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Research Article

Exact Traveling Wave Solutions of Explicit Type, Implicit Type, and Parametric Type for $K(m,n)$ Equation

Xianbin Wu,[1] Weiguo Rui,[2] and Xiaochun Hong[3]

[1] *Junior College, Zhejiang Wanli University, Ningbo 315100, China*
[2] *College of Mathematics, Honghe University, Mengzi, Yunnan 661100, China*
[3] *College of Statistics and Mathematics, Yunnan University of Finance and Economics, Kunming, Yunnan 650221, China*

Correspondence should be addressed to Weiguo Rui, weiguorhhu@yahoo.com.cn

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By using the integral bifurcation method, we study the nonlinear $K(m,n)$ equation for all possible values of m and n. Some new exact traveling wave solutions of explicit type, implicit type, and parametric type are obtained. These exact solutions include peculiar compacton solutions, singular periodic wave solutions, compacton-like periodic wave solutions, periodic blowup solutions, smooth soliton solutions, and kink and antikink wave solutions. The great parts of them are different from the results in existing references. In order to show their dynamic profiles intuitively, the solutions of $K(n,n)$, $K(2n-1,n)$, $K(3n-2,n)$, $K(4n-3,n)$, and $K(m,1)$ equations are chosen to illustrate with the concrete features.

1. Introduction

In this paper, we will investigate some new traveling-wave phenomena of the following nonlinear dispersive $K(m,n)$ equation [1]:

$$u_t + \sigma(u^m)_x + (u^n)_{xxx} = 0, \quad m > 1,\ n \geq 1, \tag{1.1}$$

where m and n are integers and σ is a real parameter. This is a family of fully KdV equations. When $\sigma = 1$, (1.1) as a role of nonlinear dispersion in the formation of patterns in liquid drops was studied by Rosenau and Hyman [1]. In [2–6], the studies show that the model equation (1.1) supports compact solitary structure. In [3], especially Rosenau's study shows that the branch + (i.e., $\sigma = 1$) supports compact solitary waves and the branch − (i.e., $\sigma = -1$) supports motion of kinks, solitons with spikes, cusps or peaks. In [7, 8], Wazwaz developed new solitary wave solutions of (1.1) with compact support and solitary patterns with cusps or infinite slopes under $\sigma = \pm 1$, respectively. In [9], by using the extend decomposition method, Zhu and Lü obtained exact special solutions with solitary patterns for (1.1). In [10], by using homotopy perturbation method (HPM), Domairry et al. studied the (1.1); under particular cases, they obtained some numerical and exact compacton solutions of the nonlinear dispersive $K(2,2)$ and $K(3,3)$ equations with initial conditions. In [11], by variational iteration method, Tian and Yin obtained new solitary solutions for nonlinear dispersive equations $K(m,n)$; under particular values of m and n, they obtained shock-peakon solutions for $K(2,2)$ equation and shock-compacton solutions for $K(3,3)$ equation. In [12], the nonlinear equation $K(m,n)$ is studied by Wazwaz for all possible values of m and n. In [13], by using Adomian decomposition method, Zhu and Gao obtained new solitary-wave special solutions with compact support for (1.1). In [14], by using a new method which is different from the Adomian decomposition method, Shang studied (1.1) and obtained new exact solitary-wave solutions with compact. In [15, 16], 1-soliton solutions of the $K(m,n)$ equation with generalized evolution are obtained by Biswas. In [17], the bright and dark soliton solutions for $K(m,n)$ equation with t-dependent coefficients are obtained by Triki and Wazwaz, especially, when $m = n$, the $K(n,n)$ equation was studied by many authors; see [18–24] and references cited therein. Defocusing branch, Deng et al. [25] obtained exact solitary and periodic traveling wave solutions of $K(2,2)$ equation. Also, under some particular values of m and n, many authors considered some particular cases of $K(m,n)$ equation. Ismail and Taha [26] implemented a finite difference method and a finite element method to study two types of equations $K(2,2)$ and $K(3,3)$. A single compacton as well as the interaction of compactons has been numerically studied. Then, Ismail [27] made an extension to the work in [26], applied a finite difference method on $K(2,3)$ equation, and obtained numerical solutions of $K(2,3)$ equation [28]. Frutos and Lopez-Marcos [29] presented a finite difference method for the numerical integration of $K(2,2)$ equation. Zhou and Tian [30] studied soliton solution of $K(2,2)$ equation. Xu and Tian [31] investigated the peaked wave solutions of $K(2,2)$ equation. Zhou et al. [32] obtained kink-like wave solutions and antikink-like wave solutions of $K(2,2)$ equation. He and Meng [33] obtain some new exact explicit peakon and smooth periodic wave solutions of the $K(3,2)$ equation by the bifurcation method of planar systems and qualitative theory of polynomial differential system.

