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*Research Article*

# Robust Exponential Stability for LPD Discrete-Time System with Interval Time-Varying Delay

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This paper investigates the problem of robust exponential stability for uncertain linear-parameter dependent (LPD) discrete-time system with delay. The delay is of an interval type, which means that both lower and upper bounds for the time-varying delay are available. The uncertainty under consideration is norm-bounded uncertainty. Based on combination of the linear matrix inequality (LMI) technique and the use of suitable Lyapunov-Krasovskii functional, new sufficient conditions for the robust exponential stability are obtained in terms of LMI. Numerical examples are given to demonstrate the effectiveness and less conservativeness of the proposed methods.

## 1. Introduction

Over the past decades, the problem of stability analysis of delay discrete-time systems has been widely investigated by many researchers. Because the existence of time delay is frequent, a source of oscillation instability performances degradation of systems. Stability criteria for discrete-time systems with time delay is generally divided into two classes: delay-independent ones and delay-dependent ones. Delay-independent stability criteria tend to be more conservative, especially for small-size delay; such criteria do not give any information on the size of the delay. On the other hand, delay-dependent stability criteria is concerned with the size of the delay and usually provide a maximal delay size. Moreover, robust stability of linear continuous-time and discrete-time systems subject to time-invariant parametric uncertainty has received considerable attention. An important class of linear time-invariant parametric uncertain system is linear parameter-dependent (LPD) system in which the uncertain state matrices are in the polytope consisting of all convex combination of known matrices. To address this problem, several results have been obtained in terms of sufficient (or necessary and sufficient) conditions; see [1–15] and references cited therein. Most of these conditions have been obtained via the Lyapunov theory approaches in which parameter dependent Lyapunov functions have been employed. These conditions are always expressed in terms of linear matrix inequalities (LMIs) which can be solved numerically by using available tools such as LMI toolbox in MATLAB. The results have been obtained for robust stability for LPD systems in which time delay occur in state variable such as [6, 11, 14] present sufficient conditions for robust stability of LPD continuous-time system with delays. However, a few results have been obtained for robust stability for LPD discrete-time systems with delay.

In this paper, we deal with the problem of robust exponential stability for uncertain LPD discrete-time system with interval time-varying delay. Combined with the linear matrix inequality technique and the use of suitable Lyapunov-Krasovskii functional, new sufficient conditions for the robust exponential stability are obtained in terms of LMI. Finally, numerical examples have demonstrated the effectiveness of the criteria.

## 2. Problem Formulation and Preliminaries

We introduce some notations and definitions that will be used throughout the paper. $\mathbb{Z}^+$ denotes the set of non negative integer numbers; $\mathbb{R}^n$ denotes the $n$-dimensional space with the vector norm $\|\cdot\|$; $\|x\|$ denotes the Euclidean vector norm of $x \in \mathbb{R}^n$; that is, $\|x\|^2 = x^T x$; $M^{n\times r}$ denotes the space of all matrices of $(n \times r)$-dimensions; $A^T$ denotes transpose of the Matrix $A$; $A$ is symmetric if $A = A^T$; $I$ denotes the identity matrix; $\lambda(A)$ denotes the set of all eigenvalues of $A$; $\lambda_{\max}(A) = \max\{\mathrm{Re}\lambda : \lambda \in \lambda(A)\}$; Matrix $A$ is called semi positive definite $(A \geq 0)$ if $x^T A x \geq 0$, for all $x \in \mathbb{R}^n$; $A$ is positive definite $(A > 0)$ if $x^T A x > 0$ for all $x \neq 0$; Matrix $B$ is called semi-negative definite $(B \leq 0)$ if $x^T B x \leq 0$, for all $x \in \mathbb{R}^n$; $B$ is negative definite $(B < 0)$ if $x^T B x < 0$ for all $x \neq 0$; $A > B$ means $A - B > 0$; $A \geq B$ means $A - B \geq 0$; $*$ represents the elements below the main diagonal of a symmetric matrix.

