$A(x, \eta^q g + \eta^q du) = \eta^p h + \eta^p d^* v. \tag{3.38}$$

Since

$$\begin{aligned} \eta^q du &= d(\eta^q u) - d(\eta^q) \wedge u, \\ \eta^p d^* v &= d^*(\eta^p v) - (-1)^{nl+1} \star d(\eta^p) \wedge \star v, \end{aligned} \tag{3.39}$$

we have

$$A(x, \eta^q g - d(\eta^q) \wedge u + d(\eta^q u)) = \eta^p h - (-1)^{nl+1} \star d(\eta^p) \wedge \star v + d^*(\eta^p v). \tag{3.40}$$

Applying the operator d^* to the above equation leads to a new nonhomogeneous A-harmonic equation as follows:

$$d^* A(x, \eta^q g - d(\eta^q) \wedge u + d(\eta^q u)) = d^*\left(\eta^p h - (-1)^{nl+1} \star d(\eta^p) \wedge \star v\right), \tag{3.41}$$

where

$$\begin{aligned} \eta^q u &\in W_0^{1,p}\left(\Omega, \wedge^{l-1}\right), \\ \eta^q g - d(\eta^q) \wedge u &\in L^p\left(\Omega, \wedge^l\right), \\ \eta^p h - (-1)^{nl+1} \star d(\eta^p) \wedge \star v &\in W^{1,q}\left(\Omega, \wedge^{l+1}\right). \end{aligned} \tag{3.42}$$

Then by applying Proposition 3.2 to (3.41) we have

$$\begin{aligned} \int_\Omega |d(\eta^q u)|^p \leq c(p,\alpha,\beta)\bigg(&\int_\Omega |\eta^q g - d(\eta^q) \wedge u|^p \\ &+ \int_\Omega \left|\eta^p h - (-1)^{nl+1} \star d(\eta^p) \wedge \star v\right|^q\bigg), \end{aligned} \tag{3.43}$$

which implies

$$
\begin{aligned}
\int_\Omega \left|\eta^q du\right|^p &= \int_\Omega \left|d(\eta^q u) \ - d(\eta^q)\wedge u\right|^p \\
&\le 2^p \int_\Omega \left|d(\eta^q)\wedge u\right|^p + 2^p \int_\Omega \left|d(\eta^q u)\right|^p \\
&\le 2^p \int_\Omega \left|d(\eta^q)\wedge u\right|^p + c(p,\alpha,\beta)\left(\int_\Omega \left|\eta^q g\right|^p \right. \\
&\qquad\qquad + \int_\Omega \left|d(\eta^q)\wedge u\right|^p + \int_\Omega \left|\eta^p h\right|^q \\
&\qquad\qquad \left. + \int_\Omega \left|d(\eta^p)\right|^q |v|^q \right).
\end{aligned}
\tag{3.44}
$$

Therefore, we obtain

$$
\begin{aligned}
\int_\Omega \eta^{p+q}|du|^p \le c(p,\alpha,\beta)\biggl(&\int_\Omega |u|^p\left|d(\eta^q)\right|^p + \int_\Omega \eta^{p+q}\left(|g|^p + |h|^q\right) \\
&+ \int_\Omega |v|^q\left|d(\eta^p)\right|^q\biggr).
\end{aligned}
\tag{3.45}
$$

For an arbitrary cube $Q\subset\mathbb{R}^n$, we can choose a cut-off function $\eta\in C_0^\infty(2Q)$ such that $0\le\eta\le 1$, $|\nabla\eta|\le c(n)|Q|^{-1/n}$ and $\eta\equiv 1$ on Q, where $|Q|$ denotes the Lebesgue measure of Q. Now inequality (3.45) yields

$$
\int_Q |du|^p \le c(n,p,\alpha,\beta)\left(\frac{1}{|Q|^{p/n}}\int_{2Q}|u|^p + \frac{1}{|Q|^{q/n}}\int_{2Q}|v|^q + \int_{2Q}\left(|g|^p + |h|^q\right)\right)
\tag{3.46}
$$

Note that (3.35) is not affected if a closed form u_0 is subtracted from u and a coclosed form v_0 is subtracted from v. Therefore, the above calculation shows

$$
\int_Q |du|^p \le c(n,p,\alpha,\beta)\left(\frac{1}{|Q|^{p/n}}\int_{2Q}|u-u_0|^p + \frac{1}{|Q|^{q/n}}\int_{2Q}|v-v_0|^q + \int_{2Q}\left(|g|^p + |h|^q\right)\right).
\tag{3.47}
$$

Now, a Poincaré-Sobolev inequality for differential forms is needed.

