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Research Article

Bounded Positive Solutions for a Third Order Discrete Equation

Zeqing Liu,[1] Ming Jia,[2] Shin Min Kang,[3] and Young Chel Kwun[4]

[1] *Liaoning Normal University, Department of Mathematics, Dalian, Liaoning 116029, China*
[2] *Dalian Hongwen Middle School, Dalian, Liaoning 116029, China*
[3] *Gyeongsang National University, Department of Mathematics and RINS, Jinju 660-701, Republic of Korea*
[4] *Dong-A University, Department of Mathematics, Pusan 614-714, Republic of Korea*

Correspondence should be addressed to Young Chel Kwun, yckwun@dau.ac.kr

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This paper studies the following third order neutral delay discrete equation $\Delta(a_n\Delta^2(x_n+p_nx_{n-\tau}))+f(n,x_{n-d_{1n}},\dots,x_{n-d_{ln}})=g_n$, $n\ge n_0$, where $\tau,l\in\mathbb{N}$, $n_0\in\mathbb{N}\cup\{0\}$, $\{a_n\}_{n\in\mathbb{N}_{n_0}}$, $\{p_n\}_{n\in\mathbb{N}_{n_0}}$, $\{g_n\}_{n\in\mathbb{N}_{n_0}}$ are real sequences with $a_n\neq 0$ for $n\ge n_0$, $\{d_{in}\}_{n\in\mathbb{N}_{n_0}}\subseteq\mathbb{Z}$ with $\lim_{n\to\infty}(n-d_{in})=+\infty$ for $i\in\{1,2,\dots,l\}$ and $f\in C(\mathbb{N}_{n_0}\times\mathbb{R}^l,\mathbb{R})$. By using a nonlinear alternative theorem of Leray-Schauder type, we get sufficient conditions which ensure the existence of bounded positive solutions for the equation. Three examples are given to illustrate the results obtained in this paper.

1. Introduction and Preliminaries

The oscillatory, nonoscillatory and asymptotic behaviors and existence of solutions for various difference equations have received more and more attentions in recent years. For details, we refer the reader to [1–11] and the references therein.

In 2005, M. Migda and J. Migda [10] studied the asymptotic behavior of solutions for the second order neutral difference equation

$$\Delta^2(x_n+px_{n-k})+f(n,x_n)=0,\quad n\ge 1,\tag{1.1}$$

where $p\in\mathbb{R}$, k is a nonnegative integer and $f:\mathbb{N}\times\mathbb{R}\to\mathbb{R}$. In 2008, Cheng and Chu [7] established sufficientand necessary conditions of oscillation for the second order difference equation

$$\Delta(r_{n-1}\Delta x_{n-1}) + p_n x_n^{\gamma} = 0, \quad n \geq 1, \tag{1.2}$$

where γ is the quotient of two odd positive integers and $p_n, r_n \in (0, +\infty)$ for $n \in \mathbb{N}$. In 2000, Li et al. [9] gave several necessary and/or sufficient conditions of the existence of unbounded positive solution for the nonlinear difference equation

$$\Delta(r_n \Delta x_n) + f(n, x_n) = 0, \quad n \geq n_0, \tag{1.3}$$

where n_0 is a fixed nonnegative integer, $r : \mathbb{N}_{n_0} \to (0, +\infty)$ and $f : \mathbb{N}_{n_0} \times \mathbb{R} \to \mathbb{R}$. In 2003, using the Leray-Schauder's nonlinear alternative theorem, Agarwal et al. [1] presented the existence of nonoscillatory solutions for the discrete equation

$$\Delta\left(a_n \Delta\left(x_n + p x_{n-\tau}\right)\right) + F(n+1, x_{n+1-\sigma}) = 0, \quad n \geq 1, \tag{1.4}$$

where τ, σ are fixed nonnegative integers, $p \in \mathbb{R}, a : \mathbb{N} \to (0, +\infty)$ and $F : \mathbb{N} \times (0, +\infty) \to [0, +\infty)$ is continuous. In 1995, Yan and Liu [11] proved the existence of a bounded nonoscillatory solution for the third order difference equation

$$\Delta^3 x_n + f(n, x_n, x_{n-r}) = 0, \quad n \geq n_0 \tag{1.5}$$

by utilizing the Schauder's fixed point theorem. In 2005, Andruch-Sobiło and Migda [2] studied the third order linear difference equations of neutral type

$$\Delta^3\left(x_n - p_n x_{\sigma_n}\right) \pm q_n x_{\tau_n} = 0, \quad n \geq n_0 \tag{1.6}$$

and obtained sufficient conditions under which all solutions of (1.6) are oscillatory.