Consider the following uncertain LPD discrete-time system with interval time-varying delay in the state

$$
\begin{gathered}
x(k+1) = [A(\alpha) + \Delta A(k)]x(k) + [B(\alpha) + \Delta B(k)]x(k - h(k)), \\
x(s) = \phi(s), \quad s = -h_2, \ldots, -1, 0,
\end{gathered}
\tag{2.1}
$$

where $k \in \mathbb{Z}^+$, $x(k) \in \mathbb{R}^n$ is the system state and $\phi(s)$ is a initial value at $s$. $A(\alpha), B(\alpha) \in M^{n\times n}$ are uncertain matrices belonging to the polytope of the form

$$
\begin{gathered}
A(\alpha) = \sum_{i=1}^{N} \alpha_i A_i, \qquad B(\alpha) = \sum_{i=1}^{N} \alpha_i B_i, \\
\sum_{i=1}^{N} \alpha_i = 1, \quad \alpha_i \geq 0, \quad A_i, B_i \in M^{n\times n}, \quad i = 1, \ldots, N.
\end{gathered}
\tag{2.2}
$$

$\Delta A(k)$ and $\Delta B(k)$ are unknown matrices representing time-varying parameter uncertainties, we assumed to be of the form

$$\Delta A(k) = K(\alpha)\Delta(k)A_1(\alpha), \qquad \Delta B(k) = K(\alpha)\Delta(k)B_1(\alpha),$$
$$A_1(\alpha) = \sum_{i=1}^{N}\alpha_i A_i^1, \quad B_1(\alpha) = \sum_{i=1}^{N}\alpha_i B_i^1, \quad K(\alpha) = \sum_{i=1}^{N}\alpha_i K_i, \tag{2.3}$$
$$\sum_{i=1}^{N}\alpha_i = 1, \quad \alpha_i \geq 0, \quad A_i^1, B_i^1 \in M^{n\times n}, \quad i = 1,\ldots,N.$$

The class of parametric uncertainties $\Delta(k)$, which satisfies

$$\Delta(k) = F(k)[I - JF(k)]^{-1}, \tag{2.4}$$

is said to be admissible where $J$ is a known matrix satisfying

$$I - JJ^T > 0, \tag{2.5}$$

and $F(k)$ is uncertain matrix satisfying

$$F(k)^T F(k) \leq I. \tag{2.6}$$

In addition, we assume that the time-varying delay $h(k)$ is upper and lower bounded. It satisfies the following assumption of the form

$$h_1 \leq h(k) \leq h_2, \tag{2.7}$$

where $h_1$ and $h_2$ are known positive integers.

*Definition 2.1.* The uncertain LPD discrete-time-delayed system in (2.1) is said to be robustly exponentially stable if there exist constant scalars $0 < a < 1$ and $b > 0$ such that

$$\|x(k)\|^2 \leq ba^k \sup_{-h_2\leq l\leq 0} \|\phi(l)\|^2, \tag{2.8}$$

for all admissible uncertainties.

**Lemma 2.2** (see [5] (Schur complement lemma)). *Given constant matrices $X, Y, Z$ of appropriate dimensions with $Y > 0$. Then $X + Z^T Y^{-1} Z < 0$ if and only if*

$$\begin{pmatrix} X & Z^T \\ Z & -Y \end{pmatrix} < 0 \quad \text{or} \quad \begin{pmatrix} -Y & Z \\ Z^T & X \end{pmatrix} < 0. \tag{2.9}$$

**Lemma 2.3** (see [2]). *Given constant matrices $M_1, M_2$, and $M_3$ of appropriate dimensions with $M_1 = M_1^T$. Then,*

$$M_1 + M_2\Delta(k)M_3 + M_3^T\Delta(k)^T M_2^T < 0, \tag{2.10}$$

*where $\Delta(k) = F(k)[I - JF(k)]^{-1}$, $F(k)^T F(k) \le I$, for all $k \in \mathbb{Z}^+$ if and only if*

$$M_1 + \left[\epsilon^{-1}M_3^T \;\; \epsilon M_2\right]\begin{bmatrix} I & -J \\ -J^T & I \end{bmatrix}^{-1}\left[\epsilon^{-1}M_3^T \;\; \epsilon M_2\right]^T < 0, \tag{2.11}$$

*for some scalar $\epsilon > 0$.*

## 3. Main Results

In this section, we present our main results on the robust exponential stability criteria for uncertain LPD discrete-time system with interval time-varying delays. We introduce the following notation for later use:

$$\widehat{A}(\alpha) = A(\alpha) + \Delta A(k), \qquad \widehat{B}(\alpha) = B(\alpha) + \Delta B(k), \qquad \widehat{h} = h_2 - h_1 + 1. \tag{3.1}$$

**Lemma 3.1.** *For any $\widehat{A}(\alpha), \widehat{B}(\alpha), \widehat{h}$ in* (3.1)*, $P(\alpha)$ and $Q(\alpha)$ given by*

$$P(\alpha) = \sum_{i=1}^{N}\alpha_i P_i, \qquad Q(\alpha) = \sum_{i=1}^{N}\alpha_i Q_i, \quad \sum_{i=1}^{N}\alpha_i = 1,\ \alpha_i \ge 0,\ i = 1,\dots,N, \tag{3.2}$$

