

## Research Article

# Model Reference Control of Hyperchaotic Systems

**Pengfei Zhao,<sup>1,2,3</sup> Cai Liu,<sup>1</sup> and Xuan Feng<sup>1</sup>**

<sup>1</sup> College of Geoexploration Science and Technology, Jilin University, Changchun 130026, China

<sup>2</sup> College of Computer Science and Technology, Jilin University, Changchun 130012, China

<sup>3</sup> Department of Scientific Computing, Florida State University, Tallahassee, FL 32306, USA

Correspondence should be addressed to Pengfei Zhao, zhaopf@jlu.edu.cn

Received 20 July 2012; Revised 1 November 2012; Accepted 15 November 2012

Academic Editor: Jong Hae Kim

Copyright © 2012 Pengfei Zhao et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

We have applied a famous engineering method, called model reference control, to control hyperchaos. We have proposed a general description of the hyperchaotic system and its reference system. By using the Lyapunov stability theorem, we have obtained the expression of the controller. Four examples for the both certain case and the uncertain case show that our method is very effective for controlling hyperchaotic systems with both certain parameters and uncertain parameters.

## 1. Introduction

Chaos has received increasing attentions in the last thirty years. Compared with the ordinary chaotic systems, the hyperchaotic systems hold at least two positive Lyapunov exponents and then possess more complicated attractors. Hyperchaotic systems have the characteristics of high capacity, high security, and high efficiency and have been studied in many fields, such as secure communication [1, 2], cellular neural network [3, 4], chemical processing [5], nonlinear circuit [6–8], and other fields [9, 10].

Controlling chaos (or hyperchaos) is very meaningful. The research has been started since the pioneering work of OGY method [11] was published. As the development of computational technique and controlling theory, many methods have been proposed for controlling chaos such as, LMI-based approach [12, 13], sliding control [14], active control [15], optimal control [16, 17], and passivity-based control [18, 19]. However, the hyperchaotic systems are more complex than the ordinary chaotic ones. To obtain the satisfying effect of control, we should focus on more advantage algorithms and ideas of controlling techniques. There have been some results on controlling hyperchaotic systems, such as feedback control [20, 21], adaptive control [22], backstepping control [23], and impulsive control [24].

In this paper, we will show that although a hyperchaotic system is complex, its dynamics can still be controlled along an expected trajectory. The controller is generated by an advantage control method, called model reference control (MRC). Nowadays, MRC is widely used in engineering, such as control of robots [25], mechanical oscillators [26], economic cycle [27], and disease spread [28]. Our aim is to control a hyperchaotic system to track with an expected trajectory with the aid of MRC. In the following section, the formulation of the problem will be presented. In Section 3, we will propose the framework of MRC on hyperchaotic systems with certain parameters and the systems with uncertain parameters. In Section 4, we will give two numerical examples to show the effectiveness of MRC on hyperchaotic systems with certain and uncertain parameters and also two examples for the hyperchaotic systems with uncertain parameters. Finally, the conclusion will be given in Section 5.

## 2. Problem Formulation

The first example of the hyperchaotic systems was presented by Rössler in 1979 [29]. Since then, other hyperchaotic systems have been reported [30], and many researchers are focusing on the discovery of new hyperchaotic systems and their control. In this section, we will describe the basic formulation of generalized hyperchaotic system and the reference system. Recently, nonlinear scientists are focusing on the control problem of chaotic and hyperchaotic systems with uncertain parameters [31–37]. Hence, we will give the formulation of both the hyperchaotic systems with certain parameters and the systems with uncertain parameters.

Since a hyperchaotic system with certain parameters should have quadratic terms at least, we may formulate the system as follows:

$$\dot{X} = X^T AX + BX + u, \quad (2.1)$$

where  $X$  is an  $n$  dimensional column vector,  $X^T$  is its transposed matrix,  $A$  is an  $n^2 \times n$  matrix including  $n$  matrices  $A_1, A_2, \dots, A_n$  in the form of  $n \times n$ ,  $B$  is an  $n \times n$  matrix, and  $u$  is an  $n$  dimensional input control column vector. It is worth noting that  $X^T AX$  is only the nominal expression of matrix multiplication, and it describes the quadratic terms in blocks as follows:

$$\left( X^T A_1 X, X^T A_2 X, \dots, X^T A_n X \right)^T. \quad (2.2)$$

Equation (2.1) can cover a great many hyperchaotic systems, though it may not fit for the systems with higher order terms or even a fractional order term. The uncertain form of system (2.1) is as follows:

$$\dot{X} = X^T AX + BX + C(X)\alpha + u, \quad (2.3)$$

where  $C(X)$  is an  $n \times m$  matrix, uncertain parameter  $\alpha$  is an  $m$  dimensional column vector, and other variables as above. Here,  $m$  might be equal to  $n$ , also smaller or bigger than  $n$ .

In this section, we will also introduce a reference system with only linear terms and constants. Our aim is to control the hyperchaotic system track along with the reference model system that exhibits asymptotic stability as follows

$$\dot{X}_r = \bar{B}X_r + \bar{u}, \quad (2.4)$$

where the matrix  $\bar{B}$  is a known constant matrix with appropriate dimensions, the eigenvalues of the matrix  $\bar{B}$  have negative real part such that the system is asymptotically stable. By letting the system (2.4) be the reference system, we will control the hyperchaotic system (2.1) track along with the system (2.4). Here,  $X_r$  is called state output vector of the reference model, and  $\bar{u}$  the reference input vector.

Particularly, if the reference system has a two-order term, we may write it as

$$\dot{X}_r = X^T \bar{A} X + \bar{B} X_r + \bar{u}, \quad (2.5)$$

where  $\bar{A}$  is a similar matrix as  $A$ .

### 3. Model Reference Control of Hyperchaotic Systems

In this section, MRC is applied to control a hyperchaotic system with both certain parameters and uncertain parameters. Our objective is to obtain the exact control law  $u$  such that the original system follows the dynamical behavior of the reference model. We will review the Lyapunov stability theorem for autonomous systems and then give the explicit expression of the controller.

**Theorem 3.1** (Lyapunov stability theorem for autonomous systems). *Let  $x = 0$  be an equilibrium point for a dynamical system described by*

$$\dot{x} = f(x), \quad (3.1)$$

*where  $f : D \rightarrow \mathbb{R}^n$  is a locally Lipschitz and  $D \subset \mathbb{R}^n$  a domain that contains the origin. Let  $V : D \rightarrow \mathbb{R}$  be a continuously differentiable, positive definite function in  $D$ . Then  $x = 0$  is an asymptotically stable equilibrium point, if  $\dot{V}(x)$  is negative definite. The scalar function  $V$  is a Lyapunov function if  $\dot{V}(x)$  is negative semidefinite in the region  $D : \dot{V}(x) \leq 0$ .*

Assume that the error system (3.1) satisfies the conditions of Theorem 3.1, and its Lyapunov function has the form as

$$V(e) = e^T P e, \quad (3.2)$$

where  $P$  is a symmetric positive definite matrix. The derivative of  $V(e)$  with respect to time is

$$\dot{V}(e) = \dot{e}^T P e + e^T P \dot{e}. \quad (3.3)$$

In the whole process of MRC, the output of reference model and that of the controlled system are compared, and the error vector  $e$  is

$$e = X_r - X. \quad (3.4)$$

If we want to control the system (2.1) to track along with system (2.4), we may have the following results. By using (2.1) and (2.4), we obtain the following error system:

$$\dot{e} = \dot{X}_r - \dot{X} = \bar{B}e - X^T A X + (\bar{B} - B)X + \bar{u} - u. \quad (3.5)$$

We will try to design a control vector  $u$  such that the objective equation

$$\lim_{t \rightarrow \infty} \|e\| = 0. \quad (3.6)$$

From (3.1) and (3.2), the MRC problem is converted to the asymptotic stability of zero vector of the error system (3.1). Here, we will use the Lyapunov stability theory to determine the proper control law  $u$ . The theorem is as follows.

