Research Article

On a Fixed Point for Generalized Contractions in Generalized Metric Spaces

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Lakzian and Samet (2010) studied some fixed-point results in generalized metric spaces in the sense of Branciari. In this paper, we study the existence of fixed-point results of mappings satisfying generalized weak contractive conditions in the framework of a generalized metric space in sense of Branciari. Our results modify and generalize the results of Laksian and Samet, as well as, our results generalize several well-known comparable results in the literature.

1. Introduction and Preliminaries

Branciari in [1] initiated the notion of a generalized metric space as a generalization of a metric space in such a way that the triangle inequality is replaced by the “quadrilateral inequality,” \( d(x, y) \leq d(x, a) + d(a, b) + d(b, y) \) for all pairwise distinct points \( x, y, a, \) and \( b \) of \( X \). Afterwards, many authors initiated and studied many existing fixed-point theorems in such spaces. For more details about fixed-point theory in generalized metric spaces, we refer the reader to [1–13].

The following definitions will be needed in the sequel.

Definition 1.1 (see [1]). Let \( X \) be a nonempty set and \( d : X \times X \to [0, +\infty) \) such that for all \( x, y \in X \) and for all distinct points \( u, v \in X \) each of them different from \( x \) and \( y \), one has
(p1): \(x = y \iff d(x, y) = 0\),

(p2): \(d(x, y) = d(y, x)\),

(p3): \(d(x, y) \leq d(x, u) + d(u, v) + d(v, y)\).

Then, \((X, d)\) is called a generalized metric space (or shortly g.m.s).

Any metric space is a generalized metric space, but the converse is not true [1].

Definition 1.2 (see [1]). Let \((X, d)\) be a g.m.s, \(\{x_n\}\) a sequence in \(X\), and \(x \in X\). We say that \(\{x_n\}\) is g.m.s convergent to \(x\) if and only if \(d(x_n, x) \to 0\) as \(n \to +\infty\). We denote this by \(x_n \to x\).

Definition 1.3 (see [1]). Let \((X, d)\) be a g.m.s and \(\{x_n\}\) a sequence in \(X\). We say that \(\{x_n\}\) is a g.m.s Cauchy sequence if and only if for each \(\varepsilon > 0\) there exists a natural number \(N\) such that \(d(x_n, x_m) < \varepsilon\) for all \(n > m > N\).

Definition 1.4 (see [1]). Let \((X, d)\) be a g.m.s. Then, \((X, d)\) is called a complete g.m.s if every g.m.s Cauchy sequence is g.m.s convergent in \(X\).

Very recently, Lakzian and Samet [9] proved the following nice result.

**Theorem 1.5.** Let \((X, d)\) be a Hausdorff and complete generalized metric space. Suppose that \(T : X \to X\) is such that for all \(x, y \in X\)

\[
\psi(d(Tx, Ty)) \leq \psi(d(x, y)) - \phi(d(x, y)),
\]

where \(\psi : [0, \infty) \to [0, \infty)\) is continuous and nondecreasing with \(\psi(t) = 0\) if and only if \(t = 0\), and \(\phi : [0, \infty) \to [0, \infty)\) is continuous and \(\phi(t) = 0\) if and only if \(t = 0\). Then, there exists a unique point \(u \in X\) such that \(u = Tu\).

Note that Theorem 1.5 extends a result of Dutta and Choudhury [14] to the set of generalized metric spaces. Moreover, its proof is more technical compared with that of [9].

In this paper, we generalize in some cases Theorem 1.5 by replacing in (1.1) the term \(d(x, y)\) by the quantity \(\max\{d(x, y), d(x, Tx), d(y, Ty)\}\) and the continuity of \(\phi\) by lower semicontinuity. Also, we derive some useful corollaries of this result.