The aim of this paper is to study the following third order neutral delay discrete equation

$$\Delta\Big(a_n \Delta^2\left(x_n + p_n x_{n-\tau}\right)\Big) + f(n, x_{n-d_{1n}}, \ldots, x_{n-d_{ln}}) = g_n, \quad n \geq n_0, \tag{1.7}$$

where $\tau, l \in \mathbb{N}, n_0 \in \mathbb{N} \cup \{0\}$, $\{a_n\}_{n \in \mathbb{N}_{n_0}}$, $\{p_n\}_{n \in \mathbb{N}_{n_0}}$, $\{g_n\}_{n \in \mathbb{N}_{n_0}}$ are real sequences with $a_n \neq 0$ for $n \geq n_0$, $\{d_{in}\}_{n \in \mathbb{N}_{n_0}} \subseteq \mathbb{Z}$ with $\lim_{n \to \infty}(n - d_{in}) = +\infty$ for $i \in \{1, 2, \ldots, l\}$ and $f \in C(\mathbb{N}_{n_0} \times \mathbb{R}^l, \mathbb{R})$. By making use of the Leray-Schauder's nonlinear alternative theorem, we establish the existence results of bounded positive solutions for (1.7), which extend substantially Theorem 2 in [11]. Three nontrivial examples are given to illustrate the superiority and applications of the results presented in this paper.

Let us recall and introduce the below concepts, signs and lemmas. Let $\mathbb{R}, \mathbb{Z}$ and $\mathbb{N}$ denote the sets of all real numbers, integers and positive integers, respectively,

$$\mathbb{N}_{n_0} = \{n : n \in \mathbb{N} \text{ with } n \geq n_0\}, \qquad \mathbb{Z}_\beta = \{n : n \in \mathbb{Z} \text{ with } n \geq \beta\},$$
$$\beta = \min\{n_0 - \tau, \inf\{n - d_{in} : 1 \leq i \leq l, \ \ n \in \mathbb{N}_{n_0}\}\} \tag{1.8}$$

and l_β^∞ stand for the Banach space of all bounded sequences on $\mathbb{Z}_\beta$ with norm

$$\|x\| = \sup_{n \in \mathbb{Z}_\beta} |x_n| \quad \text{for } x = \{x_n\}_{n \in \mathbb{Z}_\beta} \in l_\beta^\infty. \tag{1.9}$$

For any constants $M > N > 0$, put

$$E(N) = \left\{x = \{x_n\}_{n \in \mathbb{Z}_\beta} \in l_\beta^\infty : x_n \geq N \text{ for } n \in \mathbb{Z}_\beta\right\},$$
$$U(M) = \{x \in E(N) : \|x\| < M\}. \tag{1.10}$$

It is easy to verify that $E(N)$ is a nonempty closed convex subset of l_β^∞ and $U(M)$ is a nonempty open subset of $E(N)$.

By a solution of (1.7), we mean a sequence $\{x_n\}_{n \in \mathbb{Z}_\beta}$ with a positive integer $T \geq \tau + |\beta|$ such that (1.7) holds for all $n \geq T$.

For any subset U of a Banach space X, let $\overline{U}$ and ∂U denote the closure and boundary of U in X, respectively.

Lemma 1.1 (see [8]). *A bounded, uniformly Cauchy subset D of l_β^∞ is relatively compact.*

Lemma 1.2 (Leray-Schauder's Nonlinear Alternative Theorem [1]). *Let E be a nonempty closed convex subset of a Banach space X and U be an open subset of E with $p^* \in U$. Also $G : \overline{U} \to E$ is a continuous, condensing mapping with $G(\overline{U})$ bounded. Then either*

(A_1) *G has a fixed point in $\overline{U}$; or*

(A_2) *there are $x \in \partial U$ and $\lambda \in (0, 1)$ with $x = (1 - \lambda)p^* + \lambda Gx$.*

2. Existence of Bounded Positive Solutions

Now we investigate sufficient conditions of the existence of bounded positive solutions for (1.7) by using the Leray-Schauder's Nonlinear Alternative Theorem.

Theorem 2.1. *Assume that there exist constants $k_0 \in \mathbb{N}_{n_0}$ and M, N, $\overline{p}$ and $\underline{p}$ satisfying*

$$\sum_{s=k_0}^{\infty}\sum_{k=s}^{\infty}\frac{1}{|a_k|}\sum_{j=k}^{\infty}\sup\{|f(j,w_1,\ldots,w_l)| : w_i \in [N,M],\ 1 \le i \le l\} < +\infty; \tag{2.1}$$

$$\sum_{s=k_0}^{\infty}\sum_{k=s}^{\infty}\frac{1}{|a_k|}\sum_{j=k}^{\infty}|g_j| < +\infty; \tag{2.2}$$

$$0 < N < \left(1-\underline{p}-\overline{p}\right)M, \quad \min\left\{\underline{p},\overline{p}\right\} \ge 0, \quad \underline{p}+\overline{p} < 1 \tag{2.3}$$

$$-\underline{p} \le p_n \le \overline{p}, \quad n \ge k_0. \tag{2.4}$$

Then (1.7) *possesses a bounded positive solution in* $\overline{U(M)}$.