*are parameter-dependent positive definite Lyapunov matrices such that*

$$\begin{bmatrix} \widehat{A}^T(\alpha)P(\alpha)\widehat{A}(\alpha) - P(\alpha) + \widehat{h}Q(\alpha) & \widehat{A}^T(\alpha)P(\alpha)\widehat{B}(\alpha) \\ \widehat{B}^T(\alpha)P(\alpha)\widehat{A}(\alpha) & \widehat{B}^T(\alpha)P(\alpha)\widehat{B}(\alpha) - Q(\alpha) \end{bmatrix} < 0, \tag{3.3}$$

*if and only if*

$$\begin{bmatrix} -P(\alpha) + \widehat{h}Q(\alpha) & 0 & A(\alpha)^T P(\alpha) & \epsilon^{-1}A_1(\alpha)^T & 0 \\ * & -Q(\alpha) & B(\alpha)^T P(\alpha) & \epsilon^{-1}B_1(\alpha)^T & 0 \\ * & * & -P(\alpha) & 0 & \epsilon P(\alpha)K(\alpha) \\ * & * & * & -I & J \\ * & * & * & * & -I \end{bmatrix} < 0. \tag{3.4}$$

*Proof.* Consider

$$
\begin{aligned}
&\begin{bmatrix} \widehat{A}^T(\alpha)P(\alpha)\widehat{A}(\alpha) - P(\alpha) + \widehat{h}Q(\alpha) & \widehat{A}^T(\alpha)P(\alpha)\widehat{B}(\alpha) \\ \widehat{B}^T(\alpha)P(\alpha)\widehat{A}(\alpha) & \widehat{B}^T(\alpha)P(\alpha)\widehat{B}(\alpha) - Q(\alpha) \end{bmatrix} \\
&\quad = \begin{bmatrix} -P(\alpha) + \widehat{h}Q(\alpha) & 0 \\ 0 & -Q(\alpha) \end{bmatrix} + \begin{bmatrix} \widehat{A}^T(\alpha)P(\alpha)\widehat{A}(\alpha) & \widehat{A}^T(\alpha)P(\alpha)\widehat{B}(\alpha) \\ \widehat{B}^T(\alpha)P(\alpha)\widehat{A}(\alpha) & \widehat{B}^T(\alpha)P(\alpha)\widehat{B}(\alpha) \end{bmatrix} \\
&\quad = \begin{bmatrix} -P(\alpha) + \widehat{h}Q(\alpha) & 0 \\ 0 & -Q(\alpha) \end{bmatrix} + \begin{bmatrix} \widehat{A}^T(\alpha) \\ \widehat{B}^T(\alpha) \end{bmatrix} P(\alpha) \begin{bmatrix} \widehat{A}(\alpha) & \widehat{B}(\alpha) \end{bmatrix}.
\end{aligned} \tag{3.5}
$$

We assume that

$$
\begin{bmatrix} -P(\alpha) + \widehat{h}Q(\alpha) & 0 \\ 0 & -Q(\alpha) \end{bmatrix} + \begin{bmatrix} \overline{A}^T(\alpha) \\ \overline{B}^T(\alpha) \end{bmatrix} P(\alpha) \begin{bmatrix} \overline{A}(\alpha) & \overline{B}(\alpha) \end{bmatrix} < 0. \tag{3.6}
$$

Using Lemma 2.2, we obtain

$$
\begin{bmatrix} -P(\alpha) + \widehat{h}Q(\alpha) & 0 & A(\alpha)^T + [K(\alpha)\Delta(k)A_1(\alpha)]^T \\ * & -Q(\alpha) & B(\alpha)^T + [K(\alpha)\Delta(k)B_1(\alpha)]^T \\ * & * & -P(\alpha)^{-1} \end{bmatrix} < 0. \tag{3.7}
$$

We rewrite the latter inequality as

$$
\begin{aligned}
&\begin{bmatrix} -P(\alpha) + \widehat{h}Q(\alpha) & 0 & A(\alpha)^T \\ * & -Q(\alpha) & B(\alpha)^T \\ * & * & -P(\alpha)^{-1} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ K(\alpha) \end{bmatrix} \Delta(k) \begin{bmatrix} A_1(\alpha) & B_1(\alpha) & 0 \end{bmatrix} \\
&\quad + \begin{bmatrix} A_1(\alpha) & B_1(\alpha) & 0 \end{bmatrix}^T \Delta(k)^T \begin{bmatrix} 0 \\ 0 \\ K(\alpha) \end{bmatrix}^T < 0.
\end{aligned} \tag{3.8}
$$