### 2. Main Results

Let \(X\) be a nonempty set and \(T : X \to X\) a given mapping. For all \(x, y \in X\), set

\[
M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty)\}.
\]

Also, let \(\Psi = \{\psi : [0, \infty) \to [0, \infty)\) be continuous, nondecreasing, and \(\psi(t) = 0\) if and only if \(t = 0\)\}, and \(\Phi = \{\phi : [0, \infty) \to [0, \infty)\) is lower semi continuous, \(\phi(t) > 0\) for all \(t > 0\) and \(\phi(0) = 0\)\}. Note that, if \(\psi \in \Psi\), \(\psi\) is called an altering distance function [15].

The notion of a periodic point of a given mapping \(T : X \to X\) is crucial for proving our main theorem. So we need the following definition.
Theorem 2.2. Let $X$ be a nonempty set. A given mapping $T : X \to X$ admits a periodic point if there exists $u \in X$ such that $u = T^p u$ for some $p \geq 1$. If $p = 1$, $u$ is a fixed point.

Hence, each fixed point is also a periodic point of $T$.

Now, in the following, let us prove our main result.

**Theorem 2.2.** Let $(X, d)$ be a Hausdorff and complete generalized metric space. Suppose that $T : X \to X$ is such that for all $x, y \in X$

$$
\psi(d(Tx, Ty)) \leq \psi(M(x, y)) - \phi(M(x, y)),
$$

where $\psi \in \Psi$, $\phi \in \Phi$, and $M(x, y)$ is defined by (2.1). Then, there exists a unique point $u \in X$ such that $u = Tu$.

**Proof.** First, it is obvious that $M(x, y) = 0$ if and only if $x = y$ is a fixed point of $T$. Let $x_0 \in X$ an arbitrary point. By induction, we easily construct a sequence $\{x_n\}$ such that

$$
x_{n+1} = Tx_n = T^{n+1}x_0 \quad \forall n \geq 0.
$$

**Step 1.** We claim that

$$
\lim_{n \to \infty} d(x_n, x_{n+1}) = 0.
$$

Substituting $x = x_n$ and $y = x_{n-1}$ in (2.2) and using properties of functions $\psi$ and $\phi$, we obtain

$$
\psi(d(x_{n+1}, x_n)) = \psi(d(Tx_n, Tx_{n-1})) \\
\leq \psi(M(x_n, x_{n-1})) - \phi(M(x_n, x_{n-1})) \\
\leq \psi(M(x_n, x_{n-1}))
$$

which implies that

$$
d(x_{n+1}, x_n) \leq M(x_n, x_{n-1}) \quad \forall n \geq 1.
$$

Note that

$$
M(x_n, x_{n-1}) = \max\{d(x_n, x_{n-1}), d(x_n, Tx_n), d(x_{n-1}, Tx_{n-1})\} \\
= \max\{d(x_n, x_{n-1}), d(x_n, x_{n+1})\}.
$$

If for some $n \geq 1$, $d(x_{n-1}, x_n) < d(x_n, x_{n+1})$, then $M(x_n, x_{n-1}) = d(x_n, x_{n+1}) > 0$ and $\phi(d(x_{n+1}, x_n)) > 0$ by a property of $\phi$, so (2.5) becomes

$$
0 < \psi(d(x_{n+1}, x_n)) \leq \psi(d(x_{n+1}, x_n)) - \phi(d(x_{n+1}, x_n)) < \psi(d(x_{n+1}, x_n))
$$
a contradiction. Thus, for all $n \geq 1$,

$$d(x_{n+1}, x_n) \leq d(x_{n-1}, x_n) = M(x_{n-1}, x_n). \quad (2.9)$$

From (2.9), the sequence $\{d(x_n, x_{n+1})\}$ is monotone nonincreasing and so bounded below. So there exists $r \geq 0$ such that

$$\lim_{n \to \infty} d(x_n, x_{n+1}) = \lim_{n \to \infty} M(x_{n-1}, x_n) = r. \quad (2.10)$$

Letting $\limsup_{n \to \infty}$ in (2.5) and using the above limits with the continuity of $\psi$ and the lower semicontinuity of $\phi$, we get $\psi(r) \leq \psi(r) - \phi(r)$, which implies that $\phi(r) = 0$, so $r = 0$ by a property of $\phi$. Thus, (2.4) is proved.