Proof. Let $L \in (\overline{p}M + N, M(1-\underline{p}))$. It follows from (2.1)–(2.3) that there exists a positive integer $T > 1 + \tau + k_0 + |\beta|$ sufficiently large satisfying

$$\begin{aligned}&\sum_{s=T}^{\infty}\sum_{k=s}^{\infty}\frac{1}{|a_k|}\sum_{j=k}^{\infty}\left[|g_j| + \sup\{|f(j,w_1,\ldots,w_l)| : w_i \in [N,M],\ 1 \le i \le l\}\right]\\ &\quad < \min\left\{L-\overline{p}M-N,\ M\left(1-\underline{p}\right)-L\right\}.\end{aligned} \tag{2.5}$$

Choose $p^* = M - \varepsilon_0$ with $\varepsilon_0 \in (0, \min\{L-\overline{p}M-N, M(1-\underline{p}) - L\})$ and

$$\begin{aligned}&\sum_{s=T}^{\infty}\sum_{k=s}^{\infty}\frac{1}{|a_k|}\sum_{j=k}^{\infty}\left[|g_j| + \sup\{|f(j,w_1,\ldots,w_l)| : w_i \in [N,M],\ 1 \le i \le l\}\right]\\ &\quad \le \min\left\{L-\overline{p}M-N, M\left(1-\underline{p}\right)-L\right\} - \varepsilon_0.\end{aligned} \tag{2.6}$$

Note that

$$M > M-\varepsilon_0 = p^* > M - \min\left\{L-\overline{p}M-N, M\left(1-\underline{p}\right)-L\right\} \ge N + M(1+\overline{p}) - L > N, \tag{2.7}$$

which implies that $p^* = \{p^*\}_{n\in\mathbb{Z}_\beta} \in U(M)$. Define two mappings $A_L, B_L : \overline{U(M)} \to l_\beta^\infty$

by

$$A_L x_n = \begin{cases} L - p_n x_{n-\tau}, & n \ge T+1 \\ L - p_T x_T, & \beta \le n \le T; \end{cases} \tag{2.8}$$

$$B_L x_n = \begin{cases} -\sum_{s=T}^{n-1}\sum_{k=s}^{\infty}\frac{1}{a_k}\sum_{j=k}^{\infty}\left[-g_j + f\left(j, x_{j-d_{1j}}, \dots, x_{j-d_{lj}}\right)\right], & n \ge T+1 \\ 0, & \beta \le n \le T \end{cases} \tag{2.9}$$

for all $x = \{x_n\}_{n\in\mathbb{Z}_\beta} \in \overline{U(M)}$.

We now show that

$$D_L = A_L + B_L : \overline{U(M)} \longrightarrow E(N). \tag{2.10}$$

For each $x = \{x_n\}_{n\in\mathbb{Z}_\beta} \in \overline{U(M)}$, by (2.4)–(2.9), we have

$$\begin{aligned} A_L x_n + B_L x_n &= L - p_n x_{n-\tau} - \sum_{s=T}^{n-1}\sum_{k=s}^{\infty}\frac{1}{a_k}\sum_{j=k}^{\infty}\left[-g_j + f\left(j, x_{j-d_{1j}}, \dots, x_{j-d_{lj}}\right)\right] \\ &\ge L - \overline{p}M - \sum_{s=T}^{n-1}\sum_{k=s}^{\infty}\frac{1}{|a_k|}\sum_{j=k}^{\infty}\left[|g_j| + \left|f\left(j, x_{j-d_{1j}}, \dots, x_{j-d_{lj}}\right)\right|\right] \\ &\ge L - \overline{p}M - \min\left\{L - \overline{p}M - N, M\left(1 - \underline{p}\right) - L\right\} + \varepsilon_0 \\ &\ge N + \varepsilon_0 \\ &> N, \quad n \ge T+1 \\ A_L x_n + B_L x_n &= L - p_T x_T \ge L - \overline{p}M > N, \quad \beta \le n \le T, \end{aligned} \tag{2.11}$$

□

We next assert that

$$B_L : \overline{U(M)} \longrightarrow l_\beta^\infty \text{ is a continuous, compact mapping.} \tag{2.12}$$

Let $\{x^\alpha\}_{\alpha\in\mathbb{N}} \subseteq \overline{U(M)}$ be an arbitrary sequence and $x^0 \in l_\beta^\infty$ with

$$\left\|x^\alpha - x^0\right\| \longrightarrow 0 \quad \text{as } \alpha \longrightarrow \infty. \tag{2.13}$$

Since $\overline{U(M)}$ is closed, it follows that $x^0 \in \overline{U(M)}$. Given $\varepsilon > 0$. Using (2.1), (2.13) and the continuity of f, we infer that there exists $T^{**}, T^* \in \mathbb{N}$ with $T^* > T+1$ satisfying

$$\sum_{s=T^*}^{\infty}\sum_{k=s}^{\infty}\frac{1}{|a_k|}\sum_{j=k}^{\infty}\sup\left\{\left|f(j,w_1,\ldots,w_l)\right| : w_i \in [N,M], 1 \le i \le l\right\} < \frac{\varepsilon}{16}; \tag{2.14}$$

$$\sum_{k=T^*}^{\infty}\frac{1}{|a_k|}\sum_{j=k}^{\infty}\sup\left\{\left|f(j,w_1,\ldots,w_l)\right| : w_i \in [N,M],\ 1 \le i \le l\right\} < \frac{\varepsilon}{16(T^*-T)}; \tag{2.15}$$

$$\sum_{s=T}^{T^*-1}\sum_{k=s}^{T^*-1}\frac{1}{|a_k|}\sum_{j=T^*}^{\infty}\sup\left\{\left|f(j,w_1,\ldots,w_l)\right| : w_i \in [N,M],\ 1 \le i \le l\right\} < \frac{\varepsilon}{16}; \tag{2.16}$$

$$\sum_{s=T}^{T^*-1}\sum_{k=s}^{T^*-1}\frac{1}{|a_k|}\sum_{j=k}^{T^*-1}\left|f\left(j,x^{\alpha}_{j-d_{1j}},\ldots,x^{\alpha}_{j-d_{lj}}\right)-f\left(j,x^{0}_{j-d_{1j}},\ldots,x^{0}_{j-d_{lj}}\right)\right| < \frac{\varepsilon}{2}, \quad \alpha \ge T^{**}. \tag{2.17}$$