Using Lemma 2.3, inequality (3.8) holds if and only if there exists $\epsilon > 0$ such that

$$
\begin{aligned}
&\begin{bmatrix} -P(\alpha) + \widehat{h}Q(\alpha) & 0 & A(\alpha)^T \\ * & -Q(\alpha) & B(\alpha)^T \\ * & * & -P(\alpha)^{-1} \end{bmatrix} \\
&\quad + \begin{bmatrix} \epsilon^{-1}A_1(\alpha)^T & 0 \\ \epsilon^{-1}B_1(\alpha)^T & 0 \\ 0 & \epsilon K(\alpha) \end{bmatrix} \begin{bmatrix} I & -J \\ -J & I \end{bmatrix}^{-1} \begin{bmatrix} \epsilon^{-1}A_1(\alpha)^T & 0 \\ \epsilon^{-1}B_1(\alpha)^T & 0 \\ 0 & \epsilon K(\alpha) \end{bmatrix}^T < 0.
\end{aligned} \tag{3.9}
$$

If we apply to (3.9), then we obtain

$$\begin{bmatrix} -P(\alpha)+\hat{h}Q(\alpha) & 0 & A(\alpha)^T & \epsilon^{-1}A_1(\alpha)^T & 0 \\ * & -Q(\alpha) & B(\alpha)^T & \epsilon^{-1}B_1(\alpha)^T & 0 \\ * & * & -P(\alpha)^{-1} & 0 & \epsilon K(\alpha) \\ * & * & * & -I & J \\ * & * & * & * & -I \end{bmatrix} < 0. \tag{3.10}$$

Premultiplying (3.10) by $\operatorname{diag}\{I, I, P(\alpha), I, I\}$ and postmultiplying by $\operatorname{diag}\{I, I, P(\alpha), I, I\}$, we get that (3.4) and the lemma is proved. □

**Lemma 3.2.** *If there exist positive definite symmetric matrices $P_i$, $Q_i$, $i = 1, 2, \ldots, N$, and positive real numbers $\epsilon, \zeta$ such that*

$$\begin{bmatrix} -P_i+\hat{h}Q_i & 0 & A_i^T P_i & \epsilon^{-1}A_i^{1^T} & 0 \\ * & -Q_i & B_i^T P_i & \epsilon^{-1}B_i^{1^T} & 0 \\ * & * & -P_i & 0 & \epsilon P_i K_i \\ * & * & * & -I & J \\ * & * & * & * & -I \end{bmatrix} < -\zeta I, \quad i = 1, 2, \ldots, N,$$

$$\begin{bmatrix} -P_i+\hat{h}Q_i & 0 & A_i^T P_j & \epsilon^{-1}A_i^{1^T} & 0 \\ * & -Q_i & B_i^T P_j & \epsilon^{-1}B_i^{1^T} & 0 \\ * & * & -P_i & 0 & \epsilon P_i K_j \\ * & * & * & -I & J \\ * & * & * & * & -I \end{bmatrix} + \begin{bmatrix} -P_j+\hat{h}Q_j & 0 & A_j^T P_i & \epsilon^{-1}A_j^{1^T} & 0 \\ * & -Q_j & B_j^T P_i & \epsilon^{-1}B_j^{1^T} & 0 \\ * & * & -P_j & 0 & \epsilon P_j K_i \\ * & * & * & -I & J \\ * & * & * & * & -I \end{bmatrix} < \frac{2\zeta I}{N-1},$$
$$i = 1, \ldots, N-1, \quad j = i+1, \ldots, N, \tag{3.11}$$

*then, for any $A(\alpha), A_1(\alpha), B(\alpha), B_1(\alpha), K(\alpha), \hat{h}$ in* (3.1)*, $P(\alpha)$ and $Q(\alpha)$ are parameter-dependent positive definite Lyapunov matrices in Lemma 3.1 such that* (3.4) *holds.*