Lemma 3.9 (see [2]). *Let Q be a cube in $\mathbb{R}^n$. Suppose that $\xi \in W^{1,r}(Q,\wedge^l)$ and $\zeta \in W^{1,s}(\Omega,\wedge^l)$, where $1 < r$, $s < n$. Then there exist a closed form $\xi_0 \in L^r(Q,\wedge^l)$ and a coclosed form $\zeta_0^* \in L^s(Q,\wedge^l)$ such that*

$$\begin{aligned} \|\xi - \xi_0\|_{nr/(n-r)} &\le c(n,r)\|d\xi\|_r, \\ \left\|\zeta - \zeta_0^*\right\|_{ns/(n-s)} &\le c(n,s)\|d^*\zeta\|_s. \end{aligned} \tag{3.48}$$

As a consequence of Lemma 3.9, we obtain the following by proceeding as in the proof of Corollary 3 in [5].

Corollary 3.10. *Let Q be a cube in $\mathbb{R}^n$. Suppose that $\xi \in W^{1,r}(Q,\wedge^l)$ and $\zeta \in W^{1,s}(\Omega,\wedge^l)$, where $1 < r$, $s < \infty$. Then there exist a closed form $\xi_0 \in L^r(Q,\wedge^l)$ and a coclosed form $\zeta_0^* \in L^s(Q,\wedge^l)$ such that*

$$\frac{1}{\operatorname{diam}(Q)}\left(⨍_Q|\xi - \xi_0|^r\right)^{1/r} \le c(n,r)\left(⨍_Q|d\xi|^{nr/(n+r-1)}\right)^{(n+r-1)/nr}, \tag{3.49}$$

$$\frac{1}{\operatorname{diam}(Q)}\left(⨍_Q\left|\zeta - \zeta_0^*\right|^s\right)^{1/s} \le c(n,s)\left(⨍_Q|d^*\zeta|^{ns/(n+s-1)}\right)^{(n+s-1)/ns}, \tag{3.50}$$

where $⨍_Q$ denotes the integral mean over Q, that is,

$$⨍_Q = \frac{1}{|Q|}\int_Q. \tag{3.51}$$

It follows from (3.49) that

$$\int_{2Q}|u - u_0|^p \mathrm{d}x \le c(n,p)|Q|^{1/n}\left(\int_{2Q}|du|^{np/(n+p-1)}\mathrm{d}x\right)^{(n+p-1)/n}, \tag{3.52}$$

and we have by (3.50), Hölder inequality, and (3.37) that

$$\begin{aligned} \|v - v_0\|_{q;2Q} &\le c(n,p)|Q|^{1/nq}\|d^*v\|_{nq/(n+q-1)\ ;2Q} \\ &\le c(n,p)|Q|^{1/nq}|2Q|^{(n+q-1)/nq-1/q}\|d^*v\|_{q;2Q} \\ &\le c(n,p,\alpha,\beta)|Q|^{1/n}\left(\|g\|_{p;2Q}^{p/q} + \|h\|_{q;2Q}\right). \end{aligned} \tag{3.53}$$

Combining (3.52) and (3.53) with (3.47), we obtain that

$$\int_Q|du|^p \le c(n,p,\alpha,\beta)\left(|Q|^{(1-p)/n}\left(\int_{2Q}|du|^{np/(n+p-1)}\right)^{(n+p-1)/n} + \int_{2Q}\left(|g|^p + |h|^q\right)\right). \tag{3.54}$$

Therefore, we obtain that

$$\fint_Q |du|^p \le c(n,p,\alpha,\beta)\left(\left(\fint_{2Q}|du|^{np/(n+p-1)}\right)^{(n+p-1)/n}+\fint_{2Q}\left(|g|^p+|h|^q\right)\right). \tag{3.55}$$

This is the desired estimate.

In conclusion, we summarize the above results in the following theorem.

Theorem 3.11. *For (p,q) a Hölder conjugate pair. Given $(g,h)\in L^p(\Omega,\wedge^l)\times W^{1,q}(\Omega,\wedge^l)$, suppose that $u\in W_T^{1,p}(\Omega,\wedge^{l-1})$ is a solution of (1.6). Then one has*

$$\fint_Q |du|^p \le c(n,p,\alpha,\beta)\left(\left(\fint_{2Q}|du|^{np/(n+p-1)}\right)^{(n+p-1)/n}+\fint_{2Q}\left(|g|^p+|h|^q\right)\right), \tag{3.56}$$

where $\fint_Q$ denotes the integral mean over Q, that is,

$$\fint_Q = \frac{1}{|Q|}\int_Q. \tag{3.57}$$

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