Combining (2.9) and (2.14)–(2.17), we conclude that

$$\begin{aligned}
&\left\|B_Lx^{\alpha}-B_Lx^{0}\right\| \\
&= \max\left\{\sup_{\beta\le n\le T}\left|B_Lx^{\alpha}_n-B_Lx^{0}_n\right|, \sup_{n\ge T+1}\left|B_Lx^{\alpha}_n-B_Lx^{0}_n\right|\right\} \\
&\le \max\left\{0, \sup_{n\ge T+1}\sum_{s=T}^{n-1}\sum_{k=s}^{\infty}\frac{1}{|a_k|}\sum_{j=k}^{\infty}\left|f\left(j,x^{\alpha}_{j-d_{1j}},\ldots,x^{\alpha}_{j-d_{lj}}\right)-f\left(j,x^{0}_{j-d_{1j}},\ldots,x^{0}_{j-d_{lj}}\right)\right|\right\} \\
&\le \sum_{s=T}^{T^*-1}\sum_{k=s}^{\infty}\frac{1}{|a_k|}\sum_{j=k}^{\infty}\left|f\left(j,x^{\alpha}_{j-d_{1j}},\ldots,x^{\alpha}_{j-d_{lj}}\right)-f\left(j,x^{0}_{j-d_{1j}},\ldots,x^{0}_{j-d_{lj}}\right)\right| \\
&\quad+\sum_{s=T^*}^{\infty}\sum_{k=s}^{\infty}\frac{1}{|a_k|}\sum_{j=k}^{\infty}\left[\left|f\left(j,x^{\alpha}_{j-d_{1j}},\ldots,x^{\alpha}_{j-d_{lj}}\right)\right|+\left|f\left(j,x^{0}_{j-d_{1j}},\ldots,x^{0}_{j-d_{lj}}\right)\right|\right] \\
&\le \sum_{s=T}^{T^*-1}\sum_{k=s}^{T^*-1}\frac{1}{|a_k|}\sum_{j=k}^{\infty}\left|f\left(j,x^{\alpha}_{j-d_{1j}},\ldots,x^{\alpha}_{j-d_{lj}}\right)-f\left(j,x^{0}_{j-d_{1j}},\ldots,x^{0}_{j-d_{lj}}\right)\right| \\
&\quad+\sum_{s=T}^{T^*-1}\sum_{k=T^*}^{\infty}\frac{1}{|a_k|}\sum_{j=k}^{\infty}\left|f\left(j,x^{\alpha}_{j-d_{1j}},\ldots,x^{\alpha}_{j-d_{lj}}\right)\right|+\left|f\left(j,x^{0}_{j-d_{1j}},\ldots,x^{0}_{j-d_{lj}}\right)\right| \\
&\quad+2\sum_{s=T^*}^{\infty}\sum_{k=s}^{\infty}\frac{1}{|a_k|}\sum_{j=k}^{\infty}\sup\left\{\left|f(j,w_1,\ldots,w_l)\right| : w_i \in [N,M],\ 1 \le i \le l\right\} \\
&\le \sum_{s=T}^{T^*-1}\sum_{k=s}^{T^*-1}\frac{1}{|a_k|}\sum_{j=k}^{T^*-1}\left|f\left(j,x^{\alpha}_{j-d_{1j}},\ldots,x^{\alpha}_{j-d_{lj}}\right)-f\left(j,x^{0}_{j-d_{1j}},\ldots,x^{0}_{j-d_{lj}}\right)\right| \\
&\quad+\sum_{s=T}^{T^*-1}\sum_{k=s}^{T^*-1}\frac{1}{|a_k|}\sum_{j=T^*}^{\infty}\left[\left|f\left(j,x^{\alpha}_{j-d_{1j}},\ldots,x^{\alpha}_{j-d_{lj}}\right)\right|+\left|f\left(j,x^{0}_{j-d_{1j}},\ldots,x^{0}_{j-d_{lj}}\right)\right|\right] \\
&\quad+2(T^*-T)\sum_{k=T^*}^{\infty}\frac{1}{|a_k|}\sum_{j=k}^{\infty}\sup\left\{\left|f(j,w_1,\ldots,w_l)\right| : w_i \in [N,M], 1 \le i \le l\right\}+2\cdot\frac{\varepsilon}{16} \\
&< \frac{\varepsilon}{2}+2\cdot\frac{\varepsilon}{16}+2(T^*-T)\cdot\frac{\varepsilon}{16(T^*-T)}+\frac{\varepsilon}{8} \\
&< \varepsilon, \quad \alpha \ge T^{**},
\end{aligned} \tag{2.18}$$

which means that B_L is continuous in $\overline{U(M)}$. On the other hand, in light of (2.6) and (2.9), we get that for each $x = \{x_n\}_{n\in\mathbb{Z}_\beta} \in \overline{U(M)}$