From the aforementioned references, and references cited therein, it has been shown that (1.1) is a very important physical and engineering model. This is a main reason for us to study it again. In this paper, by using the integral bifurcation method [34–36], we mainly investigate some new exact solutions such as explicit solutions of Jacobian elliptic function type with low-power, implicit solutions of Jacobian elliptic function type, periodic solutions of parametric type, and so forth. We also investigate some new traveling wave phenomena and their dynamic properties.

The rest of this paper is organized as follows. In Section 2, we will derive the equivalent two-dimensional planar system of (1.1) and its first integral. In Section 3, by using the integral bifurcation method, we will obtain some new traveling wave solutions and discuss their dynamic properties; some phenomena of new traveling waves are illustrated with the concrete features.

2. The Equivalent Two-Dimensional Planar System to (1.1) and Its First Integral Equations

We make a transformation $u(t,x) = \phi(\xi)$ with $\xi = x - vt$, where the v is a nonzero constant as wave velocity. Thus, (1.1) can be reduced to the following ODE:

$$-v\phi' + \sigma(\phi^m)' + (\phi^n)''' = 0. \tag{2.1}$$

Integrating (2.1) once and setting the integral constant as zero yields

$$-v\phi + \sigma\phi^m + n(n-1)\phi^{n-2}(\phi')^2 + n\phi^{n-1}\phi'' = 0. \tag{2.2}$$

Let $\phi' = (d\phi/d\xi) = y$. Equation (2.2) can be reduced to a 2D planar system:

$$\frac{d\phi}{d\xi} = y, \qquad \frac{dy}{d\xi} = \frac{v\phi - \sigma\phi^m - n(n-1)\phi^{n-2}y^2}{n\phi^{n-1}}, \tag{2.3}$$

where $\phi \neq 0$. Obviously, the solutions of (2.2) include the solutions of (2.3) and constant solution $\phi = 0$. We notice that the second equation in (2.3) is not continuous when $\phi = 0$; that is, the function $\phi''(\xi)$ is not defined by the singular line $\phi = 0$. Therefore, we make the following transformation:

$$d\xi = n\phi^{n-1}d\tau, \tag{2.4}$$

where τ is a free parameter. Under the transformation (2.4), (2.3), and $\phi = 0$ combine to make one 2D system as follows:

$$\frac{d\phi}{d\tau} = n\phi^{n-1}y, \qquad \frac{dy}{d\tau} = v\phi - \sigma\phi^m - n(n-1)\phi^{n-2}y^2. \tag{2.5}$$

Clearly, (2.5) is equivalent to (2.2). It is easy to know that (2.3) and (2.5) have the same first integral as follows:

$$y^2 = \frac{h + (2v/(n+1))\phi^{n+1} - (2\sigma/(n+m))\phi^{n+m}}{n\phi^{2n-2}}, \tag{2.6}$$

where h is an integral constant. From (2.6), we define a function as follows:

$$H(\phi, y) = n\phi^{2n-2}y^2 + \frac{2\sigma}{n+m}\phi^{m+n} - \frac{2v}{n+1}\phi^{n+1} = h. \tag{2.7}$$

It is easy to verify that (2.5) satisfies

$$\frac{d\phi}{d\tau} = \frac{1}{2\phi^{n-1}}\frac{\partial H}{\partial y}, \qquad \frac{dy}{d\tau} = -\frac{1}{2\phi^{n-1}}\frac{\partial H}{\partial \phi}. \tag{2.8}$$

Therefore, (2.5) is a Hamiltonian system and $1/2\phi^{n-1}$ is an integral factor. In fact, (2.7) can be rewritten as the form $H = E + T$, where $E = (1/2)My^2 = (1/2)M(\phi')^2$ and $T = (2\sigma/(n+m))\phi^{m+n} - (2v/(n+1))\phi^{n+1}$ with $M = 2n\phi^{2n-2}$. E denotes kinetic energy, and T denotes potential energy. Especially, when $n = 1$, M becomes a constant 2. In this case, the kinetic energy E only depends on movement velocity ϕ' of particle; it does not depend on potential function ϕ. So, according to Theorem 3.2 in [37], it is easy to know that (2.5) is a stable and nonsingular system when $n = 1$; in this case its solutions have not singular characters. When $n > 1$, (2.5) becomes a singular system; in this case some solutions of (2.5) have singular characters.

For the equilibrium points of the system (2.5), we have the following conclusion.

Case 1. When m is even number, (2.5) has two equilibrium points $O(0,0)$ and $A_0((v/\sigma)^{1/(m-1)},0)$. From (2.7), we obtain

$$h_O = H(0,0) = 0, \qquad h_{A_0} = -\frac{2v(m-1)}{(m+n)(n+1)}\left(\frac{v}{\sigma}\right)^{(n+1)/(m-1)}. \tag{2.9}$$

Case 2. When m is odd number and $\sigma v > 0$, (2.5) has three equilibrium points $O(0,0)$ and $A_{1,2}(\pm(v/\sigma)^{1/(m-1)},0)$. From (2.7), we also obtain $h_O = H(0,0) = 0$ and

$$h_{A_1} = -\frac{2(m-1)v}{(m+n)(n+1)}\left(\frac{v}{\sigma}\right)^{(n+1)/(m-1)}, \qquad h_{A_2} = (-1)^{n+2}\frac{2(m-1)v}{(m+n)(n+1)}\left(\frac{v}{\sigma}\right)^{(n+1)/(m-1)}. \tag{2.10}$$

Obviously, if n is odd, then $h_{A_1} = h_{A_2}$. If n is even, then $h_{A_1} \neq h_{A_2}$. Then $h_O = H(0,0) = 0$ whether m is odd number or even number.

