Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 239695, 9 pages
doi:10.1155/2012/239695

Research Article

Some New Estimates for the Error of Simpson Integration Rule

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Received 9 September 2012; Accepted 10 October 2012

Academic Editor: Mohammad Mursaleen

We obtain some new estimates for the error of Simpson integration rule, which develop available results in the literature. Indeed, we introduce three main estimates for the residue of Simpson integration rule in $L^1[a,b]$ and $L^\infty[a,b]$ spaces where the compactness of the interval $[a,b]$ plays a crucial role.

1. Introduction

A general $(n+1)$-point-weighted quadrature formula is denoted by

$$\int_a^b w(x)f(x)dx = \sum_{k=0}^{n} w_k f(x_k) + R_{n+1}[f], \tag{1.1}$$

where $w(x)$ is a positive weight function on $[a,b]$, $\{x_k\}_{k=0}^n$ and $\{w_k\}_{k=0}^n$ are, respectively, nodes and weight coefficients, and $R_{n+1}[f]$ is the corresponding error [1].

Let $\mathbf{\Pi}_d$ be the set of algebraic polynomials of degree at most d. The quadrature formula (1.1) has degree of exactness d if for every $p \in \mathbf{\Pi}_d$ we have $R_{n+1}[p] = 0$. In addition, if $R_{n+1}[p] \neq 0$ for some $\mathbf{\Pi}_{d+1}$, formula (1.1) has precise degree of exactness d.

The convergence order of quadrature rule (1.1) depends on the smoothness of the function f as well as on its degree of exactness. It is well known that for given $n+1$ mutually different nodes $\{x_k\}_{k=0}^n$ we can always achieve a degree of exactness $d = n$ by interpolating at these nodes and integrating the interpolated polynomial instead of f. Namely, taking the node polynomia

$$\Psi_{n+1}(x) = \prod_{k=0}^{n}(x - x_k), \tag{1.2}$$

by integrating the Lagrange interpolation formula

$$f(x) = \sum_{k=0}^{n} f(x_k)L(x; x_k) + r_{n+1}(f; x), \tag{1.3}$$

where

$$L(x; x_k) = \frac{\Psi_{n+1}(x)}{\Psi'_{n+1}(x_k)(x - x_k)} \quad (k = 0, 1, \ldots, n), \tag{1.4}$$

we obtain (1.1), with

$$\begin{gathered} w_k = \frac{1}{\Psi'_{n+1}(x_k)} \int_a^b \frac{\Psi_{n+1}(x)w(x)}{x - x_k} dx \quad (k = 0, 1, \ldots, n), \\ R_{n+1}[f] = \int_a^b r_{n+1}(f; x)w(x)dx. \end{gathered} \tag{1.5}$$

Note that for each $f \in \mathbf{\Pi}_n$ we have $r_{n+1}(f; x) = 0$, and therefore $R_{n+1}[f] = 0$.

Quadrature formulae obtained in this way are known as interpolatory. Usually the simplest interpolatory quadrature formula of type (1.1) with predetermined nodes $\{x_k\}_{k=0}^{n} \in [a, b]$ is called a weighted Newton-Cotes formula. For $w(x) = 1$ and the equidistant nodes $\{x_k\}_{k=0}^{n} = \{a + kh\}_{k=0}^{n}$ with $h = (b - a)/n$, the classical Newton-Cotes formulas are derived. One of the important cases of the classical Newton-Cotes formulas is the well-known Simpson's rule:

$$\int_a^b f(t)dt = \frac{b - a}{6}\left(f(a) + 4f\left(\frac{a + b}{2}\right) + f(b)\right) + E(f). \tag{1.6}$$

In this direction, Simpson inequality [2–7] gives an error bound for the above quadrature rule. There are few known ways to estimate the residue value in (1.6). The main aim of this paper is to give three new estimations for $E(f)$ in $L^1[a, b]$ and $L^\infty[a, b]$ spaces.

Let $L^p[a, b]$ $(1 \le p < \infty)$ denote the space of p-power integrable functions on the interval $[a, b]$ with the standard norm

$$\|f\|_p = \left(\int_a^b |f(t)|^p dt\right)^{1/p}, \tag{1.7}$$

and $L^\infty[a, b]$ the space of all essentially bounded functions on $[a, b]$ with the norm

$$\|f\|_\infty = \operatorname*{ess\,sup}_{x \in [a,b]} |f(x)|. \tag{1.8}$$

If $f \in L^1[a,b]$ and $g \in L^\infty[a,b]$, then the following inequality is well known:

$$\left|\int_a^b f(x)g(x)dx\right| \le \|f\|_1 \|g\|_\infty. \tag{1.9}$$

Recently in [8], a main inequality has been introduced, which can estimate the error of Simpson quadrature rule too.