**Theorem 3.2.** *Let  $P$  be any symmetric positive definite matrix,  $\lambda_{\max}$  is the largest eigenvalue of the matrix  $(\bar{B}^T P + P\bar{B})$ , the controller  $u = \bar{B}X + \bar{u} - X^T A X - BX - \lambda I(X_r - X)$ . The hyperchaotic system (2.1) will asymptotically follow the desired dynamical system (2.4), if  $\lambda < -\lambda_{\max}/2$  and  $\lambda \neq 0$ .*

*Proof.* The fact that the hyperchaotic system (2.1) asymptotically follows the system (2.4) is equivalent to asymptotical stability of the error system. Hence, we will try to prove the inequality of  $\dot{V}(e) \leq 0$ .

For the symmetric property, the scalar equation (3.3) may have the following description:

$$\begin{aligned} \dot{V}(e) &= e^T P \dot{e} + e^T P \dot{e} \\ &= \left[ e^T \bar{B}^T - X^T A^T X - X^T (B^T - \bar{B}^T) - u^T + \bar{u}^T \right] P e \\ &\quad + e^T P (\bar{B}e - X^T A X - (B - \bar{B})X - (u - \bar{u})) \\ &= e^T (\bar{B}^T P + P\bar{B}) e + 2e^T P [-X^T A X - (B - \bar{B})X - (u - \bar{u})] \\ &= e^T (\bar{B}^T P + P\bar{B}) e + 2e^T (\lambda I) e \\ &= e^T (\bar{B}^T P + P\bar{B} + 2\lambda I) e. \end{aligned} \quad (3.7)$$

For  $\lambda < -\lambda_{\max}/2$ , the matrix  $(\bar{B}^T P + P\bar{B} + 2\lambda I)$  is negative definite. Also, the scalar  $\dot{V} < 0$  is obtained.

Using the result of Theorem 3.1, it can be concluded that the error  $e$  will converge to 0 asymptotically. Equivalently, the controller  $u$  will make the hyperchaotic system (2.1) asymptotically follow the desired dynamical system (2.4).  $\square$

If we want to control the system (2.1) to track along with system (2.5), we may have the following results. By using (2.1) and (2.4), we obtain the following error system:

$$\dot{e} = \dot{X}_r - \dot{X} = X_r^T \bar{A} X_r + \bar{B} e - X^T A X + (\bar{B} - B) X + \bar{u} - u. \quad (3.8)$$

It is easy to obtain the control law in the following theorem with similar process of Theorem 3.2.

**Theorem 3.3.** *Let  $P$  be any symmetric positive definite matrix,  $\lambda_{\max}$  is the largest eigenvalue of the matrix  $(\bar{B}^T P + P \bar{B})$ , and the controller  $u = X_r^T \bar{A} X_r + \bar{B} X + \bar{u} - X^T A X - B X - \lambda I (X_r - X)$ . The hyperchaotic system (2.1) will asymptotically follow the desired dynamical system (2.4), if  $\lambda < -\lambda_{\max}/2$  and  $\lambda \neq 0$ .*

We can omit the proof of Theorem 3.3, for it is similar to that of Theorem 3.2.

Theorems 3.2 and 3.3 are fit for the hyperchaotic systems with certain parameters. In the following, we will give two similar theorems for the systems with uncertain parameters. To simplify the control process, we will make  $P = I$ .

**Theorem 3.4.** *Let  $\lambda_{\max}$  is the largest eigenvalue of the matrix  $\bar{B}^T$ , the controller  $u = \bar{B} X + \bar{u} - X^T A X - B X - \lambda I (X_r - X) + C(X) \hat{\alpha}$ , where  $\hat{\alpha}$  is the estimation of uncertain parameter  $\alpha$ , and satisfies the differential equations  $\dot{\hat{\alpha}} = C^T(X) e$ . The hyperchaotic system (2.3) will asymptotically follow the desired dynamical system (2.4), if  $\lambda < -\lambda_{\max}$  and  $\lambda \neq 0$ .*

*Proof.* For the uncertain property of system (2.3), we should redefine the Lyapunov function as follows

$$V_1(e) = e^T e + (\alpha - \hat{\alpha})^T (\alpha - \hat{\alpha}). \quad (3.9)$$

Similar to the proof of Theorem 3.2, we have to prove the inequality of  $\dot{V}_1(e) \leq 0$ :

$$\begin{aligned} \dot{V}_1(e) &= 2\dot{e}^T e - 2\dot{\hat{\alpha}}^T (\alpha - \hat{\alpha}) \\ &= 2(\bar{B} X_r + \bar{u} - X^T A X - B X - u - C(X) \alpha)^T e - 2e^T C(X) (\alpha - \hat{\alpha}) \\ &= 2e^T (\bar{B}^T + \lambda I) e. \end{aligned} \quad (3.10)$$

For  $\lambda < -\lambda_{\max}$ , the matrix  $(\bar{B}^T + \lambda I)$  is negative definite. Also, the scalar  $\dot{V} < 0$  is obtained.

Hence, the controller  $u$  will make the hyperchaotic system (2.3) asymptotically follow the desired dynamical system (2.4).  $\square$

If we want to control the system (2.3) to track along with system (2.5), we may have the following results. We have the following error system:

$$\dot{e} = X_r^T \bar{A} X_r + \bar{B} e - X^T A X + (\bar{B} - B) X + \bar{u} - u - C(X) \alpha. \quad (3.11)$$

It is easy to obtain the control law in the following theorem with similar process of Theorem 3.4.

**Theorem 3.5.** *Let  $\lambda_{\max}$  is the largest eigenvalue of the matrix  $\bar{B}^T$ , the controller  $u = \bar{B}X + \bar{u} - X^T AX - BX - \lambda I(X_r - X) + C(X)\hat{\alpha}$ , where  $\hat{\alpha}$  is the estimation of uncertain parameter  $\alpha$ , and satisfies the differential equations  $\dot{\hat{\alpha}} = C^T(X)e$ . The hyperchaotic system (2.3) will asymptotically follow the desired dynamical system (2.5), if  $\lambda < -\lambda_{\max}$  and  $\lambda \neq 0$ .*

We can also omit the proof of Theorem 3.5.

## 4. Numerical Examples

This section has two parts. The first part is the numerical examples for controlling hyperchaotic systems, where all the parameters are certain. We will use the result in Theorem 3.2 to control hyperchaotic Rössler system and use that of Theorem 3.3 to control the hyperchaotic Lorenz system. The second part is the numerical examples for controlling the systems with uncertain parameters. Especially, Example III has four uncertain parameters, and Example IV has six uncertain parameters.

The whole numerical results show that MRC is very suitable and efficient for controlling hyperchaotic system with both certain parameters and uncertain parameters.

### 4.1. Example I

The four-variable hyperchaotic Rössler system is described by

$$\begin{aligned} \dot{x}_1 &= -x_2 - x_3, \\ \dot{x}_2 &= x_1 + 0.25x_2 + x_4, \\ \dot{x}_3 &= x_1x_3 + 3, \\ \dot{x}_4 &= -0.5x_3 + 0.05x_4. \end{aligned} \tag{4.1}$$

According to (2.1), we have

$$\begin{aligned} A_3 &= \begin{bmatrix} 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0 \\ 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, & B &= \begin{bmatrix} 0 & -1 & -1 & 0 \\ 1 & 0.25 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -0.5 & 0.05 \end{bmatrix}, \\ u &= \begin{bmatrix} u_1 \\ u_2 \\ u_3 + 3 \\ u_4 \end{bmatrix}, & A_1 &= A_2 = A_4 = \text{diag}[0, 0, 0, 0], \end{aligned} \tag{4.2}$$

where  $u_1, u_2, u_3$ , and  $u_4$  are equal to 0 in the initial state and need to be determined by Theorem 3.2.