3. Exact Solutions of Explicit Type, Implicit Type, and Parametric Type and Their Properties

3.1. Exact Solutions and Their Properties of (1.1) *under* $h = h_O$

Taking $h = h_O = 0$, (2.6) can be reduced to

$$y^2 = \frac{(2v/(n+1))\phi^{n+1} - (2\sigma/(n+m))\phi^{n+m}}{n\phi^{2n-2}}. \tag{3.1}$$

(i) When $m = n > 1$, (3.1) can be rewritten as

$$y = \pm\frac{\sqrt{(2nv/(n+1))\phi^{n+1} - \sigma\phi^{2n}}}{n\phi^{n-1}}. \tag{3.2}$$

Substituting (3.2) into the first expression in (2.5) yields

$$\frac{d\phi}{d\tau} = \pm\phi\sqrt{\frac{2nv}{n+1}\phi^{n-1} - \sigma\left(\phi^{n-1}\right)^2}. \tag{3.3}$$

Noticing that equation $(2nv/(n+1))\phi^{n-1} - \sigma\left(\phi^{n-1}\right)^2 = 0$ has two roots $\phi = 0$ and $\phi = [2nv/(n+1)\sigma]^{1/(n-1)}$, we take $([2nv/(n+1)\sigma]^{1/(n-1)}, 0)$ as the initial value. Using this initial value, integrating (3.2) yields

$$\int_{[2nv/(n+1)\,\sigma]^{1/(n+1)}}^{\phi} \frac{d\phi}{\phi\sqrt{(2nv/(n+1))\phi^{n-1} - \sigma\left(\phi^{n-1}\right)^2}} = \pm\int_0^{\tau} d\tau. \tag{3.4}$$

After completing the aforementioned integral, we solve this equation; thus we obtain

$$\phi = \left[\frac{2n(n+1)v}{n^2(n-1)^2 v^2\tau^2 + (n+1)^2\sigma}\right]^{1/(n-1)}. \tag{3.5}$$

Substituting (3.5) into (2.4), then integrating it yields

$$\begin{aligned} \xi &= \frac{2n}{(n-1)\sqrt{\sigma}} \arctan\left[\frac{n(n-1)v}{(n+1)\sqrt{\sigma}}\tau\right], \quad \sigma > 0, \\ \xi &= -\frac{2n}{(n-1)\sqrt{-\sigma}} \tanh^{-1}\left[\frac{n(n-1)v}{(n+1)\sqrt{-\sigma}}\tau\right], \quad \sigma < 0. \end{aligned} \tag{3.6}$$

Thus, we respectively obtain a periodic wave solution and solitary wave solution of parametric type for the equation $K(n,n)$ as follows:

$$\begin{aligned} u = \phi(\tau) &= \left[\frac{2n(n+1)v}{n^2(n-1)^2 v^2\tau^2 + (n+1)^2\sigma}\right]^{1/(n-1)}, \\ \xi &= \frac{2n}{(n-1)\sqrt{\sigma}} \arctan\left[\frac{n(n-1)v}{(n+1)\sqrt{\sigma}}\tau\right], \quad \sigma > 0, \end{aligned} \tag{3.7}$$

$$\begin{aligned} u = \phi(\tau) &= \left[\frac{2n(n+1)v}{n^2(n-1)^2 v^2\tau^2 + (n+1)^2\sigma}\right]^{1/(n-1)}, \\ \xi &= -\frac{2n}{(n-1)\sqrt{-\sigma}} \tanh^{-1}\left[\frac{n(n-1)v}{(n+1)\sqrt{-\sigma}}\tau\right], \quad \sigma < 0. \end{aligned} \tag{3.8}$$

On the other hand, (3.1) can be rewritten as

$$y = \pm \frac{\sqrt{(2nv/(n+1))\phi^{n-1} - \sigma\phi^{2(n-1)}}}{n\phi^{n-2}}. \tag{3.9}$$

Using $([2nv/(n+1)\sigma]^{1/(n-1)}, 0)$ as the initial value, substituting (3.9) into the first expression in (2.3) directly, we obtain an integral equation as follows:

$$\int_{[2nv/(n+1)\sigma]^{1/(n-1)}}^{\phi} \frac{n\phi^{n-2}d\phi}{\sqrt{(2nv/(n+1))\phi^{n-1} - \sigma\phi^{2(n-1)}}} = \pm\int_0^{\xi} d\xi. \tag{3.10}$$

