Research Article

On an Integral Transform of a Class of Analytic Functions

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Received 5 June 2012; Accepted 14 September 2012

Abstract and Applied Analysis
Hindawi Publishing Corporation
doi:10.1155/2012/259054

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For \( \alpha, \gamma \geq 0 \) and \( \beta < 1 \), let \( \mathcal{K}_\beta(\alpha, \gamma) \) denote the class of all normalized analytic functions \( f \) in the open unit disc \( E = \{ z : |z| < 1 \} \) such that \( \Re e^{i \phi} ((1 - \alpha + 2 \gamma) f(z)/z + \alpha - 2 \gamma) f'(z) + \gamma z f''(z) - \beta \) > 0, \( z \in E \) for some \( \phi \in \mathbb{R} \). It is known (Noshiro (1934) and Warschawski (1935)) that functions in \( \mathcal{K}_\beta(1,0) \) are close-to-convex and hence univalent for \( 0 \leq \beta < 1 \). For \( f \in \mathcal{K}_\beta(\alpha, \gamma) \), we consider the integral transform \( F(z) = V_\lambda(f)(z) := \int_0^1 \lambda(t) (f(tz)/t) dt \), where \( \lambda \) is a nonnegative real-valued integrable function satisfying the condition \( \int_0^1 \lambda(t) dt = 1 \). The aim of present paper is, for given \( \delta < 1 \), to find sharp values of \( \beta \) such that (i) \( V_\delta(f) \in \mathcal{K}_\delta(1,0) \) whenever \( f \in \mathcal{K}_\delta(\alpha, \gamma) \) and (ii) \( V_\lambda(f) \in \mathcal{K}_\lambda(\alpha, \gamma) \) whenever \( f \in \mathcal{K}_\lambda(\alpha, \gamma) \).

1. Introduction

Let \( \mathcal{A} \) denote the class of analytic functions \( f \) defined in the open unit disc \( E = \{ z : |z| < 1 \} \) with the normalizations \( f(0) = f'(0) - 1 = 0 \), and let \( S \) be the subclass of \( \mathcal{A} \) consisting of functions univalent in \( E \). For any two functions \( f(z) = z + \sum_{n=2}^\infty a_n z^n \) and \( g(z) = z + \sum_{n=2}^\infty b_n z^n \) in \( \mathcal{A} \), the Hadamard product (or convolution) of \( f \) and \( g \) is the function \( f \ast g \) defined by

\[
(f \ast g)(z) = z + \sum_{n=2}^\infty a_n b_n z^n.
\]

(1.1)

For \( f \in \mathcal{A} \), Fournier and Ruscheweyh [1] introduced the integral operator

\[
F(z) = V_\lambda(f)(z) := \int_0^1 \lambda(t) \frac{f(tz)}{t} dt,
\]

(1.2)
where \( \lambda \) is a nonnegative real-valued integrable function satisfying the condition \( \int_0^1 \lambda(t)dt = 1 \).

This operator contains some well-known operators such as Libera, Bernardi, and Komatu as its special cases. Fournier and Ruscheweyh [1] applied the famous duality theory to show that for a function \( f \) in the class

\[
P(\beta) := \left\{ f \in \mathcal{A} : \exists \phi \in \mathbb{R} | \Re e^{i\phi}(f'(z) - \beta) > 0, \; z \in E \right\},
\]

(1.3)

the linear integral operator \( V_\lambda(f) \) is univalent in \( E \). Since then, this operator has been studied by a number of authors for various choices of \( \lambda(t) \). In another remarkable paper, Barnard et al. in [2] obtained conditions such that \( V_\lambda(f) \in P(\beta) \) whenever \( f \) is in the class

\[
P_1(\beta) := \left\{ f \in \mathcal{A} : \exists \phi \in \mathbb{R} | \Re e^{i\phi}\left((1 - \gamma) \frac{f(z)}{z} + \gamma f'(z) - \beta \right) > 0, \; z \in E \right\},
\]

(1.4)

with \( \beta < 1, \gamma \geq 0 \). Note that for \( 0 \leq \beta < 1 \), functions in \( P_1(\beta) \equiv P(\beta) \) satisfy the condition \( \Re f'(z) > \beta \) in \( E \) and thus are close-to-convex in \( E \). A domain \( D \) in \( \mathbb{C} \) is close-to-convex if its complement in \( \mathbb{C} \) can be written as union of nonintersecting half lines.