Step 2. We shall prove that

$$\lim_{n \to \infty} d(x_n, x_{n+2}) = 0. \quad (2.11)$$

By (2.2), we have

$$\psi(d(x_{n+2}, x_n)) = \psi(d(Tx_{n+1}, Tx_{n-1})) \leq \psi(M(x_{n+1}, x_{n-1})) - \phi(M(x_{n+1}, x_{n-1})) \leq \psi(M(x_{n+1}, x_{n-1})) \quad (2.12)$$

which implies that

$$d(x_{n+2}, x_n) \leq M(x_{n+1}, x_{n-1}) \quad \forall n \geq 1, \quad (2.13)$$

where

$$M(x_{n+1}, x_{n-1}) = \max\{d(x_{n+1}, x_{n-1}), d(x_{n+1}, Tx_{n+1}), d(x_{n-1}, Tx_{n-1})\} = \max\{d(x_{n+1}, x_{n-1}), d(x_{n+1}, x_{n+2}), d(x_{n-1}, x_n)\} = \max\{d(x_{n+1}, x_{n-1}), d(x_{n-1}, x_n)\}. \quad (2.14)$$

Set $\alpha_n = d(x_{n+2}, x_n)$ and $\beta_n = d(x_n, x_{n+1})$. Thus, by (2.12), one can write

$$\psi(\alpha_n) \leq \psi(\max\{\alpha_n, \beta_n\}) - \phi(\max\{\alpha_n, \beta_n\}) \quad \forall n \geq 1 \quad (2.15)$$

which implies that

$$\alpha_n \leq \max\{\alpha_{n-1}, \beta_{n-1}\}. \quad (2.16)$$
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On the other hand, having in mind that the sequence \( \{ d(x_n, x_{n+1}) \} = \{ \beta_n \} \) is monotone nonincreasing, so

\[
\beta_n \leq \beta_{n-1} \leq \max \{ \alpha_{n-1}, \beta_{n-1} \}.
\]  

(2.17)

From (2.16) and (2.17), we have

\[
\max \{ \alpha_n, \beta_n \} \leq \max \{ \alpha_{n-1}, \beta_{n-1} \} \quad \forall n \geq 1.
\]  

(2.18)

Therefore, the sequence \( \{ \max \{ \alpha_n, \beta_n \} \} \) is monotone nonincreasing, so it converges to some \( t \geq 0 \). Assume that \( t > 0 \). Now, by (2.4), it is obvious that

\[
\limsup_{n \to \infty} \alpha_n = \limsup_{n \to \infty} \max \{ \alpha_n, \beta_n \} = \lim_{n \to \infty} \max \{ \alpha_n, \beta_n \} = t.
\]  

(2.19)

Taking the \( \limsup_{n \to \infty} \) in (2.15) and using (2.19) and the properties of \( \psi \) and \( \phi \), we obtain

\[
\psi(t) = \psi \left( \limsup_{n \to \infty} \alpha_n \right)
\]

\[
= \limsup_{n \to \infty} \psi(\alpha_n)
\]

\[
\leq \limsup_{n \to \infty} \psi \left( \max \{ \alpha_{n-1}, \beta_{n-1} \} \right) - \liminf_{n \to \infty} \phi \left( \max \{ \alpha_{n-1}, \beta_{n-1} \} \right)
\]

\[
\leq \psi \left( \lim_{n \to \infty} \max \{ \alpha_{n-1}, \beta_{n-1} \} \right) - \phi \left( \lim_{n \to \infty} \max \{ \alpha_{n-1}, \beta_{n-1} \} \right)
\]

\[
= \psi(t) - \phi(t)
\]  

(2.20)

which implies that \( \phi(t) = 0 \), so \( t = 0 \), a contradiction. Thus, from (2.19),

\[
\limsup_{n \to \infty} \alpha_n = 0,
\]  

(2.21)

and hence \( \lim_{n \to \infty} \alpha_n = 0 \), so (2.11) is proved.