Completing the aforementioned integral equation, then solving it, we obtain a periodic solution and a hyperbolic function solution as follows:

$$u(x,t) = \phi(\xi) = \left[\frac{2nv}{(n+1)\sigma}\cos^2\frac{(n-1)\sqrt{\sigma}}{2n}\xi\right]^{1/(n-1)}, \quad \sigma > 0, \tag{3.11}$$

$$u(x,t) = \phi(\xi) = \left[\frac{2nv}{(n+1)\sigma}\cosh^2\frac{(n-1)\sqrt{-\sigma}}{2n}\xi\right]^{1/(n-1)}, \quad \sigma < 0. \tag{3.12}$$

Obviously, the solution (3.7) is equal to the solution (3.11); also the solution (3.8) is equal to the solution (3.12). Similarly, taking the $(0,0)$ as initial value, substituting (3.9) into the first expression in (2.3), then integrating them, we obtain another periodic solution and another hyperbolic function solution of $K(n,n)$ equation as follows.

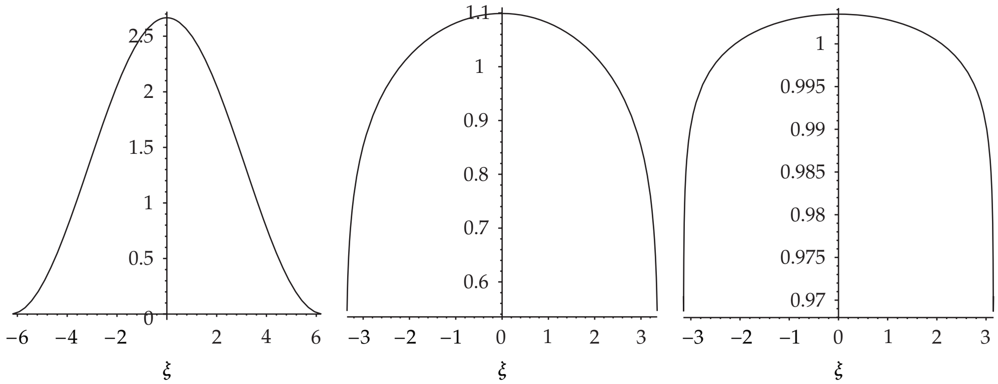
$$u(x,t) = \phi(\xi) = \left[\frac{2nv}{(n+1)\sigma}\sin^2\frac{(n-1)\sqrt{\sigma}}{2n}\xi\right]^{1/(n-1)}, \quad \sigma > 0, \tag{3.13}$$

$$u(x,t) = \phi(\xi) = \left[\frac{2nv}{(n+1)\sigma}\sinh^2\frac{(n-1)\sqrt{-\sigma}}{2n}\xi\right]^{1/(n-1)}, \quad \sigma < 0. \tag{3.14}$$

In fact, the solutions (3.11) and (3.13) have been appeared in [35], so we do not list similar solutions anymore at here. Next, we discuss a interesting problem as follows.

When $\sigma > 0$, from (3.11) and (3.13), we can construct two compacton solutions as follows:

$$\begin{cases} u(x,t) = \phi(\xi) = \left[\dfrac{2nv}{(n+1)\sigma}\cos^2\dfrac{(n-1)\sqrt{\sigma}}{2n}\xi\right]^{1/(n-1)}, & \sigma > 0, \ -\dfrac{n\pi}{n-1} \le \xi \le \dfrac{n\pi}{n-1}, \\ 0, & \text{otherwise}, \end{cases} \tag{3.15}$$



(a) $n = 2$ (b) $n = 15$ (c) $n = 400$

Figure 1: The solution u in (3.15) shows a shape of compacton for parameters $v = 2$, and $\sigma = 1$.

$$\begin{cases} u(x,t) = \phi(\xi) = \left[\dfrac{2nv}{(n+1)\sigma}\sin^2\dfrac{(n-1)\sqrt{\sigma}}{2n}\xi\right]^{1/(n-1)}, & \sigma > 0,\ 0 \le \xi \le \dfrac{2n\pi}{n-1}, \\ 0, & \text{otherwise.} \end{cases} \tag{3.16}$$

The shape of compacton solutions (3.15) and (3.16) changes gradually as the value of parameter n increases. For example, when $n = 2, 15, 400$, respectively, the shapes of compacton solution (3.15) are shown in Figure 1.