In 2008, Ponnusamy and Rønning [3] discussed the univalence of \( V_\lambda(f) \) for the functions in the class

\[
R_\gamma(\beta) := \left\{ f \in \mathcal{A} : \exists \phi \in \mathbb{R} | \Re e^{i\phi}(f'(z) + \gamma zf''(z) - \beta) > 0, \; z \in E \right\}.
\]

(1.5)

In a very recent paper, Ali et al. [4] studied the class

\[
\mathcal{K}_\beta(\alpha, \gamma) := \left\{ f \in \mathcal{A} : \exists \phi \in \mathbb{R} | \Re e^{i\phi}\left((1 - \alpha + 2\gamma) \frac{f(z)}{z} + (\alpha - 2\gamma)f'(z) + \gamma zf''(z) - \beta \right) > 0, \; z \in E \right\},
\]

(1.6)

where \( \alpha, \gamma \geq 0 \) and \( \beta < 1 \). In this paper, they obtained sufficient conditions so that the integral transform \( V_\lambda(f) \) maps normalized analytic functions \( f \in \mathcal{K}_\beta(\alpha, \gamma) \) into the class of starlike functions. It is evident that \( \mathcal{K}_\beta(1, 0) \equiv P(\beta) \), \( \mathcal{K}_\beta(\alpha, 0) \equiv P_\alpha(\beta) \) and \( \mathcal{K}_\beta(1 + 2\gamma, \gamma) \equiv R_\gamma(\beta) \).

In the present paper, we shall mainly tackle the following problems.

(1) For given \( \delta < 1 \), find sharp values of \( \beta = \beta(\delta, \alpha) \) such that \( V_\lambda(f) \in \mathcal{K}_\delta(1, 0) \) whenever \( f \in \mathcal{K}_\beta(\alpha, \gamma) \).

(2) For given \( \delta < 1 \), find sharp values of \( \beta = \beta(\delta) \) such that \( V_\lambda(f) \in \mathcal{K}_\delta(\alpha, \gamma) \) whenever \( f \in \mathcal{K}_\beta(\alpha, \gamma) \).

To prove one of our results, we shall need the generalized hypergeometric function \( \pFq \), so we define it here.
Let $\alpha_j (j = 1, 2, \ldots, p)$ and $\beta_j (j = 1, 2, \ldots, q)$ be complex numbers with $\beta_j \neq 0, -1, -2, \ldots (j = 1, 2, \ldots, q)$. Then the generalized hypergeometric function $pFq$ is defined by

$$pFq(z) = pFq(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_p)_n z^n}{(\beta_1)_n \cdots (\beta_q)_n n!} \quad (p \leq q + 1),$$

where $(a)_n$ is the Pochhammer symbol, defined in terms of the Gamma function, by

$$(a)_n := \frac{\Gamma(a + n)}{\Gamma(a)} = \begin{cases} 1, & n = 0, \\ a(a + 1) \cdots (a + n - 1), & n \in \mathbb{N}. \end{cases}$$

In particular, $zF1$ is called the Gaussian hypergeometric function. We note that the $pFq$ series in (1.7) converges absolutely for $|z| < \infty$ if $p < q + 1$ and for $z \in E$ if $p = q + 1$.

We shall also need the following lemma.

**Lemma 1.1** (see [5]). Let $\beta_1 < 1$, $\beta_2 < 1$, and $\eta \in \mathbb{R}$. Then, for $p, q$ analytic in $E$ with $p(0) = q(0) = 1$, the conditions $\Re p(z) > \beta_1$ and $\Re e^{\eta}(q(z) - \beta_2) > 0$ imply $\Re e^{\eta}(p * q)(z) > 2(1 - \beta_1)(1 - \beta_2)$.