Step 3. We claim that \( T \) has a periodic point.

We argue by contradiction. Assume that \( T \) has no periodic point. Then, \( \{ x_n \} \) is a sequence of distinct points, that is, \( x_n \neq x_m \) for all \( m \neq n \). We will show that, in this case, \( \{ x_n \} \) is g.m.s Cauchy. Suppose to the contrary. Then, there is a \( \varepsilon > 0 \) such that for an integer \( k \) there exist integers \( m(k) > n(k) > k \) such that

\[
d(x_n(k), x_{m(k)}) > \varepsilon.
\]  

(2.22)
For every integer \( k \), let \( m(k) \) be the least positive integer exceeding \( n(k) \) satisfying (2.22) and such that

\[
d(x_{n(k)}, x_{m(k)-1}) \leq \varepsilon. \tag{2.23}
\]

Now, using (2.22), (2.23), and the rectangular inequality (because \( \{x_n\} \) is a sequence of distinct points), we find that

\[
\varepsilon < d(x_{m(k)}, x_{n(k)}) \leq d(x_{m(k)}, x_{m(k)-2}) + d(x_{m(k)-2}, x_{m(k)-1}) + d(x_{m(k)-1}, x_{n(k)})
\]

\[
\leq d(x_{m(k)}, x_{m(k)-2}) + d(x_{m(k)-2}, x_{m(k)-1}) + \varepsilon. \tag{2.24}
\]

Then, by (2.4) and (2.11), it follows that

\[
\lim_{k \to +\infty} d(x_{n(k)}, x_{m(k)}) = \varepsilon. \tag{2.25}
\]

Now, by rectangular inequality, we have

\[
d(x_{m(k)}, x_{n(k)}) \leq d(x_{m(k)}, x_{m(k)-1}) + d(x_{m(k)-1}, x_{n(k)-1}) + d(x_{n(k)-1}, x_{n(k)})
\]

\[
d(x_{m(k)-1}, x_{n(k)-1}) \leq d(x_{m(k)-1}, x_{m(k)}) + d(x_{m(k)}, x_{n(k)}) + d(x_{n(k)}, x_{n(k)-1}). \tag{2.26}
\]

Letting \( k \to \infty \) in the above inequalities, using (2.4) and (2.25), we obtain

\[
\lim_{k \to \infty} d(x_{m(k)-1}, x_{n(k)-1}) = \varepsilon. \tag{2.27}
\]

Therefore, by (2.4) and (2.27), we get that

\[
M(x_{m(k)-1}, x_{n(k)-1}) = \max\{d(x_{m(k)-1}, x_{n(k)-1}), d(x_{m(k)-1}, x_{m(k)}), d(x_{n(k)-1}, x_{n(k)})\} \to \varepsilon
\]

as \( k \to \infty. \tag{2.28}
\]

Applying (2.2) with \( x = x_{m(k)-1} \) and \( y = x_{n(k)-1} \), we have

\[
\psi(d(x_{m(k)}, x_{n(k)})) = \psi(Tx_{m(k)-1}, Tx_{n(k)-1}) \leq \psi(M(x_{m(k)-1}, x_{n(k)-1})) - \phi(M(x_{m(k)-1}, x_{n(k)-1})). \tag{2.29}
\]