The reference model with asymptotical stability is as follows:

$$\begin{aligned}\dot{x}_{r1} &= -x_{r2} - x_{r3}, \\ \dot{x}_{r2} &= x_{r1} + 0.25x_{r2} + x_{r4}, \\ \dot{x}_{r3} &= 20x_{r1} + 10x_{r2} - 50x_{r3} + 100x_{r4} + 3, \\ \dot{x}_{r4} &= -0.5x_{r3} + 0.05x_{r4}.\end{aligned}\tag{4.3}$$

According to (2.4), the matrix  $\bar{B}$  and the control vector  $\bar{u}$  of the 4-dimensional reference system (4.3) have the following description:

$$\bar{B} = \begin{bmatrix} 0 & -1 & -1 & 0 \\ 1 & 0.25 & 0 & 1 \\ 20 & 10 & -50 & 100 \\ 0 & 0 & -0.5 & 0.05 \end{bmatrix}, \quad \bar{u} = \begin{bmatrix} 0 \\ 0 \\ 3 \\ 0 \end{bmatrix}.\tag{4.4}$$

According to Theorem 3.2, we may let matrix  $P = \text{diag}[1, 1, 1, 1]$ , then the matrix

$$\left(\bar{B}^T P + P\bar{B}\right) = \begin{bmatrix} 0 & 0 & 19 & 0 \\ 0 & 0.5 & 10 & 1 \\ 19 & 10 & -100 & 99.5 \\ 0 & 1 & 99.5 & 0.1 \end{bmatrix},\tag{4.5}$$

and its largest eigenvalue is  $\lambda_{\max} = 63.6297$ . By considering the conditions  $\lambda < -\lambda_{\max}/2$  and  $\lambda \neq 0$ , we should obtain  $\lambda = -40$ . Then, the controller should be determined by Theorem 3.2,

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} 40x_{r1} - 40x_1 \\ 40x_{r2} - 40x_2 \\ (20 - x_3)x_1 + 10x_2 - 90x_3 + 100x_4 + 40x_{r3} + 3 \\ 40x_{r4} - 40x_4 \end{bmatrix}.\tag{4.6}$$

Figure 1 shows that the controller can make the hyperchaotic Rössler system (4.1) track along with its reference system (4.3). Here, the initials of these two systems are  $[-10, 6, 0, 10]$  and  $[10, -6, 10, -10]$ , respectively.

## 4.2. Example II

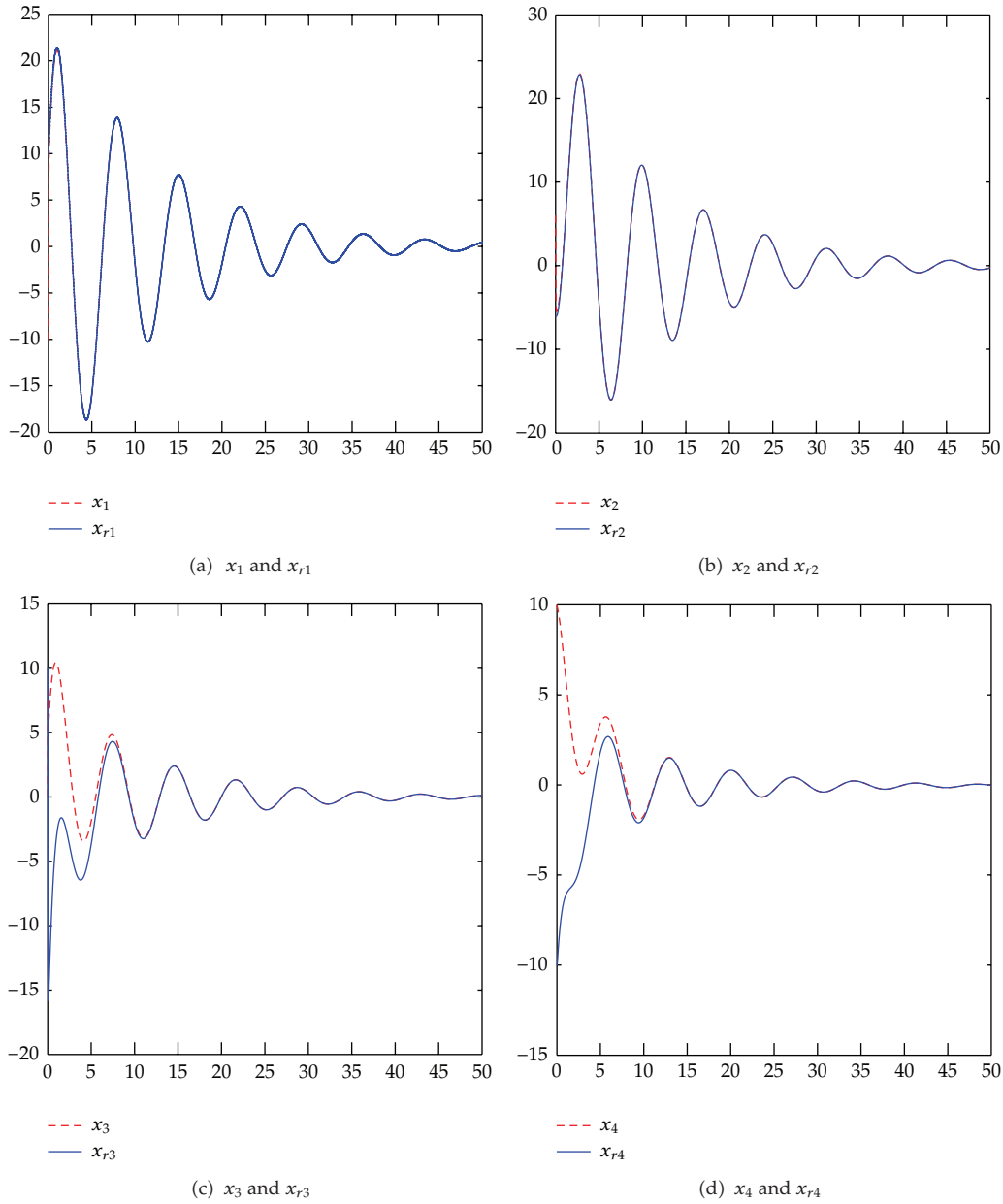
The four-variable hyperchaotic Lorenz system [30] is described by

$$\dot{x}_1 = 10(x_2 - x_1),\tag{4.7}$$

$$\dot{x}_2 = 28x_1 - x_2 - x_1x_3 + x_4,\tag{4.8}$$

$$\dot{x}_3 = x_1x_2 - \frac{8}{3}x_3,\tag{4.9}$$

$$\dot{x}_4 = -5x_1.\tag{4.10}$$



**Figure 1:** Comparison of the corresponding variables in the controlled hyperchaotic Rössler system and its reference system: (a)  $x_1$  and  $x_{r1}$ ; (b)  $x_2$  and  $x_{r2}$ ; (c)  $x_3$  and  $x_{r3}$ ; (d)  $x_4$  and  $x_{r4}$ .

According to (2.1), we have

$$A_2 = \begin{bmatrix} 0 & 0 & -0.5 & 0 \\ 0 & 0 & 0 & 0 \\ -0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & 0.5 & 0 & 0 \\ 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$



$$B = \begin{bmatrix} -10 & 10 & 0 & 0 \\ 28 & -1 & 0 & 1 \\ 0 & 0 & -\frac{8}{3} & 0 \\ -5 & 0 & 0 & 0 \end{bmatrix}, \quad u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix},$$

$$A_1 = A_4 = \text{diag}[0, 0, 0, 0],$$
(4.11)

where  $u_1, u_2, u_3,$  and  $u_4$  are equal to 0 in the initial state and need to be determined by Theorem 3.3.