### 2. Main Results

We use the notations introduced in [4]. Let $\mu \geq 0$ and $\nu \geq 0$ satisfy

$$\mu + \nu = \alpha - \gamma, \quad \mu \nu = \gamma. \quad (2.1)$$

When $\gamma = 0$, then $\mu$ is chosen to be 0, in which case, $\nu = \alpha \geq 0$. When $\alpha = 1 + 2\gamma$, (2.1) yields $\mu + \nu = 1 + \gamma = 1 + \mu \nu$ or $(\mu - 1)(1 - \nu) = 0$.

(i) For $\gamma > 0$, then choosing $\mu = 1$ gives $\nu = \gamma$.

(ii) For $\gamma = 0$, then $\mu = 0$ and $\nu = \alpha = 1.$

**Theorem 2.1.** Let $\mu \geq 0, \nu \geq 0$ satisfy (2.1). Further, let $\delta < 1$ be given, and define $\beta = \beta(\delta, \mu, \nu)$ by

$$1 - \frac{1 - \delta}{2} \left\{ 1 - \frac{1}{\nu} \int_0^1 \lambda(t) \left( \int_0^1 \frac{ds}{1 + ts^\mu} \right) dt + \frac{1}{\nu - 1} \int_0^1 \lambda(t) \left( \int_0^1 \frac{d\eta d\zeta}{1 + t\eta^\nu \zeta^\mu} \right) dt \right\}^{-1}, \quad \gamma \neq 0,$$

$$1 - \frac{1 - \delta}{2} \left\{ 1 - \frac{1}{\alpha} \int_0^1 \frac{\lambda(t)}{1 + t^\mu} dt + \frac{1}{\alpha - 1} \int_0^1 \lambda(t) \left( \int_0^1 \frac{d\eta}{1 + t^\nu} \right) dt \right\}^{-1}, \quad \gamma = 0 \quad (\mu = 0, \nu = \alpha > 0). \quad (2.2)$$

If $f \in \mathcal{W}_p(\alpha, \gamma)$, then $F = V_1(f) \in \mathcal{W}_\delta(1, 0) \subset S$. The value of $\beta$ is sharp.

**Proof.** The case $\gamma = 0$ ($\mu = 0, \nu = \alpha > 0$) corresponds to Theorem 1.5 in [2]. So we assume that $\gamma > 0.$
Define
\[
(1 - \alpha + 2\gamma) \frac{f(z)}{z} + (\alpha - 2\gamma) f'(z) + \gamma zf''(z) = H(z). \quad (2.3)
\]

Writing \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \), it follows that
\[
H(z) = 1 + \sum_{n=1}^{\infty} a_{n+1} (n\nu + 1)(n\mu + 1) z^n. \quad (2.4)
\]

It is a simple exercise to see that
\[
f'(z) = H(z) \ast _3F_2 \left( 2, \frac{1}{\nu}, \frac{1}{\mu}; \frac{1}{\nu + 1}, \frac{1}{\mu + 1}; z \right). \quad (2.5)
\]

Let \( F(z) = V_\lambda (f)(z) \), where \( V_\lambda (f) \) is defined by (1.2). Then for \( \gamma \neq 0 \), we can write
\[
F'(z) = f'(z) \ast \int_0^1 \frac{\lambda(t)}{1 - tz} dt
= H(z) \ast _3F_2 \left( 2, \frac{1}{\nu}, \frac{1}{\mu}; \frac{1}{\nu + 1}, \frac{1}{\mu + 1}; z \right) \ast \int_0^1 \frac{\lambda(t)}{1 - tz} dt
= H(z) \ast \int_0^1 \lambda(t) _3F_2 \left( 2, \frac{1}{\nu}, \frac{1}{\mu}; \frac{1}{\nu + 1}, \frac{1}{\mu + 1}; tz \right) dt. \quad (2.6)
\]