(ii) When $n = 1,\ m > 1$, (3.1) can be directly reduced to

$$y = \pm\phi\sqrt{v - \frac{2\sigma}{m+1}\phi^{m-1}}. \tag{3.17}$$

Equation (3.17) is a nonsingular equation. Using $([2\sigma/(m+1)v]^{n-1}, 0)$ as initial value and then substituting (3.17) into the first expression in (2.3) directly, we obtain a smooth solitary wave solution and a periodic wave solution of $K(m,1)$ equation as follows:

$$u(x,t) = \phi(\xi) = \left[\frac{(m+1)v}{2\sigma}\operatorname{sech}^2\frac{(m-1)\sqrt{v}}{2}\xi\right]^{1/(m-1)}, \quad v > 0, \tag{3.18}$$

$$u(x,t) = \phi(\xi) = \left[\frac{(m+1)v}{2\sigma}\sec^2\frac{(m-1)\sqrt{-v}}{2}\xi\right]^{1/(m-1)}, \quad v < 0. \tag{3.19}$$

Also, the shape of solitary wave solution (3.18) changes gradually as the value of parameter m increases. When $m = 2, 20, 200$, respectively, its shapes of compacton solution (3.18) are shown in Figure 2.

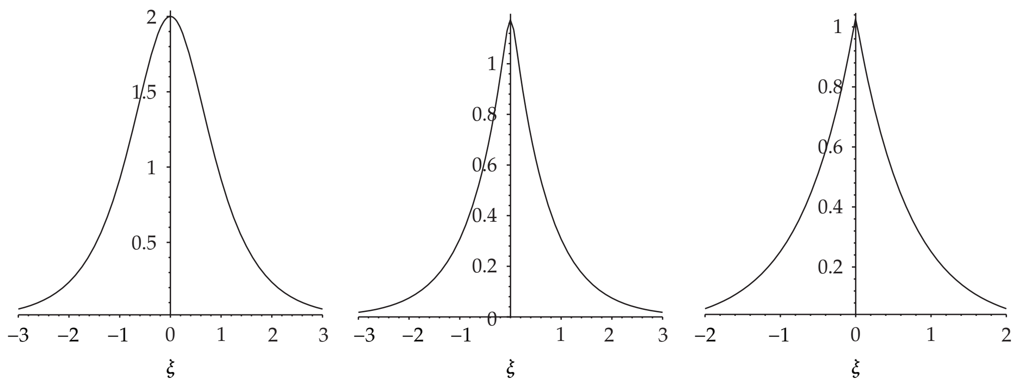


Figure 2: The solution u in (3.18) shows a shape of compacton for parameters $v = 2$, and $\sigma = 1$.

(iii) When n is even number and $m = 2n - 1$, (3.1) can be reduced to

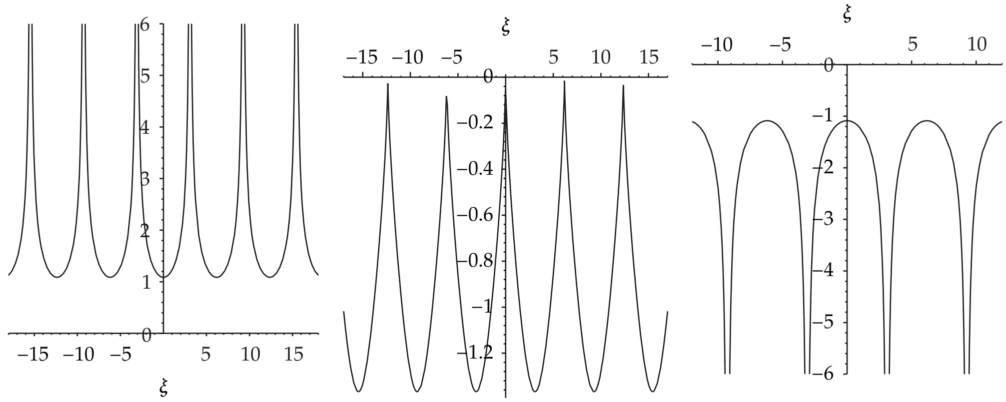
$$y = \pm \frac{\sqrt{(2nv/(n+1))\phi^{n-1} - (2n\sigma/(3n-1))\phi^{3(n-1)}}}{n\phi^{n-2}}. \tag{3.20}$$

It is easy to know that $(2nv/(n+1))\phi^{n-1} - (2n\sigma/(3n-1))\phi^{3(n-1)} = 0$ has three roots $\phi = 0$ and $\phi = \alpha, \gamma$ with $\alpha, \gamma = \pm[\sqrt{(3n-1)v/(n+1)\sigma}]^{1/(n-1)}$ when $\sigma v > 0$. In fact, $\gamma = -\alpha$. Using these three roots as initial value, respectively, then substituting (3.20) into the first expression in (2.3), we obtain three integral equations as follows:

$$\begin{aligned}
\int_\alpha^\phi \frac{n\phi^{n-2}d\phi}{\sqrt{(2nv/(n+1))\phi^{n-1} - (2n\sigma/(3n-1))\phi^{3(n-1)}}} &= \pm\int_0^\xi d\xi, \\
\int_\phi^0 \frac{n\phi^{n-2}d\phi}{\sqrt{(2nv/(n+1))\phi^{n-1} - (2n\sigma/(3n-1))\phi^{3(n-1)}}} &= \pm\int_0^\xi d\xi. \\
\int_\phi^\gamma \frac{n\phi^{n-2}d\phi}{\sqrt{(2nv/(n+1))\phi^{n-1} - (2n\sigma/(3n-1))\phi^{3(n-1)}}} &= \pm\int_0^\xi d\xi,
\end{aligned} \tag{3.21}$$