The reference model with asymptotical stability is as follows:

$$\begin{aligned} \dot{x}_{r1} &= 10(x_{r2} - x_{r1}) - 50x_{r2}, \\ \dot{x}_{r2} &= 28x_{r1} - x_{r2} - x_{r1}x_{r3} + x_{r4}, \\ \dot{x}_{r3} &= x_{r1}x_{r2} - \frac{8}{3}x_{r3}, \\ \dot{x}_{r4} &= -5x_{r1}. \end{aligned}$$
(4.12)

According to (2.5), the matrix  $A_i$  ( $i = 1, \dots, 4$ ) and  $\bar{B}$  and the control vector  $\bar{u}$  of the 4-dimensional reference system (4.12) have the following description:

$$\bar{A}_2 = \begin{bmatrix} 0 & 0 & -0.5 & 0 \\ 0 & 0 & 0 & 0 \\ -0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \bar{A}_3 = \begin{bmatrix} 0 & 0.5 & 0 & 0 \\ 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\bar{B} = \begin{bmatrix} -10 & -40 & 0 & 0 \\ 28 & -1 & 0 & 1 \\ 0 & 0 & -\frac{8}{3} & 0 \\ -5 & 0 & 0 & 0 \end{bmatrix}, \quad \bar{u} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

$$A_1 = A_4 = \text{diag}[0, 0, 0, 0].$$
(4.13)

According to Theorem 3.3, we may let matrix  $P = \text{diag}[1, 1, 1]$ , then the matrix

$$(\bar{B}^T P + P \bar{B}) = \begin{bmatrix} -20 & -12 & 0 & -5 \\ -12 & -2 & 0 & 1 \\ 0 & 0 & -\frac{16}{3} & 0 \\ -5 & 1 & 0 & 0 \end{bmatrix},$$
(4.14)

and its largest eigenvalue is  $\lambda_{\max} = 5.8387$ . By considering the conditions  $\lambda < -\lambda_{\max}/2$  and  $\lambda \neq 0$ , we should obtain  $\lambda = -10$ . Then, the controller should be determined by Theorem 3.3,

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} -10x_2 - 40x_{r2} \\ x_1x_3 - 9x_2 - x_4 + 9x_{r2} - x_{r1}x_{r3} + x_{r4} \\ -x_1x_2 - \frac{22}{3}x_3 + x_{r1}x_{r2} + \frac{22}{3}x_{r3} \\ 5x_1 - 10x_4 - 5x_{r4} + 10x_{r4} \end{bmatrix}. \quad (4.15)$$

Figure 2 shows that the controller can make the hyperchaotic Lorenz system (4.7)–(4.12) track along with its reference system (4.12). Here, the initials of these two systems are  $[10, 0, 5, 8]$  and  $[2, -10, 1, 10]$ , respectively.

### 4.3. Example III

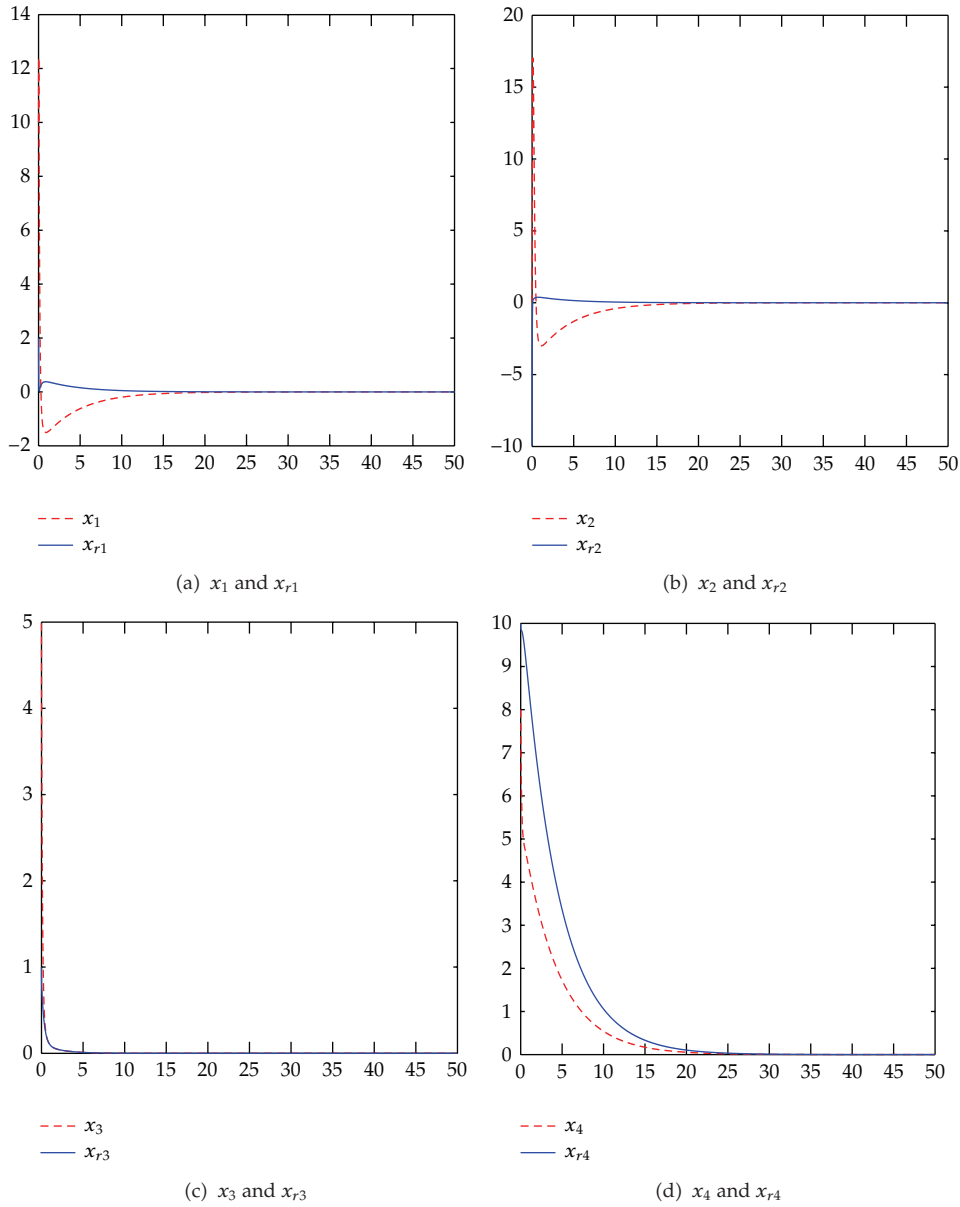
The four-variable hyperchaotic Rössler system with uncertain parameters is described by

$$\begin{aligned} \dot{x}_1 &= a(x_2 + x_3), \\ \dot{x}_2 &= x_1 + bx_2 + cx_4, \\ \dot{x}_3 &= x_1x_3 + 3, \\ \dot{x}_4 &= -0.5x_3 + dx_4. \end{aligned} \quad (4.16)$$

According to (2.3), we have

$$\begin{aligned} A_3 &= \begin{bmatrix} 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0 \\ 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, & B &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -0.5 & 0 \end{bmatrix}, \\ u &= \begin{bmatrix} u_1 \\ u_2 \\ u_3 + 3 \\ u_4 \end{bmatrix}, & C(X) &= \begin{bmatrix} x_2 + x_3 & 0 & 0 & 0 \\ 0 & x_2 & x_4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_4 \end{bmatrix}, \\ A_1 &= A_2 = A_4 = \text{diag}[0, 0, 0, 0], \end{aligned} \quad (4.17)$$