Since \( f \in \mathcal{K}_\mu (\alpha, \gamma) \), it follows that \( \Re \{ e^{i\phi} (H(z) - \beta) \} > 0 \) for some \( \phi \in \mathbb{R} \). Now, for each \( \gamma > 0 \), we first claim that
\[
\Re \left[ \int_0^1 \lambda(t) _3F_2 \left( 2, \frac{1}{\nu}, \frac{1}{\mu}; \frac{1}{\nu + 1}, \frac{1}{\mu + 1}; tz \right) dt \right] > 1 - \frac{1 - \delta}{2(1 - \beta)}, \quad z \in E, \quad (2.7)
\]
which, by Lemma 1.1, implies that \( F \in \mathcal{K}_0 (1, 0) \). Therefore, it suffices to verify the inequality (2.7). Using the identity (which can be checked by comparing the coefficients of \( z^n \) on both sides)
\[
_3F_2(2, b, c; d, e; z) = (d - 1) _3F_2(1, b, c; d - 1, e; z) - (d - 2) _3F_2(1, b, c; d, e; z), \quad (2.8)
\]
it follows that
\[
_3F_2 \left( 2, \frac{1}{\nu}, \frac{1}{\mu}; \frac{1}{\nu + 1}, \frac{1}{\mu + 1}; z \right) = \frac{1}{\nu} \int_0^1 \frac{ds}{1 - zs^\mu} + \left( 1 - \frac{1}{\nu} \right) \int_0^1 \frac{d\eta d\zeta}{1 - z\eta^\mu \zeta^\mu}. \quad (2.9)
\]
Thus,

\[
\int_0^1 \lambda(t) \, _3F_2\left( 2, \frac{1}{\nu}, \frac{1}{\mu}; \frac{1}{\nu+1}, \frac{1}{\mu+1}; tz \right) dt
= \int_0^1 \lambda(t) \left\{ \frac{1}{\nu} \int_0^1 \frac{ds}{1-tzs^\mu} + \left( 1 - \frac{1}{\nu} \right) \int_0^1 \frac{d\eta \, d\zeta}{1-tz^\eta \zeta^\mu} \right\} dt. \tag{2.10}
\]

Therefore, for \( \gamma > 0 \), we have

\[
\Re \left[ \int_0^1 \lambda(t) \, _3F_2\left( 2, \frac{1}{\nu}, \frac{1}{\mu}; \frac{1}{\nu+1}, \frac{1}{\mu+1}; tz \right) dt \right]
\geq \frac{1}{\nu} \int_0^1 \lambda(t) \left( \int_0^1 \frac{ds}{1+ts^\mu} \right) dt + \left( 1 - \frac{1}{\nu} \right) \int_0^1 \lambda(t) \left( \int \int_0^1 \frac{d\eta \, d\zeta}{1+\eta \zeta^\mu} \right) dt \tag{2.11}
\]

\[
= 1 - \frac{1-\delta}{2(1-\beta)},
\]

in the view of (2.2).

To prove the sharpness, let \( f \in \mathcal{K}_\beta(a, \gamma) \) be the function determined by

\[
(1-a + 2\gamma) \frac{f(z)}{z} + (a - 2\gamma) f'(z) + \gamma zf''(z) = \beta + (1-\beta) \frac{1+z}{1-z}. \tag{2.12}
\]

Using a series expansion, we see that we can write

\[
f(z) = z + \sum_{n=2}^{\infty} \frac{2(1-\beta)}{(nv+1-\nu)(n\mu+1-\mu)} z^n. \tag{2.13}
\]

Then,

\[
F(z) = V_1(f)(z) = z + 2(1-\beta) \sum_{n=2}^{\infty} \frac{q_n}{(nv+1-\nu)(n\mu+1-\mu)} z^n, \tag{2.14}
\]
where \( q_n = \int_0^1 \lambda(t)t^{n-1} \, dt \). Equation (2.2) can be restated as

\[
\frac{1}{1-\beta} = \frac{2}{1-\delta} \left\{ 1 - \frac{1}{\nu} \int_0^1 \lambda(t) \left( \int_0^1 \frac{ds}{1+ts^\nu} \right) \, dt + \left( \frac{1}{\nu} - 1 \right) \int_0^1 \lambda(t) \left( \int_0^1 \frac{d\eta d\zeta}{1+tn^\nu \xi^\mu} \right) \, dt \right\}
\]

\[
= \frac{2}{1-\delta} \left\{ 1 + \int_0^1 \lambda(t) \left( -\frac{1}{\nu} \int_0^1 \frac{ds}{1+ts^\nu} + \left( \frac{1}{\nu} - 1 \right) \int_0^1 \frac{d\eta d\zeta}{1+tn^\nu \xi^\mu} \right) \, dt \right\}
\]

\[
= \frac{2}{1-\delta} \int_0^1 \lambda(t) \left\{ \sum_{n=2}^{\infty} \frac{(-1)^{n-1} n^\gamma}{(n\mu + 1 - \mu)(n\nu + 1 - \nu)} \right\} \, dt
\]

\[
= - \frac{2}{1-\delta} \sum_{n=2}^{\infty} \frac{(-1)^{n-1} n^\gamma}{(n\nu + 1 - \nu)(n\mu + 1 - \mu)}.
\]