Completing the previous three integral equations, then solving them, we obtain three periodic solutions of Jacobian elliptic function for $K(2n-1, n)$ equation as follows:

$$u(x,t) = \phi(\xi) = \left[\alpha\, \mathrm{nc}^2\left(\frac{(n-1)\sqrt{2\alpha}}{2n}\xi, \frac{1}{\sqrt{2}}\right)\right]^{1/(n-1)}, \quad n = \text{even number}, \tag{3.22}$$



(a) Periodic blowup wave (b) Periodic cusp wave (c) Periodic blowup wave

Figure 3: Three periodic waves of solutions (3.22), (3.23), and (3.24) for parameters $n = 4$, $v = 2$, and $\sigma = 1$.

$$u(x,t) = \phi(\xi) = \left[-\frac{\alpha\, \mathrm{sn}^2\left(\left(\left((n-1)\sqrt{2\alpha}\right)/2n\right)\, \xi, 1/\sqrt{2}\right)}{2\mathrm{dn}^2\left(\left(\left((n-1)\sqrt{2\alpha}\right)/2n\right)\xi, 1/\sqrt{2}\right)} \right]^{1/(n-1)}, \quad n = \text{even number}, \tag{3.23}$$

$$u(x,t) = \phi(\xi) = \left[\gamma\ \mathrm{nc}^2\left(\left(\left((n-1)\sqrt{2\alpha}\right)/2n\right)\ \xi, 1/\sqrt{2}\right)\right]^{1/(n-1)}, \quad n = \text{even number}. \tag{3.24}$$

The solutions (3.22) and (3.24) show two shapes of periodic wave with blowup form, which are shown in Figures 3(a) and 3(c). The solution (3.23) shows a shape of periodic cusp wave, which is shown in Figure 3(b).

(iv) When $m = 3n - 2,\ n > 1$, (3.1) can be directly reduced to

$$y = \pm \frac{\sqrt{(2nv/(n+1))\phi^{n-1} - (2n\sigma/(4n-2))\left(\phi^{n-1}\right)^4}}{n\phi^{n-2}}. \tag{3.25}$$

It is easy to know that the function $(2n\sigma/(4n-2))(a - \phi^{n-1})(\phi^{n-1} - 0)(\phi^{n-1} - c)(\phi^{n-1} - \overline{c}) = (2n\sigma/(4n-2))(a - \phi^{n-1})(\phi^{n-1} - 0)[(\phi^{n-1} - b_1)^2 + a_1^2]$, where $b_1 = (c + \overline{c})/2 = -a/2$, $a_1^2 = -(c - \overline{c})^2/4 = 3a^2/4$. Using $(a^{1/(n-1)}, 0)$ and $(0, 0)$ as initial values, respectively, substituting (3.25) into the first expression in (2.3), we obtain four elliptic integral equations as follows.

(1) When $\sigma > 0,\ v > 0$,

$$\int_0^{\phi} \frac{d\phi^{n-1}}{\sqrt{(a-\phi^{n-1})(\phi^{n-1}-0)\left[(\phi^{n-1}-b_1)^2+a_1^2\right]}} = \pm\frac{n-1}{n}\sqrt{\frac{2n\sigma}{4n-2}}\int_0^{\xi} d\xi. \tag{3.26}$$

(2) When $\sigma > 0,\ v < 0$,

$$\int_{a^{1/(n-1)}}^{\phi} \frac{d\phi^{n-1}}{\sqrt{(\phi^{n-1}-a)(\phi^{n-1}-0)\left[(\phi^{n-1}-b_1)^2+a_1^2\right]}} = \pm\frac{n-1}{n}\sqrt{\frac{2n\sigma}{4n-2}}\int_0^{\xi} d\xi. \tag{3.27}$$