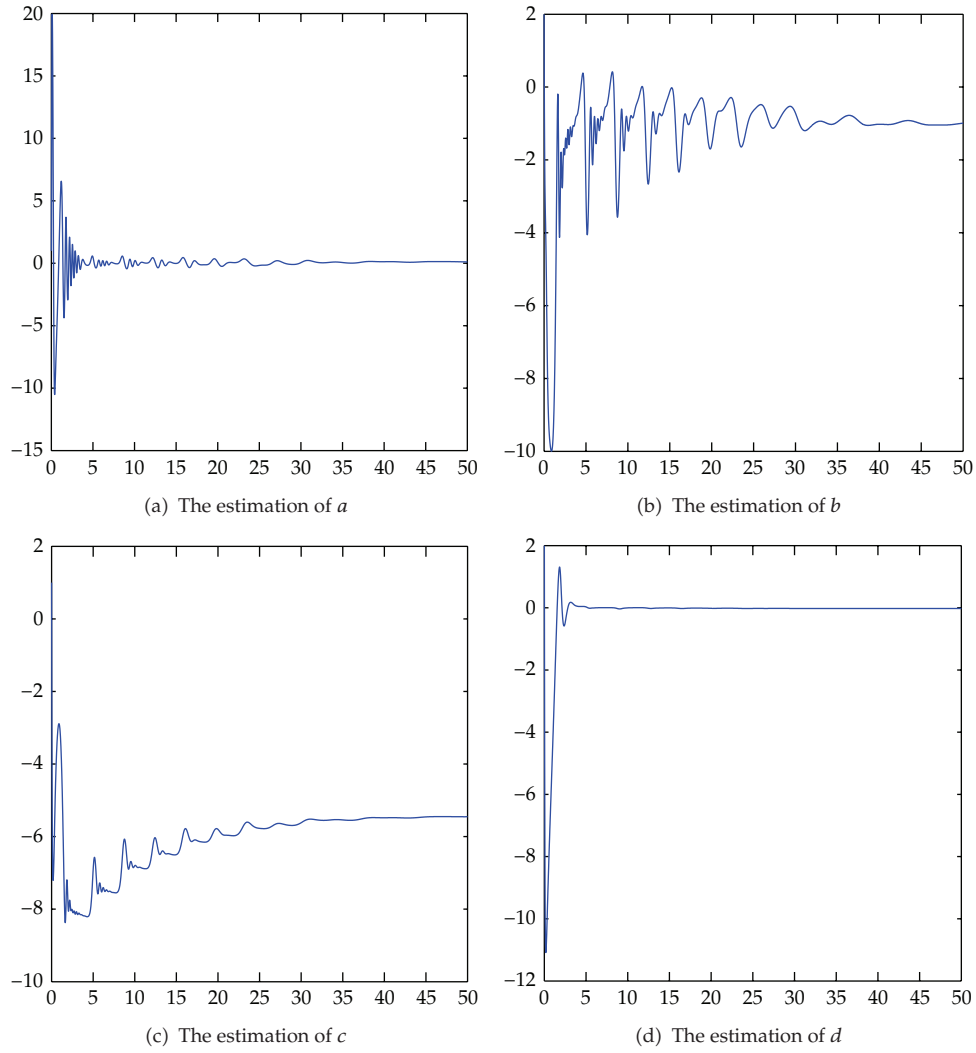
where  $u_1, u_2, u_3$ , and  $u_4$  are equal to 0 in the initial state and need to be determined by Theorem 3.4. In Figure 3,  $\hat{\alpha}$  is the estimation value of the column vector  $\alpha$ , where  $\hat{\alpha} = (\hat{a}, \hat{b}, \hat{c}, \hat{d})^T$ .



**Figure 2:** Comparison of the corresponding variables in the controlled hyperchaotic Lorenz system and its reference system: (a)  $x_1$  and  $x_{r1}$ ; (b)  $x_2$  and  $x_{r2}$ ; (c)  $x_3$  and  $x_{r3}$ ; (d)  $x_4$  and  $x_{r4}$ .

The estimation system of uncertain parameters is

$$\begin{aligned}
 \dot{\hat{a}} &= (x_2 + x_3)(x_{r1} - x_1), \\
 \dot{\hat{b}} &= x_2(x_{r2} - x_2), \\
 \dot{\hat{c}} &= x_4(x_{r2} - x_2), \\
 \dot{\hat{d}} &= x_4(x_{r4} - x_4).
 \end{aligned}
 \tag{4.18}$$



**Figure 3:** The estimation of the uncertain parameters: (a)  $\hat{a}$  is the estimation of  $a$ ; (b)  $\hat{b}$  is the estimation of  $b$ ; (c)  $\hat{c}$  is the estimation of  $c$ ; (d)  $\hat{d}$  is the estimation of  $d$ .

The reference model is the same as Example I,

$$\begin{aligned}
 \dot{x}_{r1} &= -x_{r2} - x_{r3}, \\
 \dot{x}_{r2} &= x_{r1} + 0.25x_{r2} + x_{r4}, \\
 \dot{x}_{r3} &= 20x_{r1} + 10x_{r2} - 50x_{r3} + 100x_{r4} + 3, \\
 \dot{x}_{r4} &= -0.5x_{r3} + 0.05x_{r4},
 \end{aligned} \tag{4.19}$$

$$\bar{B} = \begin{bmatrix} 0 & -1 & -1 & 0 \\ 1 & 0.25 & 0 & 1 \\ 20 & 10 & -50 & 100 \\ 0 & 0 & -0.5 & 0.05 \end{bmatrix}, \quad \bar{u} = \begin{bmatrix} 0 \\ 0 \\ 3 \\ 0 \end{bmatrix}.$$

We have the largest eigenvalue  $\lambda_{\max} = -0.9675$ . By considering the conditions  $\lambda < -\lambda_{\max}$  and  $\lambda \neq 0$ , we should obtain  $\lambda = -1$ .

Figure 4 shows that the controller can make the hyperchaotic Rössler system with four uncertain parameters (4.1) track along with its reference system (4.3). Here, the initials of the hyperchaotic system, reference system, and estimation system of uncertain parameters are  $[-10, 6, 0, 10]$ ,  $[10, -6, 10, -10]$ , and  $[1, 2, 1, 2]$ , respectively.

Comparing with Figures 1(c), 1(d), 4(c), and 4(d), we find that our method has even a better performance in the uncertain case than the certain case. In Figure 4(a), we can see some waves in time domain  $[0, 2]$ , and it is not better than the certain case. In Figures 4(b) and 1(b), the controlling performance is similar.