Finally,

\[
F'(z) = 1 + 2(1-\bar{\beta}) \sum_{n=2}^{\infty} \frac{nq_n}{(n\nu + 1 - \nu)(n\mu + 1 - \mu)} z^{n-1},
\]

which for \( z = -1 \) takes the value

\[
F'(-1) = 1 + 2(1-\bar{\beta}) \sum_{n=2}^{\infty} \frac{(-1)^{n-1} nq_n}{(n\nu + 1 - \nu)(n\mu + 1 - \mu)} = 1 + 2(1-\bar{\beta}) \left\{ \frac{-(1- \delta)}{2(1-\beta)} \right\} = \delta.
\]

This shows that the result is sharp.

Letting \( \gamma = 0 \) and \( \alpha = 1 \) in Theorem 1.1, we obtain the following result of Ruscheweyh [6].

**Corollary 2.2.** Let \( \delta < 1 \), and define \( \beta = \beta(\delta, 1) < 1 \) by

\[
\frac{\beta}{1 - \beta} = 1 - \frac{1 - \delta}{2} \left\{ 1 - \int_0^1 \frac{\lambda(t)}{1+t} \, dt \right\}^{-1}.
\]

If \( f \in \mathcal{W}_\beta(1, 0) \equiv \mathcal{P}_1(\beta) \), then \( F = V_\lambda(f) \in \mathcal{W}_\delta(1, 0) \subset S \). The value of \( \beta \) is sharp.

**Theorem 2.3.** Let \( \delta < 1 \) and \( \alpha, \gamma \geq 0 \), and define \( \beta = \beta(\delta) < 1 \) by

\[
\frac{\beta}{1 - \beta} = - \int_0^1 \lambda(t) \frac{(1 - ((1+\delta)/(1-\delta))t)}{(1+t)} \, dt.
\]

If \( f \in \mathcal{W}_\beta(\alpha, \gamma) \), then \( V_\lambda(f) \in \mathcal{W}_\delta(\alpha, \gamma) \). The value of \( \beta \) is sharp.
Proof. The idea of the proof is similar to the one used to prove Theorem 2 in [1]. Let \( F(z) = V_{\lambda}(f)(z) = \int_0^1 \lambda(t)(f(tz)/t)dt \). Clearly,

\[
F'(z) = \int_0^1 \frac{\lambda(t)}{1-tz} dt \ast f'(z). \tag{2.20}
\]

Since, \( f \in \mathcal{H}_\beta(\alpha, \gamma) \), so with

\[
g(z) = \frac{(1 - \alpha + 2\gamma)(f(z)/z) + (\alpha - 2\gamma)f'(z) + \gamma zf''(z) - \beta}{1 - \beta}, \tag{2.21}
\]

we have \( \Re[e^{i\phi}g(z)] > 0 \), where \( \phi \in \mathbb{R} \).

For \( \gamma \neq \alpha/2 \),

\[
f'(z) = \frac{1}{\alpha - 2\gamma} \left( \beta + (1 - \beta)g(z) \right) - \frac{1 - \alpha + 2\gamma f(z)}{\alpha - 2\gamma} - \frac{\gamma}{\alpha - 2\gamma} zf''(z). \tag{2.22}
\]

Putting this value in (2.20),

\[
F'(z) = \int_0^1 \frac{\lambda(t)}{1-tz} dt \ast \left( \frac{1}{\alpha - 2\gamma} \left( \beta + (1 - \beta)g(z) \right) - \frac{1 - \alpha + 2\gamma f(z)}{\alpha - 2\gamma} - \frac{\gamma}{\alpha - 2\gamma} zf''(z) \right). \tag{2.23}
\]