$$\begin{aligned}
\|B_L x\| &= \max\left\{\sup_{\beta\le n\le T}|B_L x_n|, \sup_{n\ge T+1}|B_L x_n|\right\} \\
&\le \sup_{n\ge T+1}\sum_{s=T}^{n-1}\sum_{k=s}^{\infty}\frac{1}{|a_k|}\sum_{j=k}^{\infty}\left[|g_j| + \left|f\left(j, x_{j-d_{1j}}, \dots, x_{j-d_{lj}}\right)\right|\right] \\
&\le \sum_{s=T}^{\infty}\sum_{k=s}^{\infty}\frac{1}{|a_k|}\sum_{j=k}^{\infty}\left[|g_j| + \sup\{|f(j, w_1, \dots, w_l)| : w_i \in [N, M],\ 1 \le i \le l\}\right] \\
&\le \min\left\{L - \overline{p}M - N, M\left(1 - \underline{p}\right) - L\right\} - \varepsilon_0 \\
&\le M,
\end{aligned} \tag{2.19}$$

which yields that $B_L(\overline{U(M)})$ is a bounded subset of l_β^∞. By virtue of (2.1) and (2.2), we deduce that for any $\varepsilon > 0$, there exists $T_0 > T$ satisfying

$$\sum_{s=T_0}^{\infty}\sum_{k=s}^{\infty}\frac{1}{|a_k|}\sum_{j=k}^{\infty}\left[|g_j| + \sup\{|f(j, w_1, \dots, w_l)| : w_i \in [N, M], 1 \le i \le l\}\right] < \varepsilon, \tag{2.20}$$

which together with (2.9) gives that for any $x = \{x_n\}_{n\in\mathbb{Z}_\beta} \in \overline{U(M)}$

$$\begin{aligned}
&|B_L x_n - B_L x_m| \\
&= \left|\sum_{s=T}^{n-1}\sum_{k=s}^{\infty}\frac{1}{a_k}\sum_{j=k}^{\infty}\left[-g_j + f\left(j, x_{j-d_{1j}}, \dots, x_{j-d_{lj}}\right)\right] - \sum_{s=T}^{m-1}\sum_{k=s}^{\infty}\frac{1}{a_k}\sum_{j=k}^{\infty}\left[-g_j + f\left(j, x_{j-d_{1j}}, \dots, x_{j-d_{lj}}\right)\right]\right| \\
&\le \sum_{s=n}^{m-1}\sum_{k=s}^{\infty}\frac{1}{|a_k|}\sum_{j=k}^{\infty}\left[|g_j| + \left|f\left(j, x_{j-d_{1j}}, \dots, x_{j-d_{lj}}\right)\right|\right] \\
&< \sum_{s=T_0}^{\infty}\sum_{k=s}^{\infty}\frac{1}{|a_k|}\sum_{j=k}^{\infty}\left[|g_j| + \sup\{|f(j, w_1, \dots, w_l)| : w_i \in [N, M], 1 \le i \le l\}\right] \\
&< \varepsilon, \quad m > n \ge T_0,
\end{aligned} \tag{2.21}$$

which means that $B_L(\overline{U(M)})$ is uniformly Cauchy. Thus Lemma 1.1 ensures that $B_L(\overline{U(M)})$ is a relatively compact subset of l_β^∞.

Let $x = \{x_n\}_{n\in\mathbb{Z}_\beta}, y = \{y_n\}_{n\in\mathbb{Z}_\beta} \in \overline{U(M)}$. In view of (2.3), (2.4) and (2.8), we know that

$$\begin{aligned}
|A_L x_n - A_L y_n| &= |L - p_n x_{n-\tau} - L + p_n y_{n-\tau}| \le |p_n|\|x - y\| \le \left(\underline{p} + \overline{p}\right)\|x - y\|, \quad n \ge T + 1, \\
|A_L x_n - A_L y_n| &= |L - p_T x_{T-\tau} - L + p_T y_{T-\tau}| \le |p_T|\|x - y\| \le \left(\underline{p} + \overline{p}\right)\|x - y\|, \quad \beta \le n \le T,
\end{aligned} \tag{2.22}$$

which implies that

$$\|Ax - Ay\| \le \left(\underline{p} + \overline{p}\right)\|x - y\|, \tag{2.23}$$

which together with (2.10) and (2.12) guarantees that $D_L : \overline{U(M)} \to E(N)$ is a continuous, condensing mapping.

In order to show the existence of a fixed point of D_L, we need to prove that (A_2) in Lemma 1.2 does not hold. Otherwise there exist $x = \{x_n\}_{n\in\mathbb{Z}_\beta} \in \partial U(M)$ and $\lambda \in (0,1)$ such that $x = (1-\lambda)p^* + \lambda D_L x$. Let

$$\begin{gathered} S_1 = \left\{x \in l_\beta^\infty : N \le x_n \le M,\ \ \forall n \ge \beta,\ \|x\| = M\right\}, \\ S_2 = \left\{x \in l_\beta^\infty : N \le x_n \le M,\ \ \forall n \ge \beta,\ \text{ and there exists } n^* \ge \beta \text{ satisfying } x_{n^*} = N\right\}. \end{gathered} \tag{2.24}$$

It is easy to verify that $\partial U(M) = S_1 \cup S_2$. Now we have to discuss two possible cases as follows:

Case 1. Let $x \in S_1$. It follows from (2.3), (2.4), (2.8) and (2.9) that

$$\begin{aligned} x_n &= (1-\lambda)p^* + \lambda[A_L x_n + B_L x_n] \\ &= (1-\lambda)p^* + \lambda\left[L - p_n x_{n-\tau} - \sum_{s=T}^{n-1}\sum_{k=s}^{\infty}\frac{1}{a_k}\sum_{j=k}^{\infty}\left[-g_j + f\left(j, x_{j-d_{1j}}, \dots, x_{j-d_{lj}}\right)\right]\right] \\ &\le (1-\lambda)p^* + \lambda\left[L + \underline{p}M + \sum_{s=T}^{n-1}\sum_{k=s}^{\infty}\frac{1}{|a_k|}\sum_{j=k}^{\infty}\left[|g_j| + \left|f\left(j, x_{j-d_{1j}}, \dots, x_{j-d_{lj}}\right)\right|\right]\right] \\ &\le (1-\lambda)(M-\varepsilon_0) + \lambda\left[L + \underline{p}M + \min\left\{L - \overline{p}M - N,\ M\left(1-\underline{p}\right) - L\right\} - \varepsilon_0\right] \\ &\le (1-\lambda)(M-\varepsilon_0) + \lambda(M-\varepsilon_0) \\ &= M - \varepsilon_0, \quad n \ge T+1, \end{aligned} \tag{2.25}$$

$$\begin{aligned} x_n &= (1-\lambda)p^* + \lambda\left[L - p_T x_T\right] \le (1-\lambda)p^* + \lambda\left(L + \underline{p}M\right) \\ &\le (1-\lambda)(M-\varepsilon_0) + \lambda(M-\varepsilon_0) = M - \varepsilon_0, \quad \beta \le n \le T, \end{aligned} \tag{2.26}$$

which yield that

$$M = \|x\| \le M - \varepsilon_0 < M, \tag{2.27}$$

which is a contradiction;

Case 2. Let $x \in S_2$. If $n^* \geq T+1$, by (2.3), (2.4), (2.8) and (2.9), we deduce that

$$\begin{aligned} N = x_{n^*} &= (1-\lambda)p^* + \lambda[A_L x_{n^*} + B_L x_{n^*}] \\ &= (1-\lambda)p^* + \lambda\left[L - p_{n^*}x_{n^*-\tau} - \sum_{s=T}^{n^*-1}\sum_{k=s}^{\infty}\frac{1}{a_k}\sum_{j=k}^{\infty}\left[-g_j + f\left(j, x_{j-d_{1j}}, \ldots, x_{j-d_{lj}}\right)\right]\right] \\ &\geq (1-\lambda)(M-\varepsilon_0) + \lambda\left[L - \overline{p}M - \sum_{s=T}^{n^*-1}\sum_{k=s}^{\infty}\frac{1}{|a_k|}\sum_{j=k}^{\infty}\left[|g_j| + \left|f\left(j, x_{j-d_{1j}}, \ldots, x_{j-d_{lj}}\right)\right|\right]\right] \\ &> (1-\lambda)N + \lambda\left[L - \overline{p}M - \min\left\{L - \overline{p}M - N, \quad M\left(1-\underline{p}\right) - L\right\} + \varepsilon_0\right] \\ &\geq (1-\lambda)N + \lambda(N+\varepsilon_0) \\ &= N + \varepsilon_0 \\ &> N, \end{aligned} \tag{2.28}$$

which is impossible; if $n^* \leq T$, by (2.3), (2.4), (2.8) and (2.9), we arrive at

$$\begin{aligned} N = x_{n^*} &= (1-\lambda)p^* + \lambda[L - p_T x_T] \geq (1-\lambda)p^* + \lambda(L - \overline{p}M) \\ &\geq (1-\lambda)\mathrm{N} + \lambda(N+\varepsilon_0) = N + \varepsilon_0 \\ &> N, \end{aligned} \tag{2.29}$$

which is absurd.

Consequently Lemma 1.2 ensures that there is $x = \{x_n\}_{n\in\mathbb{Z}_\beta} \in \overline{U(M)}$ such that $D_L x = A_L x + B_L x = x$, which is a bounded positive solution of (1.7). This completes the proof.

Remark 2.2. Under the conditions of Theorem 2.1 we prove also that (1.7) has uncountably many bounded positive solutions in $\overline{U(M)}$.

In fact, as in the proof of Theorem 2.1, for any different $L_1, L_2 \in (\overline{p}M + N, M(1-\underline{p}))$ we conclude that for each $r \in \{1,2\}$, there exist a constant $T_r > 1 + \tau + k_0 + |\beta|$ and two mappings $A_r, B_r : \overline{U(M)} \to l_\beta^\infty$ satisfying (2.6)–(2.9) and

$$\sum_{s=\min\{T_1,T_2\}}^{\infty}\sum_{k=s}^{\infty}\frac{1}{|a_k|}\sum_{j=k}^{\infty}\left[|g_j| + \sup\{|f(j, w_1, \ldots, w_l)| : w_i \in [N, M],\ 1 \leq i \leq l\}\right] \leq \frac{|L_1 - L_2|}{4}, \tag{2.30}$$