#### 4.4. Examples IV

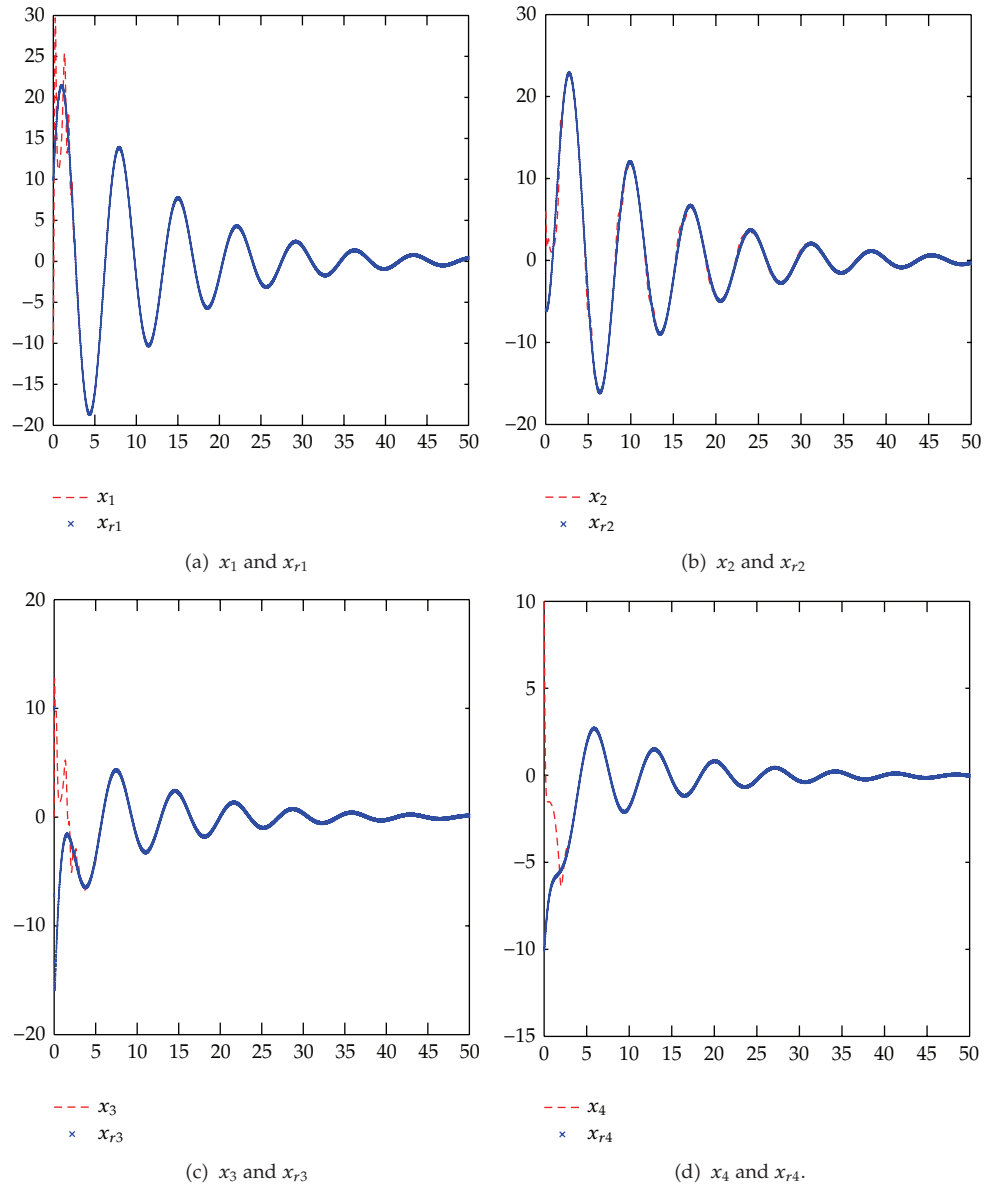
The four-variable hyperchaotic Lorenz system with uncertain parameters is described by

$$\begin{aligned}\dot{x}_1 &= a(x_2 - x_1), \\ \dot{x}_2 &= bx_1 - cx_2 - x_1x_3 + dx_4, \\ \dot{x}_3 &= x_1x_2 - fx_3, \\ \dot{x}_4 &= gx_1,\end{aligned}\tag{4.20}$$

where

$$\begin{aligned}A_2 &= \begin{bmatrix} 0 & 0 & -0.5 & 0 \\ 0 & 0 & 0 & 0 \\ -0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, & A_3 &= \begin{bmatrix} 0 & 0.5 & 0 & 0 \\ 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \\ B &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, & u &= \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}, & \alpha &= \begin{bmatrix} a \\ b \\ c \\ d \\ f \\ g \end{bmatrix}, \\ C(X) &= \begin{bmatrix} x_2 - x_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & x_1 & -x_2 & x_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & -x_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & x_1 \end{bmatrix}, \\ A_1 &= A_4 = \text{diag}[0, 0, 0, 0],\end{aligned}\tag{4.21}$$

where  $u_1, u_2, u_3$ , and  $u_4$  are equal to 0 in the initial state and need to be determined by Theorem 3.5. Figure 5 shows the estimation of  $\alpha$ , and  $\hat{\alpha}$  is the estimation value of the column vector  $\alpha$ , where  $\hat{\alpha} = (\hat{a}, \hat{b}, \hat{c}, \hat{d}, \hat{f}, \hat{g})^T$ .



**Figure 4:** Comparison of the corresponding variables in the controlled hyperchaotic Rössler system and its reference system: (a)  $x_1$  and  $x_{r1}$ ; (b)  $x_2$  and  $x_{r2}$ ; (c)  $x_3$  and  $x_{r3}$ ; (d)  $x_4$  and  $x_{r4}$ .

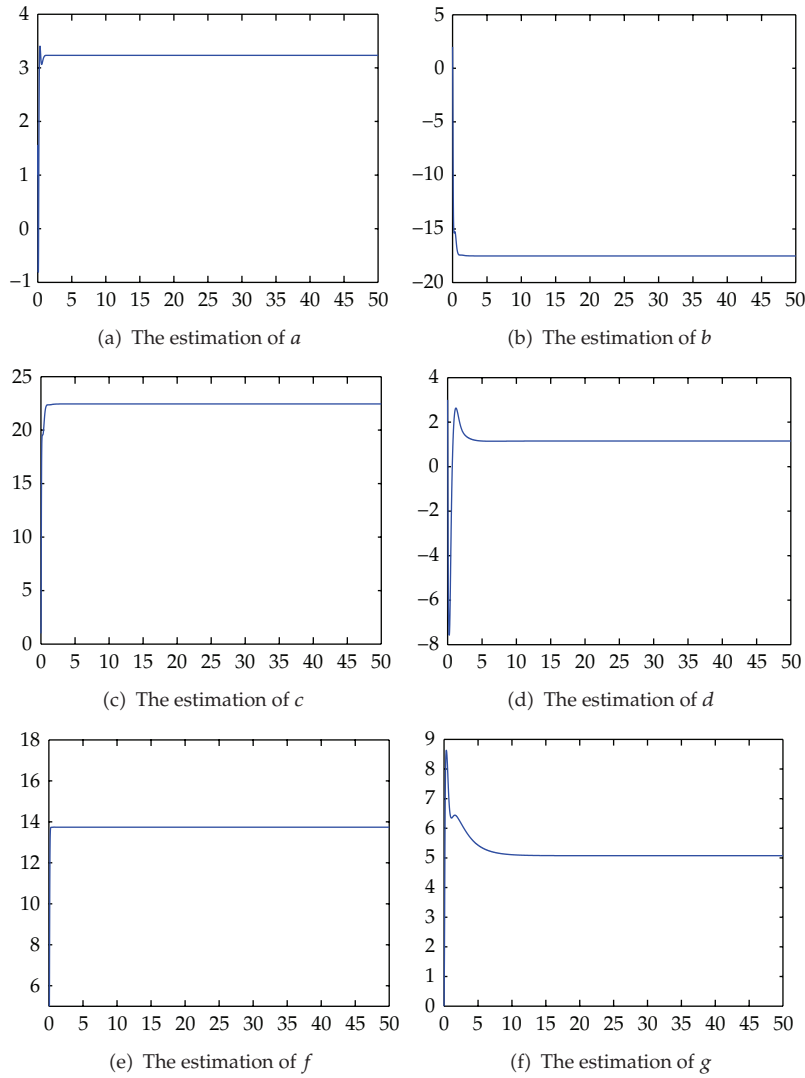
The estimation system of uncertain parameters is