Theorem A. *Let $f : \mathbf{I} \to \mathbf{R}$, where $\mathbf{I}$ is an interval, be a differentiable function in the interior $\mathbf{I}^0$ of $\mathbf{I}$, and let $[a,b] \subset \mathbf{I}^0$. If α_0, β_0 are two real constants such that $\alpha_0 \le f'(t) \le \beta_0$ for all $t \in [a,b]$, then for any $\lambda \in [1/2, 1]$ and all $x \in [(a+(2\lambda-1)b)/2\lambda, (b+(2\lambda-1)a)/2\lambda] \subseteq [a,b]$ we have*

$$\begin{aligned}&\left|f(x) - \frac{1}{\lambda(b-a)}\int_a^b f(t)dt - \frac{f(b)-f(a)}{b-a}x + \frac{(2\lambda-1)a+b}{2\lambda(b-a)}f(b) - \frac{a+(2\lambda-1)b}{2\lambda(b-a)}f(a)\right| \\ &\quad \le \frac{\beta_0-\alpha_0}{4(b-a)}\frac{\lambda^2+(1-\lambda)^2}{\lambda}\left((x-a)^2+(b-x)^2\right).\end{aligned} \tag{1.10}$$

As is observed, replacing $x = (a+b)/2$ and $\lambda = 2/3$ in (1.10) gives an error bound for the Simpson rule as

$$\left|\int_a^b f(t)dt - \frac{b-a}{6}\left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right)\right| \le \frac{5}{72}(b-a)^2(\beta_0-\alpha_0). \tag{1.11}$$

To introduce three new error bounds for the Simpson quadrature rule in $L^1[a,b]$ and $L^\infty[a,b]$ spaces we first consider the following kernel on $[a,b]$:

$$K(t) = \begin{cases} t - \dfrac{5a+b}{6}, & t \in \left[a, \dfrac{a+b}{2}\right], \\ t - \dfrac{a+5b}{6}, & t \in \left(\dfrac{a+b}{2}, b\right]. \end{cases} \tag{1.12}$$

After some calculations, it can be directly concluded that

$$\int_a^b f'(t)K(t)dt = \frac{b-a}{6}\left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right) - \int_a^b f(t)dt, \tag{1.13}$$

$$\max_{t\in[a,b]}|K(t)| = \frac{1}{3}(b-a). \tag{1.14}$$

2. Main Results

Theorem 2.1. *Let* $f : \mathbf{I} \to \mathbf{R}$*, where* $\mathbf{I}$ *is an interval, be a function differentiable in the interior* $\mathbf{I}^0$ *of* $\mathbf{I}$*, and let* $[a,b] \subset \mathbf{I}^0$*. If* $\alpha(x) \le f'(x) \le \beta(x)$ *for any* $\alpha, \beta \in C[a,b]$ *and* $x \in [a,b]$*, then the following inequality holds:*

$$\begin{aligned}
m_1 &= \int_a^{(5a+b)/6}\left(t-\frac{5a+b}{6}\right)\beta(t)dt + \int_{(5a+b)/6}^{(a+b)/2}\left(t-\frac{5a+b}{6}\right)\alpha(t)dt \\
&\quad + \int_{(a+b)/2}^{(a+5b)/6}\left(t-\frac{a+5b}{6}\right)\beta(t)dt + \int_{(a+5b)/6}^{b}\left(t-\frac{a+5b}{6}\right)\alpha(t)dt \\
&\le \frac{b-a}{6}\left(f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right) - \int_a^b f(t)dt \le \\
M_1 &= \int_a^{(5a+b)/6}\left(t-\frac{5a+b}{6}\right)\alpha(t)dt + \int_{(5a+b)/6}^{(a+b)/2}\left(t-\frac{5a+b}{6}\right)\beta(t)dt \\
&\quad + \int_{(a+b)/2}^{(a+5b)/6}\left(t-\frac{a+5b}{6}\right)\alpha(t)dt + \int_{(a+5b)/6}^{b}\left(t-\frac{a+5b}{6}\right)\beta(t)dt.
\end{aligned} \tag{2.1}$$