*Proof.* Consider

$$\begin{aligned} &\begin{bmatrix} -P(\alpha)+\hat{h}Q(\alpha) & 0 & A(\alpha)^T P(\alpha) & \epsilon^{-1}A_1(\alpha)^T & 0 \\ * & -Q(\alpha) & B(\alpha)^T P(\alpha) & \epsilon^{-1}B_1(\alpha)^T & 0 \\ * & * & -P(\alpha) & 0 & \epsilon P(\alpha)K(\alpha) \\ * & * & * & -I & J \\ * & * & * & * & -I \end{bmatrix} \\ &= \sum_{i=1}^{N}\sum_{j=1}^{N}\alpha_i\alpha_j \begin{bmatrix} -P_i+\hat{h}Q_i & 0 & A_i^T P_j & \epsilon^{-1}A_i^{1^T} & 0 \\ * & -Q_i & B_i^T P_j & \epsilon^{-1}B_i^{1^T} & 0 \\ * & * & -P_i & 0 & \epsilon P_i K_j \\ * & * & * & -I & J \\ * & * & * & * & -I \end{bmatrix}. \end{aligned} \tag{3.12}$$

Using the fact that $\sum_{i=1}^{N}\alpha_i = 1$, we obtain the following identities:

$$\begin{aligned}
&\sum_{i=1}^{N}\sum_{j=1}^{N}\alpha_i\alpha_j A_iB_j = \sum_{i=1}^{N}\alpha^2 A_iB_i + \sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\alpha_i\alpha_j\left[A_iB_j + A_jB_i\right],\\
&(N-1)\sum_{i=1}^{N}\alpha_i^2\zeta - 2\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\alpha_i\alpha_j\zeta = \sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\left[\alpha_i-\alpha_j\right]^2\zeta \geq 0.
\end{aligned} \tag{3.13}$$

Then, it follows from (3.11), (3.12), and (3.13) that (3.4) holds. The proof of the lemma is complete. □

**Theorem 3.3.** *The system* (2.1) *is robustly exponentially stable if the LMI conditions* (3.11) *are feasible.*

*Proof.* Consider the following Lyapunov-Krasovskii function for system (2.1) of the form

$$V(x(k)) = V_1(x(k)) + V_2(x(k)) + V_3(x(k)), \tag{3.14}$$

where

$$\begin{aligned}
V_1(x(k)) &= x^T(k)P(\alpha)x(k), \qquad V_2(x(k)) = \sum_{i=k-h(k)}^{k-1} x^T(i)Q(\alpha)x(i),\\
V_3(x(k)) &= \sum_{j=-h_2+2}^{-h_1+1}\sum_{l=k+j-1}^{k-1} x^T(l)Q(\alpha)x(l).
\end{aligned} \tag{3.15}$$

A Lyapunov-Krasovskii difference for the system (2.1) is defined as

$$\Delta V(x(k)) = \Delta V_1(x(k)) + \Delta V_2(x(k)) + \Delta V_3(x(k)). \tag{3.16}$$

Taking the difference of $V_1(x(k))$ and $V_2(x(k))$, the increments of $V_1(x(k))$ and $V_2(x(k))$ are

$$\begin{aligned}
\Delta V_1(x(k)) &= V_1(x(k+1)) - V_1(x(k))\\
&= x^T(k+1)P(\alpha)x(k+1) - x^T(k)P(\alpha)x(k)\\
&= x^T(k)\widehat{A}^T(\alpha)P(\alpha)\widehat{A}(\alpha)x(k) + x^T(k-h(k))\widehat{B}^T(\alpha)P(\alpha)\widehat{A}(\alpha)x(k)\\
&\quad + x^T(k-h(k))\widehat{B}^T(\alpha)P(\alpha)\widehat{B}(\alpha)x(k-h(k))\\
&\quad + x^T(k)\widehat{A}^T(\alpha)P(\alpha)\widehat{B}(\alpha)x(k-h(k)) - x^T(k)P(\alpha)x(k),
\end{aligned} \tag{3.17}$$

$$\begin{aligned}
\Delta V_2(x(k)) &= V_2(x(k+1)) - V_2(x(k)) \\
&= \sum_{i=k+1-h(k+1)}^{k} x^T(i)Q(\alpha)x(i) - \sum_{i=k-h(k)}^{k-1} x^T(i)Q(\alpha)x(i) \\
&= x^T(k)Q(\alpha)x(k) - x^T(k-h(k))Q(\alpha)x(k-h(k)) \\
&\quad + \sum_{i=k+1-h(k+1)}^{k-h_1} x^T(i)Q(\alpha)x(i) - \sum_{i=k+1-h(k+1)}^{k-1} x^T(i)Q(\alpha)x(i) \\
&\quad + \sum_{i=k+1-h_1}^{k-1} x^T(i)Q(\alpha)x(i).
\end{aligned} \tag{3.18}$$

Form $h(k) \geq h_1$, the two last terms of the right-hand side of the latter equality yield

$$\sum_{i=k+1-h_1}^{k-1} x^T(i)Q(\alpha)x(i) - \sum_{i=k+1-h(k+1)}^{k-1} x^T(i)Q(\alpha)x(i) \leq 0. \tag{3.19}$$