$$\dot{\hat{a}} = (x_2 - x_1)(x_{r1} - x_1),$$

$$\dot{\hat{b}} = x_1(x_{r2} - x_2),$$

$$\dot{\hat{c}} = -x_2(x_{r2} - x_2),$$

$$\dot{\hat{d}} = x_4(x_{r2} - x_2),$$

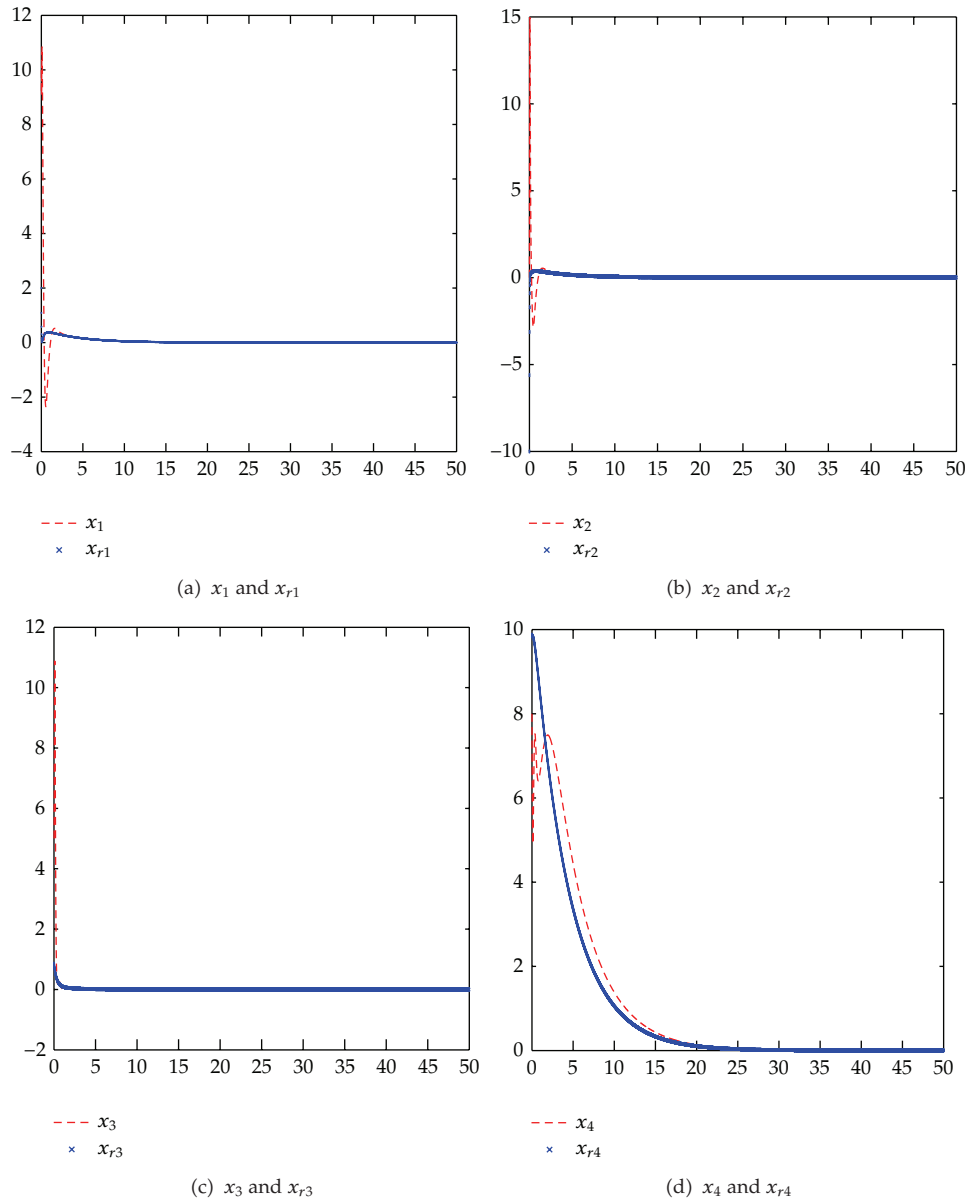


**Figure 5:** The estimation of the uncertain parameters: (a)  $\hat{a}$  is the estimation of  $a$ ; (b)  $\hat{b}$  is the estimation of  $b$ ; (c)  $\hat{c}$  is the estimation of  $c$ ; (d)  $\hat{d}$  is the estimation of  $d$ ; (e)  $\hat{f}$  is the estimation of  $f$ ; (f)  $\hat{g}$  is the estimation of  $g$ .

$$\begin{aligned} \dot{\hat{f}} &= -x_3(x_{r3} - x_{r3}), \\ \dot{\hat{g}} &= x_1(x_{r4} - x_4). \end{aligned} \tag{4.22}$$

The reference model is the same as Example II with asymptotical stability is as follows:

$$\begin{aligned} \dot{x}_{r1} &= 10(x_{r2} - x_{r1}) - 50x_{r2}, \\ \dot{x}_{r2} &= 28x_{r1} - x_{r2} - x_{r1}x_{r3} + x_{r4}, \end{aligned}$$



**Figure 6:** Comparison of the corresponding variables in the controlled hyperchaotic Lorenz system and its reference system: (a)  $x_1$  and  $x_{r1}$ ; (b)  $x_2$  and  $x_{r2}$ ; (c)  $x_3$  and  $x_{r3}$ ; (d)  $x_4$  and  $x_{r4}$ .

$$\begin{aligned}\dot{x}_{r3} &= x_{r1}x_{r2} - \frac{8}{3}x_{r3}, \\ \dot{x}_{r4} &= -5x_{r1},\end{aligned}\tag{4.23}$$

and the largest eigenvalue of  $\bar{B}^{-T}$  is  $\lambda_{\max} = 0.1767$ . By considering the conditions  $\lambda < -\lambda_{\max}$  and  $\lambda \neq 0$ , we should obtain  $\lambda = -1$ .



Figure 6 shows that the controller can make the hyperchaotic Lorenz system with uncertain parameters (4.20) track along with its reference system (4.23). Here, the initials of the hyperchaotic system, the reference system, and the estimation system of uncertain parameters are  $[10, 0, 5, 8]$ ,  $[2, -10, 1, 10]$ , and  $[1, 2, 1, 3, 1, 0]$ , respectively.

The performance in Figures 6(a)-6(b) is much better than the one in Figures 2(a)-2(b). Other parts of the two figures have almost the same performance.

## 5. Conclusion

In this paper, we have used an MRC technique to control the hyperchaotic system to track with an expected trajectory. The expression of the controller has been given. Four numerical examples show that the MRC method is very effective for controlling both the hyperchaotic system with all certain parameters and the systems with uncertain parameters. By comparing the results in the corresponding figures, we find that our method does not only fit for controlling hyperchaotic systems with certain parameters, but also is a robust for the systems with uncertain parameters.

## Acknowledgments

This paper is supported by China Scholarship Council. P. F. Zhao expresses sincere thanks to Professor Y. Li for his guidance, Professor D. Liu and Professor B. Yang for their instructions. Also, many thanks to the NSFC (grant number 11071026, 11001100, 11171131, 61133011, 61170092, 60973088, 61202308 and 11026043) and the Basic Research Program of Jilin University (450060481098). This work is supported in part by the National Basic Research Program of China (973) under Grant 2009CB219301, the National Public Benefit Scientific Research Foundation of China under Grant 201011078, the China Scholarship Council and the National Innovation Research Project for Exploration and Development of Oil Shale under Grant OSP-02 and OSR-02. The authors gratefully acknowledge the anonymous reviewers for their hard work and good patience.