Letting \( k \to \infty \) in the above inequality and using (2.25) and (2.28), we obtain

\[
\psi(\varepsilon) \leq \psi(\varepsilon) - \phi(\varepsilon) \tag{2.30}
\]

which yields that \( \phi(\varepsilon) = 0 \), so \( \varepsilon = 0 \), which is a contradiction.
Hence, \( \{x_n\} \) is g.m.s Cauchy. Since \((X, d)\) is a complete g.m.s, there exists \(u \in X\) such that \(x_n \to u\). Applying (2.2) with \(x = x_n\) and \(y = u\), we obtain

\[
\psi(d(x_{n+1}, Tu)) = \psi(d(Tx_n, Tu)) \leq \psi(M(x_n, u)) - \phi(M(x_n, u)) \leq \psi(M(x_n, u))
\]

which implies that

\[
d(x_{n+1}, Tu) \leq M(x_n, u),
\]

where

\[
M(x_n, u) = \max\{d(x_n, u), d(x_n, x_{n+1}), d(u, Tu)\}.
\]

Since \(\lim_{n \to \infty} d(x_n, u) = \lim_{n \to \infty} d(x_n, x_{n+1}) = 0\), so we obtain that

\[
\lim_{n \to \infty} M(x_n, u) = d(u, Tu).
\]

It follows that

\[
\limsup_{n \to \infty} d(x_{n+1}, Tu) \leq d(u, Tu).
\]

Next, we shall find a contradiction of the fact that \(T\) has no periodic point in each of the two following cases.

(i) If, for all \(n \geq 2\), \(x_n \neq u\) and \(x_n \neq Tu\), then by rectangular inequality

\[
d(u, Tu) \leq d(u, x_n) + d(x_n, x_{n+1}) + d(x_{n+1}, Tu),
\]

and, using (2.4), we get that

\[
d(u, Tu) \leq \limsup_{n \to \infty} d(x_{n+1}, Tu).
\]

From (2.35) and (2.37),

\[
\limsup_{n \to \infty} d(x_{n+1}, Tu) = d(u, Tu).
\]

Taking the \(\limsup_{n \to \infty}\) in (2.31) and using (2.34), (2.38), and the properties of \(\psi\) and \(\phi\), we obtain

\[
\psi(d(u, Tu)) \leq \psi(d(u, Tu)) - \phi(d(u, Tu))
\]
which implies that \( d(u, Tu) = 0 \), so \( u = Tu \), that is, \( u \) is a fixed point of \( T \), so \( u \) is a periodic point of \( T \). It contradicts the fact that \( T \) has no periodic point.

(ii) Let for some \( q \geq 2 \), \( x_q = u \) or \( x_q = Tu \). Since \( T \) has no periodic point, then obviously \( u \neq x_0 \). Indeed, if \( x_q = u = x_0 \), so \( T^q x_0 = x_q \), that is, \( x_0 \) is a periodic point of \( T \), while if \( x_q = Tu \) and \( x_0 = u \), so \( Tx_0 = Tu = x_q = T^{q+1} (Tx_0) \), that is, \( Tx_0 \) is a periodic point of \( T \).

For all \( n \geq 0 \), we have

\[
\begin{align*}
     d(T^n u, u) &= d(T^n x_q, u) = d(x_{n+q}, u) \quad \text{or} \\
     d(T^n u, u) &= d(T^{n+1} u, Tu) = d(T^{n+1} x_q, u) = d(x_{n+q+1}, u).
\end{align*}
\]

In the two precedent identities, the integer \( q \geq 2 \) is fixed, and so \( \{x_{n+q}\} \) and \( \{x_{n+q+1}\} \) are subsequences from \( \{x_n\} \), and since \( \{x_n\} \) g.m.s. converges to \( u \) in \( (X, d) \) which is assumed to be Hausdorff, so the two subsequences g.m.s. converge to same unique limit \( u \), that is,

\[
\lim_{n \to \infty} d(x_{n+q}, u) = \lim_{n \to \infty} d(x_{n+q+1}, u) = 0.
\]

Thus,

\[
\lim_{n \to \infty} d(T^n u, u) = 0.
\]

Again, since \( (X, d) \) is Hausdorff, then by (2.42),

\[
\lim_{n \to \infty} d(T^{n+2} u, u) = 0.
\]