Thus, we obtain

$$\begin{aligned}
\Delta V_2(x(k)) \leq{}& x^T(k)Q(\alpha)x(k) - x^T(k-h(k))Q(\alpha)x(k-h(k)) \\
&+ \sum_{i=k+1-h(k+1)}^{k-h_1} x^T(i)Q(\alpha)x(i).
\end{aligned} \tag{3.20}$$

The increment of $V_3(x(k))$ is easily computed as

$$\begin{aligned}
\Delta V_3(x(k)) &= V_3(x(k+1)) - V_3(x(k)) \\
&= \sum_{j=-h_2+2}^{-h_1+1} \left[ x^T(k)Q(\alpha)x(k) + \sum_{l=k+j-1}^{k-1} x^T(l)Q(\alpha)x(l) - \sum_{l=k+j-1}^{k-1} x^T(l)Q(\alpha)x(l) \right] \\
&= (h_2 - h_1)x^T(k)Q(\alpha)x(k) - \sum_{i=k+1-h_2}^{k-h_1} x^T(i)Q(\alpha)x(i).
\end{aligned} \tag{3.21}$$

It is easy to see that

$$\begin{aligned}
\Delta V_2(x(k)) + \Delta V_3(x(k)) \leq{}& (h_2 - h_1 + 1)x^T(k)Q(\alpha)x(k) - x_{k-h}^T Q(\alpha)x_{k-h} \\
&+ \sum_{i=k+1-h(k+1)}^{k-h_1} x^T(i)Q(\alpha)x(i) - \sum_{i=k+1-h_2}^{k-h_1} x^T(i)Q(\alpha)x(i),
\end{aligned} \tag{3.22}$$

for simplicity, we let $x(k-h(k)) = x_{k-h}$. Since, $h(k) \leq h_2$, we obtain that

$$\sum_{i=k+1-h(k+1)}^{k-h_1} x^T(i)Q(\alpha)x(i) - \sum_{i=k+1-h_2}^{k-h_1} x^T(i)Q(\alpha)x(i) \leq 0. \tag{3.23}$$

Therefore, we conclude that

$$\begin{aligned}\Delta V(x(k)) \leq{}& x^T(k)\hat{A}^T(\alpha)P(\alpha)\hat{A}(\alpha)x(k) + x_{k-h}^T\hat{B}^T(\alpha)P(\alpha)\hat{A}(\alpha)x(k) \\ &+ x^T(k)\hat{A}^T(\alpha)P(\alpha)\hat{B}(\alpha)x_{k-h} + x_{k-h}^T\hat{B}^T(\alpha)P(\alpha)\hat{B}(\alpha)x_{k-h} \\ &- x^T(k)P(\alpha)x(k) + (h_2 - h_1 + 1)x^T(k)Q(\alpha)x(k) \\ &- x_{k-h}^T Q(\alpha)x_{k-h}.\end{aligned} \tag{3.24}$$

It follows form (3.24) that

$$\Delta V(x(k)) \leq Y^T \begin{bmatrix} \Delta_{11}(\alpha) & \hat{A}^T(\alpha)P(\alpha)\hat{B}(\alpha) \\ \hat{B}^T(\alpha)P(\alpha)\hat{A}(\alpha) & \hat{B}^T(\alpha)P(\alpha)\hat{B}(\alpha) - Q(\alpha) \end{bmatrix} Y, \tag{3.25}$$

where $\Delta_{11}(\alpha) = \hat{A}^T(\alpha)P(\alpha)\hat{A}(\alpha) - P(\alpha) + \hat{h}Q(\alpha)$ and $Y^T = [x(k)^T \, x(k-h(k))^T]$. By (3.11), (3.25), and Lemma 3.1, and 3.2, we obtain

$$\Delta V(x(k)) < -\omega \|x\|^2, \tag{3.26}$$

where $\omega > 0$. By (3.14), it is easy to see that

$$V(x(k)) \leq \beta_1 \|x\|^2 + \beta_1 \hat{h} \sum_{i=k-h_2}^{k-1} \|x(i)\|^2, \tag{3.27}$$

where

$$\beta_1 = \max\{\lambda_{\max}(P_i), \lambda_{\max}(Q_i);\ i = 1, 2, \ldots, N\}. \tag{3.28}$$

It can be shown that there always exists a scalar $\theta > 1$ satisfying

$$(\theta - 1)\beta_1 - \lambda\theta + h_2\theta^{h_2}(\theta - 1)\beta_1\hat{h} = 0. \tag{3.29}$$