(3) When $\sigma < 0,\ v < 0$,

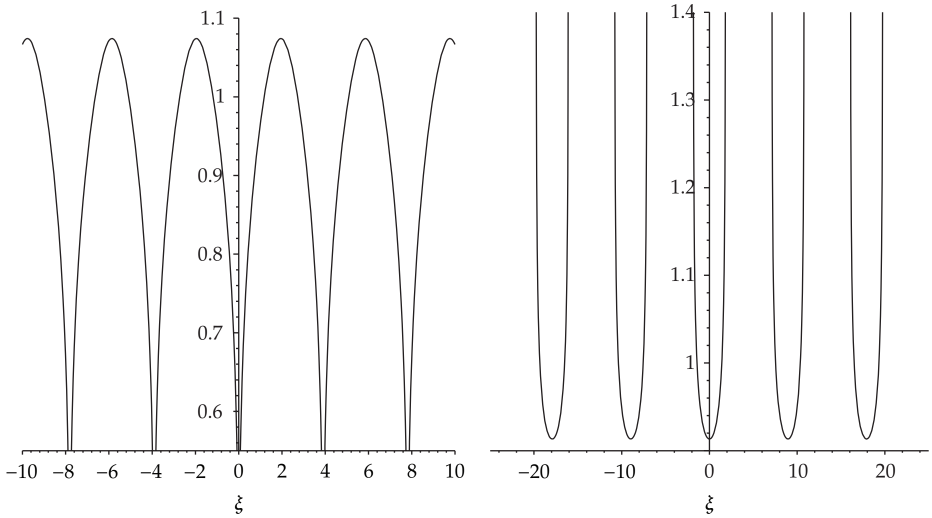
$$\int_0^{\phi} \frac{d\phi^{n-1}}{\sqrt{(a-\phi^{n-1})(\phi^{n-1}-0)\left[(\phi^{n-1}-b_1)^2+a_1^2\right]}} = \pm\frac{n-1}{n}\sqrt{\frac{2n\sigma}{2-4n}}\int_0^{\xi} d\xi. \tag{3.28}$$

(4) When $\sigma < 0,\ v > 0$,

$$\int_{a^{1/(n-1)}}^{\phi} \frac{d\phi^{n-1}}{\sqrt{(\phi^{n-1}-a)(\phi^{n-1}-0)\left[(\phi^{n-1}-b_1)^2+a_1^2\right]}} = \pm\frac{n-1}{n}\sqrt{\frac{2n\sigma}{2-4n}}\int_0^{\xi} d\xi. \tag{3.29}$$

Corresponding to (3.26), (3.27), (3.28), and (3.29), respectively, we obtain four periodic solutions of elliptic function type for $K(3n-2,n)$ equation as follows:

$$\begin{aligned} u(x,t) &= \phi(\xi) \\ &= \left[\frac{aB\left[1-\operatorname{cn}\left(((n-1)/gn)\sqrt{2n\sigma/(4n-2)}\xi,\sqrt{6}\left(3-\sqrt{3}\right)/12\right)\right]}{A+B+(A-B)\operatorname{cn}\left(((n-1)/gn)\sqrt{2n\sigma/(4n-2)}\xi,\sqrt{6}\left(3-\sqrt{3}\right)/12\right)}\right]^{1/(n-1)}, \end{aligned} \tag{3.30}$$



(a) $n = 10$, $\sigma = 1$, $v = 2$ (b) $n = 9$, $\sigma = 3$, $v = 0.1$

Figure 4: Two different periodic waves on solutions (3.30) and (3.31) for given parameters.

$$u(x,t) = \phi(\xi) = \left[\frac{aB\left[1 + \mathrm{cn}\left(((n-1)/gn)\sqrt{2n\sigma/(4n-2)}\xi, \left(\sqrt{6}-\sqrt{2}\right)/4\right)\right]}{B - A + (A+B)\mathrm{cn}\left(((n-1)/gn)\sqrt{2n\sigma/(4n-2)}\xi, \left(\sqrt{6}-\sqrt{2}\right)/4\right)}\right]^{1/(n-1)}, \tag{3.31}$$

$$u(x,t) = \phi(\xi) = \left[\frac{aB\left[1 - \mathrm{cn}\left(((n-1)/gn)\sqrt{2n\sigma/(2n-4)}\xi, \sqrt{6}\left(3-\sqrt{3}\right)/12\right)\right]}{A + B + (A-B)\mathrm{cn}\left(((n-1)/gn)\sqrt{2n\sigma/(2n-4)}\xi, \sqrt{6}\left(3-\sqrt{3}\right)/12\right)}\right]^{1/(n-1)}, \tag{3.32}$$

$$u(x,t) = \phi(\xi) = \left[\frac{aB\left[1 + \mathrm{cn}\left(((n-1)/gn)\sqrt{2n\sigma/(2-4n)}\xi, \left(\sqrt{6}-\sqrt{2}\right)/4\right)\right]}{B - A + (A+B)\mathrm{cn}\left(((n-1)/gn)\sqrt{2n\sigma/(2-4n)}\xi, \left(\sqrt{6}-\sqrt{2}\right)/4\right)}\right]^{1/(n-1)}, \tag{3.33}$$

where $A = \sqrt{(a-b_1)^2 + a_1^2} = \sqrt{3}a$, $B = \sqrt{(0-b_1)^2 + a_1^2} = a$, and $g = 1/\sqrt{AB} = \sqrt[4]{27}/3a$ with $a = \sqrt[3]{((4n-2)v)/((n+1)\sigma)}$ given previously.

The solution (3.30) shows a shape of periodic wave with blowup form, which is shown in Figure 4(a). The solution (3.31) shows s shape of compacton-like periodic wave, which is shown in Figure 4(b). The profile of solution (3.32) is similar to that of solution (3.30). Also the profile of solution (3.33) is similar to that of solution (3.31). So we omit the graphs of their profiles here.