On the other hand, since \( T \) has no periodic point, it follows that

\[
T^s u \neq T^r u \quad \text{for any} \ s, r \in \mathbb{N}, \ s \neq r.
\]  

Using (2.44) and the rectangular inequality, we may write

\[
|d(T^{n+1} u, Tu) - d(u, Tu)| \leq d(T^{n+1} u, T^{n+2} u) + d(T^{n+2} u, u).
\]

Letting \( n \to \infty \) in the above limit and proceeding as (2.4) (since the point \( x_0 \) is arbitrary), using (2.43), we obtain

\[
\lim_{n \to \infty} d(T^{n+1} u, Tu) = d(u, Tu).
\]

Now, by (2.2),

\[
\psi(d(T^{n+1} u, Tu)) \leq \psi(M(T^n u, u)) - \phi(M(T^n u, u)),
\]
where
\[
M(T^n u, u) = \max \left\{ d(T^n u, u), d(T^n u, T^{n+1} u), d(u, T u) \right\} \to d(u, T u) \quad \text{as } n \to \infty. \tag{2.48}
\]

Letting \( n \to \infty \) in (2.47) and using (2.46) and the above limit, we get that
\[
\psi(d(u, T u)) \leq \psi(d(u, T u)) - \phi(d(u, T u)) \tag{2.49}
\]
which holds only if \( d(u, T u) = 0 \), that is, \( T u = u \), which implies that \( u \) is a periodic point of \( T \). This contradicts the fact that \( T \) has no periodic point.

Consequently, \( T \) admits a periodic point, that is, there exists \( u \in X \) such that \( u = T^p u \) for some \( p \geq 1 \).

**Step 4.** Existence of a fixed point of \( T \).

If \( p = 1 \), then \( u = T u \), that is, \( u \) is a fixed point of \( T \). Suppose now that \( p > 1 \). We will prove that \( a = T^{p-1} u \) is a fixed point of \( T \). Suppose that it is not the case, that is, \( T^{p-1} u \neq T^p u \). Then, \( d(T^{p-1} u, T^p u) > 0 \) and \( \phi(d(T^{p-1} u, T^p u)) > 0 \), which implies that \( \phi(M(T^{p-1} u, T^p u)) > 0 \). Now, using inequality (2.2), we obtain
\[
\psi(d(u, T u)) = \psi \left( d \left( T^p u, T^{p+1} u \right) \right)
\]
\[
= \psi \left( d \left( T(T^{p-1} u), T(T^p u) \right) \right)
\]
\[
\leq \psi \left( M \left( T^{p-1} u, T^p u \right) \right) - \phi \left( M \left( T^{p-1} u, T^p u \right) \right)
\]
\[
< \psi \left( M \left( T^{p-1} u, T^p u \right) \right) \tag{2.50}
\]
which by the monotone nondecreasing property of \( \psi \) implies
\[
d(u, T u) < M \left( T^{p-1} u, T^p u \right), \tag{2.51}
\]
where
\[
M \left( T^{p-1} u, T^p u \right) = \max \left\{ d \left( T^{p-1} u, T^p u \right), d \left( T^{p-1} u, T^p u \right), d \left( T^p u, T^{p+1} u \right) \right\}
\]
\[
= \max \left\{ d \left( T^{p-1} u, T^p u \right), d(u, T u) \right\} = d \left( T^{p-1} u, T^p u \right) \tag{2.52}
\]
because otherwise we get a contradiction with (2.51). Thus, (2.51) becomes
\[
d(u, T u) < d \left( T^{p-1} u, T^p u \right). \tag{2.53}
\]
Again, using (2.2), we have
\[ q(d(T^{p-1}u, T^pu)) = q(d(T(T^{p-2}u), T(T^{p-1}u))) \leq q(M(T^{p-2}u, T^{p-1}u)) - \phi(M(T^{p-2}u, T^{p-1}u)) \]
\[ < q(M(T^{p-2}u, T^{p-1}u)). \tag{2.54} \]