Proof. By referring to the kernel (1.12) and identity (1.13) we first have

$$\begin{aligned}
&\int_a^b K(t)\left(f'(t)-\frac{\alpha(t)+\beta(t)}{2}\right)dt \\
&= \frac{b-a}{6}\left(f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right) - \int_a^b f(t)dt - \frac{1}{2}\left(\int_a^b K(t)(\alpha(t)+\beta(t))dt\right) \\
&= \frac{b-a}{6}\left(f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right) - \int_a^b f(t)dt \\
&\quad - \frac{1}{2}\left(\int_a^{(a+b)/2}\left(t-\frac{5a+b}{6}\right)(\alpha(t)+\beta(t))dt + \int_{(a+b)/2}^{b}\left(t-\frac{a+5b}{6}\right)(\alpha(t)+\beta(t))dt\right).
\end{aligned} \tag{2.2}$$

On the other hand, the given assumption $\alpha(t) \le f'(t) \le \beta(t)$ results in

$$\left|f'(t)-\frac{\alpha(t)+\beta(t)}{2}\right| \le \frac{\beta(t)-\alpha(t)}{2}. \tag{2.3}$$

Therefore, one can conclude from (2.2) and (2.3) that

$$\begin{aligned}
&\left|\frac{b-a}{6}\left(f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right)-\int_a^b f(t)dt\right.\\
&\qquad\left.-\frac{1}{2}\left(\int_a^{(a+b)/2}\left(t-\frac{5a+b}{6}\right)(\alpha(t)+\beta(t))dt+\int_{(a+b)/2}^b\left(t-\frac{a+5b}{6}\right)(\alpha(t)+\beta(t))dt\right)\right|\\
&\quad=\left|\int_a^b K(t)\left(f'(t)-\frac{\alpha(t)+\beta(t)}{2}\right)dt\right|\le\int_a^b|K(t)|\ \frac{\beta(t)-\alpha(t)}{2}dt\\
&\quad=\frac{1}{2}\left(\int_a^{(a+b)/2}\left|\ t-\frac{5a+b}{6}\ \right|(\beta(t)-\alpha(t))dt+\int_{(a+b)/2}^b\left|\ t-\frac{a+5b}{6}\right|(\beta(t)-\alpha(t))dt\right).
\end{aligned}\tag{2.4}$$

After rearranging (2.4) we obtain

$$\begin{aligned}
m_1&=\int_a^{(a+b)/2}\left(\left(t-\frac{5a+b}{6}-\left|\ t-\frac{5a+b}{6}\ \right|\right)\frac{\beta(t)}{2}+\left(t-\frac{5a+b}{6}+\left|\ t-\frac{5a+b}{6}\ \right|\right)\frac{\alpha(t)}{2}\right)dt\\
&\quad+\int_{(a+b)/2}^b\left(\left(t-\frac{a+5b}{6}-\left|\ t-\frac{a+5b}{6}\ \right|\right)\frac{\beta(t)}{2}+\left(t-\frac{a+5b}{6}+\left|\ t-\frac{a+5b}{6}\ \right|\right)\frac{\alpha(t)}{2}\right)dt\\
&=\int_a^{(5a+b)/6}\left(x-\frac{5a+b}{6}\right)\beta(x)dx+\int_{(5a+b)/6}^{(a+b)/2}\left(x-\frac{5a+b}{6}\right)\alpha(x)dx\\
&\quad+\int_{(a+b)/2}^{(a+5b)/6}\left(x-\frac{a+5b}{6}\right)\beta(x)dx+\int_{(a+5b)/6}^b\left(x-\frac{a+5b}{6}\right)\alpha(x)dx,\\
M_1&=\int_a^{(a+b)/2}\left(\left(t-\frac{5a+b}{6}-\left|t-\frac{5a+b}{6}\right|\right)\frac{\alpha(t)}{2}+\left(t-\frac{5a+b}{6}+\left|t-\frac{5a+b}{6}\right|\right)\frac{\beta(t)}{2}\right)dt\\
&\quad+\int_{(a+b)/2}^b\left(\left(t-\frac{a+5b}{6}-\left|t-\frac{a+5b}{6}\right|\right)\frac{\alpha(t)}{2}+\left(t-\frac{a+5b}{6}+\left|t-\frac{a+5b}{6}\right|\right)\frac{\beta(t)}{2}\right)dt\\
&=\int_a^{(5a+b)/6}\left(x-\frac{5a+b}{6}\right)\alpha(x)dx+\int_{(5a+b)/6}^{(a+b)/2}\left(x-\frac{5a+b}{6}\right)\beta(x)dx\\
&\quad+\int_{(a+b)/2}^{(a+5b)/6}\left(x-\frac{a+5b}{6}\right)\alpha(x)dx+\int_{(a+5b)/6}^b\left(x-\frac{a+5b}{6}\right)\beta(x)dx.
\end{aligned}\tag{2.5}$$

□

The advantage of Theorem 2.1 is that necessary computations in bounds m_1 and M_1 are just in terms of the preassigned functions $\alpha(t)$, $\beta(t)$ (not f').