where T, L, A_L and B_L are replaced by T_r, r, A_r and B_r, respectively, and $A_r + B_r$ has a fixed point $z^r = \{z_n^r\}_{n\in\mathbb{Z}_\beta} \in \overline{U(M)}$, which is a bounded positive solution of (1.7). In order to prove that (1.7) possesses uncountably many bounded positive solutions in $\overline{U(M)}$, we need only to prove that $z^1 \neq z^2$. It follows from (2.8), (2.9) and (2.30) that for $n \geq \min\{T_1, T_2\}$

$$
\begin{aligned}
\left|z_n^1 - z_n^2\right| = \Bigg| & L_1 - p_n z_{n-\tau}^1 - \sum_{s=T_1}^{n-1} \sum_{k=s}^{\infty} \frac{1}{a_k} \sum_{j=k}^{\infty} \left[-g_j + f\left(j, z_{j-d_{1j}}^1, \dots, z_{j-d_{lj}}^1\right)\right] \\
& -L_2 + p_n z_{n-\tau}^2 + \sum_{s=T_2}^{n-1} \sum_{k=s}^{\infty} \frac{1}{a_k} \sum_{j=k}^{\infty} \left[-g_j + f\left(j, z_{j-d_{1j}}^2, \dots, z_{j-d_{lj}}^2\right)\right] \Bigg| \\
\geq & |L_1 - L_2| - \left(\underline{p} + \overline{p}\right) \left\| z^1 - z^2 \right\| \\
& - 2 \sum_{s=\min\{T_1,T_2\}}^{\infty} \sum_{k=s}^{\infty} \frac{1}{|a_k|} \sum_{j=k}^{\infty} \left[|g_j| + \sup\{|f(j, w_1, \dots, w_l)| : w_i \in [N, M],\ 1 \leq i \leq l\}\right] \\
\geq & |L_1 - L_2| - \left(\underline{p} + \overline{p}\right) \left\| z^1 - z^2 \right\| - \frac{|L_1 - L_2|}{2},
\end{aligned}
\tag{2.31}
$$

which implies that

$$
\left\| z^1 - z^2 \right\| \geq \frac{|L_1 - L_2|}{2\left(1 + \underline{p} + \overline{p}\right)} > 0, \tag{2.32}
$$

which yields that $z^1 \neq z^2$.

Remark 2.3. If either $\underline{p} = 0$ or $\overline{p} = 0$, then Theorem 2.1 reduces to the below results, respectively.

Theorem 2.4. *Assume that there exist constants $k_0 \in \mathbb{N}_{n_0}$ and M, N and $\overline{p}$ satisfying* (2.1), (2.2) *and*

$$
0 < N < (1 - \overline{p}) M, \quad 0 \leq p_n \leq \overline{p} < 1,\ n \geq k_0. \tag{2.33}
$$

Then (1.7) *possesses a bounded positive solution in $\overline{U(M)}$.*

Theorem 2.5. *Assume that there exist constants $k_0 \in \mathbb{N}_{n_0}$ and M, N and $\underline{p}$ satisfying* (2.1), (2.2) *and*

$$
0 < N < \left(1 - \underline{p}\right) M, \quad -1 < -\underline{p} \leq p_n \leq 0, \quad n \geq k_0. \tag{2.34}
$$

Then (1.7) *possesses a bounded positive solution in $\overline{U(M)}$.*

Remark 2.6. Theorems 2.1–2.3 include Theorem 2 in [11] as special cases. Examples 3.1–3.3 in Section 3 explain that Theorems 2.1–2.3 are genuine generalizations of Theorem 2 in [11].

3. Examples and Applications

Now we construct three nontrivial examples to explain the superiority and applications of Theorems 2.1-2.3, respectively.

Example 3.1. Consider the third order neutral delay discrete equation

$$\Delta\left((-1)^n n^3 \Delta^2\left(x_n + \frac{n\sin(n^2)}{3n+1}x_{n-\tau}\right)\right) + \frac{n^3 x_{n^2-2} - \sqrt{n}x^3_{n^3-6}}{n^5+5n+nx^2_{2n+5}} = \frac{\cos(n\ln(n^2+1))}{\sqrt{n^3+1}}, \quad n \ge 1, \tag{3.1}$$

where $\tau \in \mathbb{N}$ is fixed. Let $l = 3,\ n_0 = 1,\ k_0 = 2,\ \underline{p} = 1/2,\ \overline{p} = 1/3,\ M = 7,\ N = 1,$

$$\begin{gathered} a_n = (-1)^n n^3, \qquad p_n = \frac{n\sin(n^2)}{3n+1}, \\ g_n = \frac{\cos(n\ln(n^2+1))}{\sqrt{n^3+1}}, \qquad d_{1n} = -n^2+n+2, \\ d_{3n} = -n-5, \qquad f(n,u,v,w) = \frac{n^3u - \sqrt{n}v^3}{n^5+5n+nw^2}, \quad \forall (n,u,v,w) \in \mathbb{N}\times\mathbb{R}^3. \end{gathered} \tag{3.2}$$

It is easy to verify that (2.1)–(2.4) hold. It follows from Theorem 2.1 that (3.1) has a bounded positive solution in $\overline{U(M)}$. However Theorem 2 in [11] is useless for (3.1).