## References

- [1] C. Li, X. Liao, and K.-W. Wong, "Lag synchronization of hyperchaos with application to secure communications," *Chaos, Solitons & Fractals*, vol. 23, no. 1, pp. 183–193, 2005.
- [2] T. Gao and Z. Chen, "A new image encryption algorithm based on hyper-chaos," *Physics Letters A*, vol. 372, no. 4, pp. 394–400, 2008.
- [3] J. Peng, D. Zhang, and X. Liao, "A digital image encryption algorithm based on hyper-chaotic cellular neural network," *Fundamenta Informaticae*, vol. 90, no. 3, pp. 269–282, 2009.
- [4] Q. T. Yang and T. G. Gao, "One-way hash function based on hyper-chaotic cellular neural network," *Chinese Physics B*, vol. 17, no. 7, pp. 2388–2393, 2008.
- [5] M. Eiswirtha, T. Kruelb, G. Ertla, and F. W. Schneider, "Hyperchaos in a chemical reaction," *Chemical Physics Letters*, vol. 193, no. 4, pp. 305–310, 1992.
- [6] G. Gandhi, "An improved Chua's circuit and its use in hyperchaotic circuit," *Analog Integrated Circuits and Signal Processing*, vol. 46, no. 2, pp. 173–178, 2005.
- [7] K. Thamilmaran, M. Lakshmanan, and A. Venkatesan, "Hyperchaos in a modified canonical Chua's circuit," *International Journal of Bifurcation and Chaos in Applied Sciences and Engineering*, vol. 14, no. 1, pp. 221–243, 2004.
- [8] J. M. V. Grzybowski, M. Rafikov, and J. M. Balthazar, "Synchronization of the unified chaotic system and application in secure communication," *Communications in Nonlinear Science and Numerical Simulation*, vol. 14, no. 6, pp. 2793–2806, 2009.
- [9] H. Wang, Z. Z. Han, and Z. Mo, "Synchronization of hyperchaotic systems via linear control," *Communications in Nonlinear Science and Numerical Simulation*, vol. 15, no. 7, pp. 1910–1920, 2010.

- [10] M. Rafikov, J. M. Balthazar, and H. F. von Bremen, "Mathematical modeling and control of population systems: applications in biological pest control," *Applied Mathematics and Computation*, vol. 200, no. 2, pp. 557–573, 2008.
- [11] E. Ott, C. Grebogi, and J. A. Yorke, "Controlling chaos," *Physical Review Letters*, vol. 64, no. 11, pp. 1196–1199, 1990.
- [12] G. P. Jiang and W. X. Zheng, "An LMI criterion for linear-state-feedback based chaos synchronization of a class of chaotic systems," *Chaos, Solitons and Fractals*, vol. 26, pp. 437–443, 2005.
- [13] J. H. Park, O. M. Kwon, and S. M. Lee, "LMI optimization approach to stabilization of Genesio-Tesi chaotic system via dynamic controller," *Applied Mathematics and Computation*, vol. 196, no. 1, pp. 200–206, 2008.
- [14] H. Salarieh and A. Alasty, "Control of stochastic chaos using sliding mode method," *Journal of Computational and Applied Mathematics*, vol. 225, no. 1, pp. 135–145, 2009.
- [15] R.-A. Tang, Y.-L. Liu, and J.-K. Xue, "An extended active control for chaos synchronization," *Physics Letters A*, vol. 373, no. 16, pp. 1449–1454, 2009.
- [16] M. Rafikov and J. M. Balthazar, "Optimal linear and nonlinear control of design for chaotic systems," in *Proceedings of the ASME International Design Engineering Technical Conferences and Computers and Information in Engineering Conference (IDETC '05)*, pp. 24–28, Long Beach, Calif, USA, September 2005.
- [17] M. Rafikov and J. M. Balthazar, "On an optimal control design for Rössler system," *Physics Letters A*, vol. 333, no. 3-4, pp. 241–245, 2004.
- [18] C.-I. Morărescu and B. Brogliato, "Passivity-based switching control of flexible-joint complementarity mechanical systems," *Automatica*, vol. 46, no. 1, pp. 160–166, 2010.
- [19] D. H. Ji, J. H. Koo, S. C. Won, S. M. Lee, and J. H. Park, "Passivity-based control for Hopfield neural networks using convex representation," *Applied Mathematics and Computation*, vol. 217, no. 13, pp. 6168–6175, 2011.
- [20] C. Yang, C. H. Tao, and P. Wang, "Comparison of feedback control methods for a hyperchaotic Lorenz system," *Physics Letters A*, vol. 374, no. 5, pp. 729–732, 2010.
- [21] M. Rafikov and J. M. Balthazar, "On control and synchronization in chaotic and hyperchaotic systems via linear feedback control," *Communications in Nonlinear Science and Numerical Simulation*, vol. 13, no. 7, pp. 1246–1255, 2008.
- [22] M. M. Al-Sawalha and M. S. M. Noorani, "Adaptive anti-synchronization of two identical and different hyperchaotic systems with uncertain parameters," *Communications in Nonlinear Science and Numerical Simulation*, vol. 15, no. 4, pp. 1036–1047, 2010.
- [23] H. Zhang, X. Ma k, M. Li, and J. Zou -l, "Controlling and tracking hyperchaotic Rössler system via active backstepping design," *Chaos, Solitons and Fractals*, vol. 26, pp. 353–361, 2005.
- [24] Y. Li, X. Liu, and H. Zhang, "Dynamical analysis and impulsive control of a new hyperchaotic system," *Mathematical and Computer Modelling*, vol. 42, no. 11-12, pp. 1359–1374, 2005.
- [25] A. Eklekli and D. J. Brookfield, "The practical implementation of model reference robot control," *Mechatronics*, vol. 7, pp. 549–564, 1997.
- [26] N. Hovakimyan, R. Rysdyk, and A. J. Calise, "Dynamic neural networks for output feedback control," in *Proceedings of the 38th IEEE Conference on Decision and Control (CDC '99)*, vol. 2, pp. 1685–1690, December 1999.
- [27] P. F. Zhao, C. Liu, and X. Feng, "Model reference control for an economic growth cycle model," *Journal of Applied Mathematics*, vol. 2012, Article ID 384732, 13 pages, 2012.
- [28] S. N. Chow and Y. Li, "Model reference control for sirs models," *Discrete and Continuous Dynamical Systems*, vol. 24, no. 3, pp. 675–697, 2009.
- [29] O. E. Rössler, "An equation for hyperchaos," *Physics Letters A*, vol. 71, no. 2-3, pp. 155–157, 1979.
- [30] T. Gao, Z. Chen, Q. Gu, and Z. Yuan, "A new hyper-chaos generated from generalized Lorenz system via nonlinear feedback," *Chaos, Solitons and Fractals*, vol. 35, pp. 390–397, 2008.
- [31] J. H. Park, "Adaptive modified projective synchronization of a unified chaotic system with uncertain parameter," *Chaos Solitons and Fractals*, vol. 34, pp. 1552–1559, 2007.
- [32] R. Z. Luo and Z. M. Wei, "Adaptive function projective synchronization of unified chaotic with uncertain parameters," *Chaos, Solitons and Fractals*, vol. 42, pp. 1266–1272, 2009.
- [33] X. Chen and J. Lu, "Adaptive synchronization of different chaotic systems with fully unknown parameters," *Physics Letters A*, vol. 364, no. 2, pp. 123–128, 2007.
- [34] C. Li, "Tracking control and generalized projective synchronization of a class of hyperchaotic system with unknown parameter and disturbance," *Communications in Nonlinear Science and Numerical Simulation*, vol. 17, no. 1, pp. 405–413, 2012.

- [35] W. L. Li, Z. H. Liu, and J. Miao, "Adaptive synchronization for a unified chaotic system with uncertainty," *Communication in Nonlinear Science and Numerical Simulation*, vol. 15, pp. 3015–3021, 2010.
- [36] H. Du, Q. Zeng, and C. Wang, "Function projective synchronization of different chaotic systems with uncertain parameters," *Physics Letters A*, vol. 372, no. 33, pp. 5402–5410, 2008.
- [37] J. H. Park, "Adaptive control for modified projective synchronization of a four-dimensional chaotic system with uncertain parameters," *Journal of Computational and Applied Mathematics*, vol. 213, no. 1, pp. 288–293, 2008.