Special Case 1

Substituting $\alpha(x) = \alpha_1 x + \alpha_0 \neq 0$ and $\beta(x) = \beta_1 x + \beta_0 \neq 0$ in (2.1) gives

$$\left|\int_a^b f(t)dt - \frac{b-a}{6}\left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right)\right| \leq \frac{5(b-a)^2}{144}\left((\beta_1 - \alpha_1)(a+b) + 2(\beta_0 - \alpha_0)\right). \tag{2.6}$$

In particular, replacing $\alpha_1 = \beta_1 = 0$ in above inequality leads to one of the results of [9] as

$$\left|\int_a^b f(t)dt - \frac{b-a}{6}\left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right)\right| \leq \frac{5}{72}(b-a)^2(\beta_0 - \alpha_0). \tag{2.7}$$

Remark 2.2. Although $\alpha(x) \leq f'(x) \leq \beta(x)$ is a straightforward condition in Theorem 2.1, however, sometimes one might not be able to easily obtain both bounds of $\alpha(x)$ and $\beta(x)$ for f'. In this case, we can make use of two analogue theorems. The first one would be helpful when f' is unbounded from above and the second one would be helpful when f' is unbounded from below.

Theorem 2.3. *Let $f : \mathbf{I} \to \mathbf{R}$, where $\mathbf{I}$ is an interval, be a function differentiable in the interior $\mathbf{I}^0$ of $\mathbf{I}$, and let $[a,b] \subset \mathbf{I}^0$. If $\alpha(x) \leq f'(x)$ for any $\alpha \in C[a,b]$ and $x \in [a,b]$ then*

$$\begin{aligned}
&\int_a^{(a+b)/2}\left(t - \frac{5a+b}{6}\right)\alpha(t)dt + \int_{(a+b)/2}^b\left(t - \frac{a+5b}{6}\right)\alpha(t)dt - \frac{b-a}{3}\left(f(b) - f(a) - \int_a^b \alpha(t)dt\right) \\
&\quad \leq \frac{b-a}{6}\left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right) - \int_a^b f(t)\, dt \\
&\quad \leq \int_a^{(a+b)/2}\left(t - \frac{5a+b}{6}\right)\alpha(t)dt + \int_{(a+b)/2}^b\left(t - \frac{a+5b}{6}\right)\alpha(t)dt \\
&\qquad + \frac{b-a}{3}\left(f(b) - f(a) - \int_a^b \alpha(t)dt\right).
\end{aligned} \tag{2.8}$$

Proof. Since

$$\begin{aligned}
\int_a^b K(t)(f'(t) - \alpha(t))dt &= \frac{b-a}{6}\left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right) - \int_a^b f(t)dt - \left(\int_a^b K(t)\alpha(t)dt\right) \\
&= \frac{b-a}{6}\left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right) - \int_a^b f(t)dt \\
&\quad - \left(\int_a^{(a+b)/2}\left(t - \frac{5a+b}{6}\right)\alpha(t)dt + \int_{(a+b)/2}^b\left(t - \frac{a+5b}{6}\right)\alpha(t)dt\right),
\end{aligned} \tag{2.9}$$

so we have

$$\begin{aligned}
&\left|\frac{b-a}{6}\left(f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right)-\int_a^b f(t)dt\right.\\
&\quad\left.-\left(\int_a^{(a+b)/2}\left(t-\frac{5a+b}{6}\right)\alpha(t)dt+\int_{(a+b)/2}^b\left(t-\frac{a+5b}{6}\right)\alpha(t)dt\right)\right|\\
&\quad=\left|\int_a^b K(t)\left(f'(t)-\alpha(t)\right)dt\right|\le\int_a^b|K(t)|\left(f'(t)-\alpha(t)\right)dt\\
&\quad\le\max_{t\in[a,b]}|K(t)|\int_a^b\left(f'(t)-\alpha(t)\right)dt=\frac{b-a}{3}\left(f(b)-f(a)-\int_a^b\alpha(t)dt\right).
\end{aligned}\tag{2.10}$$