For any scalar $\theta > 1$, it follows from (3.26) and (3.27) that

$$\begin{aligned}\theta^{k+1}V(x(k+1)) - \theta^{k+1}V(x(k)) &= \theta^{k+1}(V(x(k+1)) - V(x(k))) + \theta^k(\theta-1)V(x(k)) \\ &< \alpha_1(\theta)\theta^k\|x(k)\|^2 + \alpha_2(\theta)\theta^k\sum_{i=k-h_2}^{k-1}\|x(i)\|^2,\end{aligned} \tag{3.30}$$

where

$$\alpha_1(\theta) = (\theta-1)\beta_1 - \lambda\theta, \qquad \alpha_2(\theta) = (\theta-1)\beta_1\widehat{h}. \tag{3.31}$$

Therefore, for any integer $T \geq h_2 + 1$, summing up both sides of (3.30) from 0 to $T-1$ gives

$$\theta^T V(x(T)) - V(x(0)) \leq \alpha_1(\theta)\sum_{i=0}^{T-1}\theta^i\|x(i)\|^2 + \alpha_2(\theta)\sum_{i=0}^{T-1}\sum_{l=i-h_2}^{i-1}\theta^i\|x(l)\|^2. \tag{3.32}$$

For $h_2 \geq 1$,

$$\begin{aligned}\sum_{i=0}^{T-1}\sum_{l=i-h_2}^{i-1}\theta^i\|x(l)\|^2 &\leq \sum_{l=-h_2}^{-1}\sum_{i=0}^{l+h_2}\theta^i\|x(l)\|^2 + \sum_{l=0}^{T-1-h_2}\sum_{i=l+1}^{l+h_2}\theta^i\|x(l)\|^2 \\ &\quad + \sum_{l=T-h_2}^{T-1}\sum_{i=l+1}^{T-1}\theta^i\|x(l)\|^2 \\ &\leq h_2\sum_{l=-h_2}^{-1}\theta^{l+h_2}\|x(l)\|^2 + h_2\sum_{l=0}^{T-1-h_2}\theta^{l+h_2}\|x(l)\|^2 \\ &\quad + h_2\sum_{l=T-h_2}^{T-1}\theta^{l+h_2}\|x(l)\|^2 \\ &\leq h_2(h_2+1)\theta^{h_2}\sup_{-h_2\leq l\leq 0}\left\|\phi(l)\right\|^2 + h_2\theta^{h_2}\sum_{l=1}^{T-1}\theta^l\|x(l)\|^2.\end{aligned} \tag{3.33}$$

From (3.32) and (3.33), we obtain

$$\begin{aligned}\theta^T V(x(T)) &\leq V(x(0)) + h_2(h_2+1)\theta^{h_2}\alpha_2(\theta)\sup_{-h_2\leq l\leq 0}\left\|\phi(l)\right\|^2 \\ &\quad + \left[\alpha_1(\theta) + \alpha_2(\theta)h_2\theta^{h_2}\right]\sum_{l=0}^{T-1}\theta^l\|x(l)\|^2.\end{aligned} \tag{3.34}$$

Observe

$$V(x(T)) \geq \gamma \|x(T)\|^2, \qquad V(x(0)) \leq \left(\beta_1 + \beta_1 \widehat{h} h_2\right) \sup_{-h_2 \leq l \leq 0} \|\phi(l)\|^2,$$
$$\gamma = \min\{\lambda_{\min}(P_i);\ i = 1, 2, \ldots, N\}. \tag{3.35}$$

Then, it follows from (3.29), (3.33), and (3.35) that

$$\|x(T)\|^2 \leq \frac{h_2\left[(h_2+1)\theta^{h_2}\alpha_2(\theta) + \beta_1 + \beta_1 \widehat{h} h_2\right]}{\gamma} \left(\frac{1}{\theta}\right)^T \sup_{-h_2 \leq l \leq 0} \|\phi(l)\|^2. \tag{3.36}$$

By Definition 2.1, this means that the system (2.1) is robustly exponentially stable. The proof of the theorem is complete. □