Again, this implies that
\[ d(T^{p-1}u, T^pu) < M(T^{p-2}u, T^{p-1}u), \tag{2.55} \]

Where
\[ M(T^{p-2}u, T^{p-1}u) = \max\{d(T^{p-2}u, T^{p-1}u), d(T^{p-2}u, T^{p-1}u), d(T^{p-1}u, T^pu)\} \]
\[ = \max\{d(T^{p-2}u, T^{p-1}u), d(T^{p-1}u, T^pu)\} = d(T^{p-2}u, T^{p-1}u) \tag{2.56} \]

because of (2.55). Thus, from (2.55),
\[ d(T^{p-1}u, T^pu) < d(T^{p-2}u, T^{p-1}u). \tag{2.57} \]

Continuing this process as (2.53) and (2.57), we find that
\[ d(u, Tu) < d(T^{p-1}u, T^pu) < d(T^{p-2}u, T^{p-1}u) < \cdots < d(u, Tu) \tag{2.58} \]

which is a contradiction. We deduce that \( a = T^{p-1}u \) is a fixed point of \( T \).

**Step 5. Uniqueness of the fixed point of \( T \).**

Suppose that there are two distinct points \( b, c \in X \) such that \( Tb = b \) and \( Tc = c \). Then, \( M(b, c) = \max\{d(b, c), d(b, Tb), d(c, Tc)\} = d(b, c) \) and \( \phi(d(b, c)) > 0 \). By (2.2), we obtain
\[ q(d(b, c)) = q(d(Tb, Tc)) \leq q(M(b, c)) - \phi(M(b, c)) \]
\[ = q(d(b, c)) - \phi(d(b, c)) < q(d(b, c)) \tag{2.59} \]

a contradiction. Thus, \( T \) has a unique fixed point. This completes the proof of Theorem 2.2. \( \square \)

Now, we state some corollaries of Theorem 2.2, which are given in the following.

**Corollary 2.3.** Let \((X, d)\) be a Hausdorff and complete generalized metric space. Suppose that \( T : X \rightarrow X \) is such that, for all \( x, y \in X \), there exists \( k \in [0, 1) \) and
\[ d(Tx, Ty) \leq k \max\{d(x, y), d(x, Tx), d(y, Ty)\}, \tag{2.60} \]
then \( T \) has a unique fixed point.
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Proof. It suffices to take \( q(t) = t \) and \( \phi(t) = (1 - k)t \) in Theorem 2.2.

Corollary 2.4. Let \( (X, d) \) be a Hausdorff and complete generalized metric space. Suppose that \( T : X \rightarrow X \) is such that, for all \( x, y \in X \), there exists \( \alpha \in [0, 1/2) \) and

\[
\left( d(Tx, Ty) \right) \leq \alpha \left( d(x, Tx) + d(y, Ty) \right),
\]

(2.61)

then \( T \) has a unique fixed point.

Proof. Let \( k = 2\alpha \), so \( k \in [0, 1) \). Also, if (2.61) holds, so

\[
\left( d(Tx, Ty) \right) \leq \alpha \left( d(x, Tx) + d(y, Ty) \right) = k \frac{d(x, Tx) + d(y, Ty)}{2}
\]

\[
\leq k \max \{ d(x, y), d(x, Tx), d(y, Ty) \}. \tag{2.62}
\]

Then, it suffices to apply Corollary 2.3.

Another easy consequence of Corollary 2.3 (a Reich contraction type) is the following.

Corollary 2.5. Let \( (X, d) \) be a Hausdorff and complete generalized metric space. Suppose that \( T : X \rightarrow X \) is such that, for all \( x, y \in X \), there exists \( k \in [0, 1/3) \) and

\[
\left( d(Tx, Ty) \right) \leq k \left( d(x, y) + d(x, Tx) + d(y, Ty) \right),
\]

(2.63)

then \( T \) has a unique fixed point.