After rearranging (2.10), the main inequality (2.8) will be derived. □

Special Case 2

If $\alpha(x)=\alpha_1x+\alpha_0\neq 0$, then (2.8) becomes

$$\left|\int_a^b f(t)dt-\frac{b-a}{6}\left(f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right)\right|\le\frac{(b-a)^2}{3}\left(\frac{f(b)-f(a)}{b-a}-\left(\alpha_0+\frac{a+b}{2}\alpha_1\right)\right)\tag{2.11}$$

if and only if $\alpha_1x+\alpha_0\le f'(x)$ for all $x\in[a,b]$. In particular, replacing $\alpha_1=0$ in above inequality leads to [10, Theorem 1, relation (4)] as follows:

$$\left|\int_a^b f(t)dt-\frac{b-a}{6}\left(f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right)\right|\le\frac{(b-a)^2}{3}\left(\frac{f(b)-f(a)}{b-a}-\alpha_0\right).\tag{2.12}$$

Theorem 2.4. *Let $f:\mathbf{I}\to\mathbf{R}$, where $\mathbf{I}$ is an interval, be a function differentiable in the interior $\mathbf{I}^0$ of $\mathbf{I}$, and let $[a,b]\subset\mathbf{I}^0$. If $f'(x)\le\beta(x)$ for any $\beta\in C[a,b]$ and $x\in[a,b]$ then*

$$\begin{aligned}
&\int_a^{(a+b)/2}\left(t-\frac{5a+b}{6}\right)\beta(t)dt+\int_{(a+b)/2}^b\left(t-\frac{a+5b}{6}\right)\beta(t)dt-\frac{b-a}{3}\left(\int_a^b\beta(t)dt-f(b)+f(a)\right)\\
&\le\frac{b-a}{6}\left(f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right)-\int_a^b f(t)dt\\
&\le\int_a^{(a+b)/2}\left(t-\frac{5a+b}{6}\right)\beta(t)dt+\int_{(a+b)/2}^b\left(t-\frac{a+5b}{6}\right)\beta(t)dt+\frac{b-a}{3}\left(\int_a^b\beta(t)dt-f(b)+f(a)\right).
\end{aligned}\tag{2.13}$$

Proof. Since

$$\begin{aligned}
&\int_a^b K(t)\left(f'(t)-\beta(t)\right)dt \\
&\quad= \frac{b-a}{6}\left(f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right)-\int_a^b f(t)dt-\left(\int_a^b K(t)\beta(t)dt\right) \\
&\quad= \frac{b-a}{6}\left(f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right)-\int_a^b f(t)dt \\
&\qquad - \left(\int_a^{(a+b)/2}\left(t-\frac{5a+b}{6}\right)\beta(t)dt+\int_{(a+b)/2}^b\left(t-\frac{a+5b}{6}\right)\beta(t)dt\right),
\end{aligned} \tag{2.14}$$

so we have

$$\begin{aligned}
&\left|\frac{b-a}{6}\left(f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right)-\int_a^b f(t)dt\right. \\
&\qquad \left. -\left(\int_a^{(a+b)/2}\int_a\left(t-\frac{5a+b}{6}\right)\beta(t)dt+\int_{(a+b)/2}^b\left(t-\frac{a+5b}{6}\right)\beta(t)dt\right)\right| \\
&\quad= \left|\int_a^b K(t)\left(f'(t)-\beta(t)\right)dt\right| \le \int_a^b |K(t)|\left(\beta(t)-f'(t)\right)dt \\
&\quad\le \max_{t\in[a,b]}|K(t)|\int_a^b\left(\beta(t)-f'(t)\right)dt = \frac{b-a}{3}\left(\int_a^b \beta(t)dt-f(b)+f(a)\right).
\end{aligned} \tag{2.15}$$

After rearranging (2.15), the main inequality (2.13) will be derived. □

Special Case 3

If $\beta(x)=\beta_1 x+\beta_0\neq 0$ in (2.13), then

$$\left|\int_a^b f(t)\ dt-\frac{b-a}{6}\left(f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right)\right| \le \frac{(b-a)^2}{3}\left(\beta_0+\frac{a+b}{2}\ \beta_1-\frac{f(b)-f(a)}{b-a}\right) \tag{2.16}$$

if and only if $f'(x)\le\beta_1 x+\beta_0$, for all $x\in[a,b]$. In particular, replacing $\beta_1=0$ in above inequality leads to [10, Theorem 1, relation (5)] as follows:

$$\left|\int_a^b f(t)dt-\frac{b-a}{6}\left(f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right)\right| \le \frac{(b-a)^2}{3}\left(\beta_0-\frac{f(b)-f(a)}{b-a}\right). \tag{2.17}$$

Acknowledgment

The second and third authors gratefully acknowledge the support provided by the Deanship of Scientific Research (DSR), King Abdulaziz University during this research.

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