Example 3.2. Consider the third order neutral delay discrete equation

$$\Delta\left((n+1)^2\ln^3(n+2)\Delta^2\left(x_n + \frac{3n-4}{4n+2}x_{n-\tau}\right)\right) + \frac{x^2_{n(n+1)/2}x^3_{n(n-1)/2}}{\sqrt{n^3+1}} = \frac{(-1)^n}{n^2}, \quad n \ge 1, \tag{3.3}$$

where $\tau \in \mathbb{N}$ is fixed. Let $l = 2,\ n_0 = 1,\ k_0 = 2,\ \overline{p} = 3/4,\ M = 40,\ N = 8,$

$$\begin{gathered} a_n = (n+1)^2\ln^3(n+2), \qquad p_n = \frac{3n-4}{4n+2}, \qquad d_{1n} = \frac{n(1-n)}{2}, \qquad d_{2n} = \frac{n(3-n)}{2}, \\ g_n = \frac{(-1)^n}{n^2}, \qquad f(n,u,v) = \frac{u^2v^3}{\sqrt{n^3+1}}, \quad \forall (n,u,v) \in \mathbb{N}\times\mathbb{R}^2. \end{gathered} \tag{3.4}$$

It is clear that (2.1), (2.2) and (2.33) hold. Consequently Theorem 2.4 guarantees that (3.3) has a bounded positive solution in $\overline{U(M)}$. But Theorem 2 in [11] is inapplicable for (3.3).

Example 3.3. Consider the third order neutral delay discrete equation

$$\Delta\left(\sqrt{n^5+1}\Delta^2\left(x_n - \frac{n^3+1}{3n^3+4}x_{n-\tau}\right)\right) + \frac{\sqrt{n}x^6_{n^2-2n}}{n^4+n+1} + \frac{x^3_{2n+3}}{n^2+2} = \frac{\sin(n^2-n)}{n^2+1}, \quad n \ge 1, \tag{3.5}$$

where $\tau \in \mathbb{N}$ is fixed. Let $l = 2,\ n_0 = 1,\ k_0 = 3,\ \underline{p} = 1/3,\ M = 30,\ N = 19$,

$$a_n = \sqrt{n^5+1}, \qquad p_n = -\frac{n^3+1}{3n^3+4}, \qquad d_{1n} = n(3-n), \qquad d_{2n} = -n-3,$$
$$g_n = \frac{\sin(n^2-n)}{n^2+1}, \qquad f(n,u,v) = \frac{u^6\sqrt{n}}{n^4+n+1} + \frac{v^3}{n^2+2}, \quad \forall (n,u,v) \in \mathbb{N}\times\mathbb{R}^2. \tag{3.6}$$

Obviously, (2.1), (2.2) and (2.34) hold. Thus Theorem 2.5 ensures that (3.5) has a bounded positive solution in $\overline{U(M)}$. While Theorem 2 in [11] is unfit for (3.5)

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References

[1] R. P. Agarwal, S. R. Grace, and D. O'Regan, "Nonoscillatory solutions for discrete equations," *Computers & Mathematics with Applications*, vol. 45, no. 6–9, pp. 1297–1302, 2003.

[2] A. Andruch-Sobiło and M. Migda, "On the oscillation of solutions of third order linear difference equations of neutral type," *Mathematica Bohemica*, vol. 130, no. 1, pp. 19–33, 2005.

[3] J. Baštinec, L. Berezansky, J. Diblík, and Z. Šmarda, "A final result on the oscillation of solutions of the linear discrete delayed equation $\Delta x(n) = -p(n)x(n-k)$ with a positive coefficient," *Abstract and Applied Analysis*, vol. 2011, Article ID 586328, 28 pages, 2011.

[4] J. Baštinec and J. Diblík, "Subdominant positive solutions of the discrete equation $\Delta u(k+n) = -p(k)u(k)$," *Abstract and Applied Analysis*, vol. 2004, no. 6, pp. 461–470, 2004.

[5] J. Baštinec and J. Diblík, "Remark on positive solutions of discrete equation $\Delta u(k+n) = -p(k)u(k)$," *Nonlinear Analysis*, vol. 63, no. 5–7, pp. e2145–e2151, 2005.

[6] J. Baštinec, J. Diblík, and Z. Šmarda, "Existence of positive solutions of discrete linear equations with a single delay," *Journal of Difference Equations and Applications*, vol. 16, no. 9, pp. 1047–1056, 2010.

[7] J. F. Cheng and Y. M. Chu, "Oscillation theorem for second-order difference equations," *Taiwanese Journal of Mathematics*, vol. 12, no. 3, pp. 623–633, 2008.

[8] S. S. Cheng and W. T. Patula, "An existence theorem for a nonlinear difference equation," *Nonlinear Analysis*, vol. 20, no. 3, pp. 193–203, 1993.

[9] W. T. Li, X. L. Fan, and C. K. Zhong, "On unbounded positive solutions of second-order difference equations with a singular nonlinear term," *Journal of Mathematical Analysis and Applications*, vol. 246, no. 1, pp. 80–88, 2000.

[10] M. Migda and J. Migda, "Asymtotic properties of solutions of second-order neutral difference equations," *Nonlinear Analysis*, vol. 63, pp. 789–799, 2005.

[11] J. Yan and B. Liu, "Asymptotic behavior of a nonlinear delay difference equation," *Applied Mathematics Letters*, vol. 8, pp. 1–5, 1995.