Corollary 2.6. Let \( T \) satisfy the conditions of Theorem 2.2, except that condition (2.2) is replaced by the following: there exist positive Lebesgue integrable functions \( u \) and \( v \) on \( \mathbb{R}^+ \) such that \( \int_0^\epsilon u(t)dt > 0 \) and \( \int_0^\epsilon v(t)dt > 0 \) for each \( \epsilon > 0 \) and that

\[
\int_0^{\psi(d(Tx, Ty))} u(t)dt \leq \int_0^{\psi(M(x, y))} u(t)dt - \int_0^{\psi(M(x, y))} v(t)dt. \tag{2.64}
\]

Then, \( T \) has a unique fixed point.

Proof. Consider the functions

\[
\varphi_0(x) = \int_0^x u(t)dt, \quad \varphi_1(x) = \int_0^x v(t)dt. \tag{2.65}
\]

Then, (2.64) becomes

\[
(\varphi_0 \circ \varphi)(d(Tx, Ty)) \leq (\varphi_0 \circ \varphi)(M(x, y)) - (\varphi_1 \circ \varphi)(M(x, y)), \tag{2.66}
\]

And, putting \( \varphi_0 = \varphi_0 \circ \varphi \) and \( \varphi_0 = \varphi_1 \circ \varphi \) and applying Theorem 2.2, we obtain the proof of Corollary 2.6 (it is easy to verify that \( \varphi_0 \in \Psi \) and \( \varphi_0 \in \Phi \)).
Corollary 2.7. Let \((X, d)\) be a Hausdorff and complete generalized metric space. Let \(T : X \to X\). Assume there exist positive Lebesgue integrable functions \(u\) and \(v\) on \(\mathbb{R}_+\) such that \(\int_0^\infty u(t)dt > 0\) and \(\int_0^\infty v(t)dt > 0\) for each \(\epsilon > 0\) and for all \(x, y \in X\), and
\[
\int_0^\epsilon d(Tx, Ty) u(t)dt \leq \int_0^\epsilon d(x, y) u(t)dt - \int_0^\epsilon v(t)dt,
\]
then \(T\) has a unique fixed point.

Proof. It follows by taking \(\varphi(t) = \psi(t) = t\) in Corollary 2.6.

\[\Box\]

Corollary 2.8. Let \((X, d)\) be a Hausdorff and complete generalized metric space. Let \(T : X \to X\). Assume there exist \(k \in [0, 1)\) and a positive Lebesgue integrable function \(u\) on \(\mathbb{R}_+\) such that \(\int_0^\epsilon u(t)dt > 0\) for each \(\epsilon > 0\) and for all \(x, y \in X\), and
\[
\int_0^\epsilon d(Tx, Ty) u(t)dt \leq k \int_0^\epsilon \max\{d(x, y), d(x, Tx), d(y, Ty)\} u(t)dt,
\]
then \(T\) has a unique fixed point.

Proof. It suffices to take \(\varphi(t) = (1 - k)u(t)\) in Corollary 2.7.

\[\Box\]

Finally, let us finish this paper by noticing the following remark.

Remark 2.9. (i) Theorem 2.2 extends Theorem 3.1 of Lakzian and Samet [9].
(ii) Corollary 2.3 extends the results of Branciari [1], Azam and Arshad [2], and Sarma et al. [13].
(iii) Corollary 2.8 extends Theorem 2 of Samet [11].
(iv) Several publications attempting to generalize fixed-point theorems in metric spaces to g.m.s are plagued by the use of some false properties given in [1] (see, e.g., [2–5]). This was observed by Das and Dey [7] who proved a fixed-point theorem without using the false properties. Subsequently, but independently, this was also observed by Samet [12] and Sarma et al. [13] who proved fixed-point theorems assuming that the generalized metric space is Hausdorff. Here, we give a rigorous proof of Theorem 2.2 by taking the same assumption.

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References