Equivalently,

\[
F'(z) = \frac{1}{\alpha - 2\gamma} g(z) \ast \left[ \beta + (1 - \beta) \int_0^1 \frac{\lambda(t)}{1-tz} dt \right] - \frac{1 - \alpha + 2\gamma F(z)}{\alpha - 2\gamma} - \frac{\gamma}{\alpha - 2\gamma} zf''(z). \tag{2.24}
\]

Thus

\[
(1 - \alpha + 2\gamma)(F(z)/z) + (\alpha - 2\gamma)F'(z) + \gamma zf''(z) = g(z) \ast \left[ \beta + (1 - \beta) \int_0^1 \frac{\lambda(t)}{1-tz} dt \right]. \tag{2.25}
\]

In the case when \( \gamma = \alpha/2 \),

\[
g(z) = \frac{f(z)/z + \gamma zf''(z) - \beta}{1 - \beta}. \tag{2.26}
\]

Since

\[
\frac{f(z)}{z} = \beta + (1 - \beta)g(z) - \gamma zf''(z), \tag{2.27}
\]
This leads to,

$$\frac{F(z)}{z} + \gamma z F''(z) = g(z) * \left[ \beta + (1 - \beta) \int_0^1 \frac{\lambda(t)}{1 - tz} dt \right],$$  \hspace{1cm} (2.28)

which is clearly (2.25) with \( \gamma = \alpha/2 \).

Further \( F \in \mathcal{K}_0(\alpha, \gamma) \) if and only if \( G(z) := (F(z) - \delta z)/(1 - \delta) \in \mathcal{K}_0(\alpha, \gamma) \). Now using (2.25), we obtain

$$\left(1 - \alpha + 2\gamma\right) \frac{G(z)}{z} + (\alpha - \gamma) G'(z) + \gamma z G''(z) = \left[ \beta - \frac{\delta}{1 - \delta} - \frac{1 - \beta}{1 - \delta} \int_0^1 \frac{\lambda(t)}{1 - tz} dt \right].$$  \hspace{1cm} (2.29)

Since \( \Re e^{\phi} g(z) > 0 \) for some \( \phi \in \mathbb{R} \), it follows by duality principle [8, page 23] that

$$\left(1 - \alpha + 2\gamma\right) \frac{G(z)}{z} + (\alpha - 2\gamma) G'(z) + \gamma z G''(z) \neq 0$$  \hspace{1cm} (2.30)

if, and only if,

$$\Re \left[ \beta - \frac{\delta}{1 - \delta} - \frac{1 - \beta}{1 - \delta} \int_0^1 \frac{\lambda(t)}{1 - tz} dt \right] > \frac{1}{2}. \hspace{1cm} (2.31)$$

Using \( \Re(1/(1 - tz)) > 1/(1 + t) \), we get

$$\Re \left[ \beta - \frac{\delta}{1 - \delta} + \frac{1 - \beta}{1 - \delta} \int_0^1 \frac{\lambda(t)}{1 - tz} dt \right] > 1 - \beta \left[ \beta - \frac{\delta}{1 - \delta} + \int_0^1 \frac{\lambda(t)}{1 + t} dt \right]. \hspace{1cm} (2.32)$$

By using (2.19), we have

$$\frac{\beta - (1 + \delta)/2}{1 - \beta} = -\int_0^1 \frac{\lambda(t)}{(1 + t)} dt. \hspace{1cm} (2.33)$$

Thus,

$$\frac{\beta - \delta}{1 - \beta} + \int_0^1 \frac{\lambda(t)}{1 + t} dt = \frac{1}{2} \frac{1 - \delta}{1 - \beta} \hspace{1cm} (2.34)$$

which implies that

$$\Re \left[ \beta - \frac{\delta}{1 - \delta} + \frac{1 - \beta}{1 - \delta} \int_0^1 \frac{\lambda(t)}{1 - tz} dt \right] > 1 - \beta \left[ \beta - \frac{\delta}{1 - \delta} + \int_0^1 \frac{\lambda(t)}{1 + t} dt \right] = \frac{1}{2}. \hspace{1cm} (2.35)$$
Thus, we deduce, using duality principle, that \((1 - \alpha + 2\gamma)(G(z)/z) + (\alpha - \gamma)G'(z) + \gamma zG''(z)\) is contained in a half plane not containing the origin. So, \(G \in \mathcal{K}_0(\alpha, \gamma)\) and hence \(F \in \mathcal{K}_0(\alpha, \gamma)\).