## 4. Numerical Example

*Example 4.1.* Consider the following uncertain LPD discrete-time system with time-varying delays (2.1) where $h(k) = 2 + \cos(k\pi/2)$, that is,. $h_1 = 1, h_2 = 3$ and

$$A_1 = \begin{bmatrix} -0.6 & 0.02 \\ 0.02 & -0.6 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} -0.7 & 0.03 \\ 0.03 & -0.7 \end{bmatrix}, \qquad B_1 = \begin{bmatrix} -0.6 & 0.02 \\ 0.02 & -0.08 \end{bmatrix},$$
$$B_2 = \begin{bmatrix} -0.8 & 0.03 \\ 0.03 & -0.09 \end{bmatrix}, \qquad A_1^1 = \begin{bmatrix} 0.005 & 0.0001 \\ 0.0001 & 0.005 \end{bmatrix}, \qquad A_2^1 = \begin{bmatrix} 0.006 & 0.0002 \\ 0.0002 & 0.006 \end{bmatrix},$$
$$B_1^1 = \begin{bmatrix} -0.007 & 0.0005 \\ 0.0005 & -0.007 \end{bmatrix}, \qquad B_2^1 = \begin{bmatrix} -0.004 & 0.0002 \\ 0.0002 & -0.004 \end{bmatrix},$$
$$K_1 = \begin{bmatrix} 0.01 & 0.003 \\ 0.003 & 0.01 \end{bmatrix}, \qquad K_2 = \begin{bmatrix} 0.02 & 0.001 \\ 0.001 & 0.02 \end{bmatrix}, \tag{4.1}$$

and $J = \left[\begin{smallmatrix} 0.001 & 0 \\ 0 & 0.001 \end{smallmatrix}\right]$. By using LMI Toolbox in MATLAB, we use condition (3.11) in Theorem 3.3 for this example. The solutions of LMI verify as follows of the form $\epsilon = 1$, $P_1 = \left[\begin{smallmatrix} 31.3635 & 1.2365 \\ 1.2365 & 29.4763 \end{smallmatrix}\right]$, $P_2 = \left[\begin{smallmatrix} 37.6354 & 0.2543 \\ 0.2543 & 41.3745 \end{smallmatrix}\right]$, $Q_1 = \left[\begin{smallmatrix} 9.4325 & 0.5587 \\ 0.5587 & 11.4534 \end{smallmatrix}\right]$, and $Q_2 = \left[\begin{smallmatrix} 10.8564 & 1.3856 \\ 1.3856 & 11.9781 \end{smallmatrix}\right]$ (see Figure 1).

*Example 4.2.* Consider the following the LPD discrete-time system with time-varying delays (2.1) where, $\Delta A(k) = \Delta B(k) = 0$ with

$$A_1 = \begin{bmatrix} 0.60 & 0 \\ 0.01 & 0.60 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 0.80 & 0 \\ 0.05 & 0.70 \end{bmatrix},$$
$$B_1 = \begin{bmatrix} 0.10 & 0 \\ 0.20 & 0.10 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} -0.10 & 0 \\ -0.20 & -0.10 \end{bmatrix}. \tag{4.2}$$

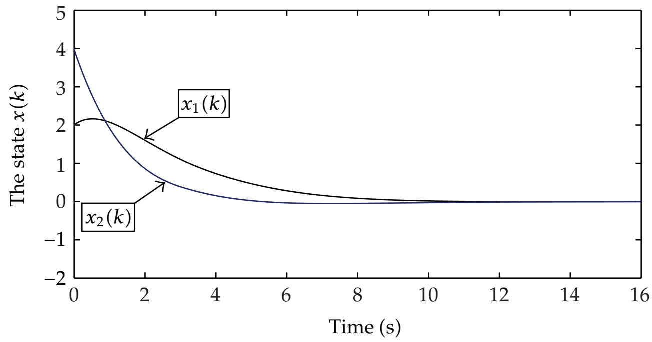


**Figure 1:** The simulation solution of the states $x_1(k)$ and $x_2(k)$ in Example 4.1 for uncertain LPD discrete-time delayed system with initial conditions $x_1(k) = 2$ and $x_2(k) = 4$, $k = -3, -2, -1, 0$, and $\alpha_1 = \alpha_2 = 1/2$ by using the method of Runge-Kutta order $4(h = 0.01)$ with Matlab.

**Table 1:** Comparison of the maximum allowed time delay $h_2$.

| Methods | $h_2$ ($h_1 = 2$) | $h_2$ ($h_1 = 4$) | $h_2$ ($h_1 = 5$) | $h_2$ ($h_1 = 7$) |
|---|---|---|---|---|
| Liuet al. [7] 2006 | 2 | 4 | 5 | 7 |
| Our results | 4 | 6 | 7 | 9 |

Table 1 lists the comparison of the upper-bound delay for asymptotic stability of system (2.1) where $\Delta A(k) = \Delta B(k) = 0$ by different method. We apply Theorem 3.3 and see from Table 1 that our result is superior to those in [7, Theorem 3.2].

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