To prove the sharpness, let \(f(z) = z + 2(1 - \beta) \sum_{n=2}^{\infty} (\zeta_n/(n\mu + 1 - \mu)(n\nu + 1 - \nu))\).

\[
F(z) = V_{1,2}(f)(z) = z + 2(1 - \beta) \sum_{n=2}^{\infty} \frac{\zeta_n/(n\mu + 1 - \mu)(n\nu + 1 - \nu)}{\omega_n}, \quad \text{where } \omega_n = \int_0^1 \lambda(t)t^{n-1} \, dt.
\]

(2.36)

Further,

\[
\frac{\beta}{1 - \beta} = -\int_0^1 \frac{\lambda(t)(1 - ((1 + \delta)/(1 - \delta))t)}{(1 + t)} \, dt
\]

(2.37)

gives

\[
\frac{\beta}{1 - \beta} = -1 + \int_0^1 \frac{\lambda(t)(1 + (1 + \delta)/(1 - \delta))t}{(1 + t)} \, dt,
\]

(2.38)

or

\[
\frac{1}{1 - \beta} = \frac{2}{1 - \delta} \int_0^1 \frac{\lambda(t)(1 + (1 + \delta)/(1 - \delta))t}{1 + t} \, dt = \frac{2}{1 - \delta} \sum_{n=2}^{\infty} (-1)^n \omega_n.
\]

(2.39)

Further, assume that

\[
H(z) = (1 - \alpha + 2\gamma) \frac{F(z)}{z} + (\alpha - \gamma)F'(z) + \gamma zF''(z).
\]

(2.40)

Since \(F(z) = z + 2(1 - \beta) \sum_{n=2}^{\infty} (\zeta_n/(n\mu + 1 - \mu)(n\nu + 1 - \nu))\),

so,

\[
H(z) = 1 + 2(1 - \beta) \sum_{n=2}^{\infty} \omega_n z^{n-1}.
\]

(2.41)

Therefore, for \(z = -1\),

\[
H(-1) = 1 - 2(1 - \beta) \sum_{n=2}^{\infty} \omega_n (-1)^n = 1 - 2(1 - \beta) \frac{1 - \delta}{2(1 - \beta)} = \delta.
\]

(2.42)

This shows that the result is sharp.

Letting \(\gamma = 0\) in Theorem 2.3 above, we obtain the following result of Kim and Rønning [9].
Corollary 2.4. Let $\delta < 1$ and $\alpha \geq 0$, and define $\beta = \beta(\delta)$ by

$$
\frac{\beta}{1 - \beta} = -\int_0^1 \lambda(t) \frac{(1 - ((1 + \delta)/(1 - \delta))t)}{(1 + t)} dt.
$$

(2.43)

If $f \in \mathcal{W}(\alpha, 0) \equiv \mathcal{P}_a(\beta)$, then $V_{\lambda}(f) \in \mathcal{W}(0, 0) \equiv \mathcal{P}_a(\delta)$. The value of $\beta$ is sharp.

Upon setting $\lambda(t) = (1 + c)t^2$ with $-1 < c$, we have the following corollary.

Corollary 2.5. Let $\delta < 1$, $\alpha, \gamma \geq 0$, and $-1 < c \leq 0$ be given, and let $G(z)$ be defined by

$$
G(z) = \frac{(1 + c)}{z^2} \int_0^z u^{c-1} f(u) du.
$$

(2.44)

Suppose that $f \in \mathcal{W}(\alpha, \gamma)$, then $G \in \mathcal{W}(\alpha, 0)$, where

$$
\beta = \frac{2(1 + c) z F_1(1, 2 + c; 3 + c, -1) - (2 + c)}{2(1 + c) z F_1(1, 2 + c; 3 + c, -1)}.
$$

(2.45)

The constant $\beta$ is sharp.

The special case of Corollary 2.5 (with $\gamma = 0$) has been obtained by Aghalary et al. [11].

References