(v) When $m = (k-1)n - k + 2,\ n > 1,\ k > 4$, (3.1) can be directly reduced to

$$y = \pm \frac{\sqrt{(2nv/(n+1))\phi^{n-1} - (2n\sigma/(k(n-1)+2))\phi^{k(n-1)}}}{n\phi^{n-2}}. \tag{3.34}$$

Suppose that $\phi_0 = \phi(0)$ is one of roots for equation $(2nv/(n+1))\phi^{n-1} - (2n\sigma/(k(n-1)+2))\phi^{k(n-1)} = 0$. Clearly, the 0 is its one root. Anyone solution of $K((k-1)n-k+2,\ n)$ equation can be obtained theoretically from the following integral equations:

$$\int_{\phi_0}^{\phi} \frac{d\phi^{n-1}}{\sqrt{(2nv/(n+1))\phi^{n-1} - (2n\sigma/(k(n-1)+2))\left(\phi^{n-1}\right)^k}} = \pm\frac{n-1}{n}\ \xi. \tag{3.35}$$

The left integral of (3.35) is called hyperelliptic integral for ϕ^{n-1} when the degree k is greater than four. Let $\phi^{n-1} = z$. Thus, (3.35) can be reduced to

$$\int_{z_0{}^{1/(n-1)}}^{z} \frac{dz}{\sqrt{(2nv/(n+1))z - (2n\sigma/(k(n-1)+2))z^k}} = \pm\frac{n-1}{n}\ \xi. \tag{3.36}$$

In fact, we cannot obtain exact solutions by (3.36) when the degree k is grater than five. But we can obtain exact solutions by (3.36) when $k = 5$, $v = -\sigma(n+1)/k(n-1)+2$, and $\sigma < 0$. Under these particular conditions, taking $\phi_0 = z_0{}^{1/(n-1)} = 0$ as initial value, (3.36) becomes

$$\int_0^Z \frac{dz}{\sqrt{z+z^5}} = \pm\frac{n-1}{n}\sqrt{-\frac{\sigma(n+1)}{5n-3}}\ \xi. \tag{3.37}$$

Let $z = (1/2)[\rho - \sqrt{\rho^2-4}]$, and $z = (1+Z^2)/Z^2$. We obtain $-dz/z\sqrt{z} = (1/2)[1/\sqrt{\rho+2} + 1/\sqrt{\rho-2}]$ and $0 < Z \le 1$. Thus, (3.37) can be transformed to

$$\frac{1}{2}\left[\int_z^{\infty} \frac{d\rho}{\sqrt{(\rho+2)(\rho^2-2)}} + \int_z^{\infty} \frac{d\rho}{\sqrt{(\rho-2)(\rho^2-2)}}\right] = \pm\frac{n-1}{n}\sqrt{-\frac{\sigma(n+1)}{5n-3}}\ \xi. \tag{3.38}$$

Completing (3.38) and refunded the variable $z = \phi^{n-1}$, we obtain two implicit solutions of elliptic function type for $K(4n-3,\ n)$ equation as follows:

$$\mathrm{sn}^{-1}\left(\sqrt{\frac{\sqrt{2}+2}{\phi^{n-1}+2}}, \sqrt{\frac{2-\sqrt{2}}{2+\sqrt{2}}}\right) + \mathrm{sn}^{-1}\left(\sqrt{\frac{\sqrt{2}+2}{\phi^{n-1}+\sqrt{2}}}, \sqrt{\frac{2\sqrt{2}}{2+\sqrt{2}}}\right) = \Omega_{1,2}\ \xi, \tag{3.39}$$

where $\Omega_{1,2} = \pm((n-1)/n)(2+\sqrt{2})\sqrt{-\sigma(n+1)/(5n-3)}$. The solutions also can be rewritten as

$$F\left(\sin^{-1}\sqrt{\frac{\sqrt{2}+2}{\phi^{n-1}+2}}, \sqrt{\frac{2-\sqrt{2}}{2+\sqrt{2}}}\right) + F\left(\sin^{-1}\sqrt{\frac{\sqrt{2}+2}{\phi^{n-1}+\sqrt{2}}}, \sqrt{\frac{2\sqrt{2}}{2+\sqrt{2}}}\right) = \Omega_{1,2}\ \xi, \tag{3.40}$$

where the function $F(\varphi, k) = \text{Elliptic}\ F(\varphi, k)$ is the incomplete Elliptic integral of the first kind.

The two solutions in (3.40) are asymptotically stable. Under $\Omega_1 = ((n-1)/n)(2+\sqrt{2})\sqrt{-\sigma(n+1)/(5n-3)}, \phi \to 0$ as $\xi \to \infty$. Under $\Omega_2 = -((n-1)/n)(2+\sqrt{2})\sqrt{-\sigma(n+1)/(5n-3)}, \phi \to 0$ as $\xi \to -\infty$. The graphs of their profiles are shown in Figure 5.

3.2. Exact Solutions and Their Properties of (1.1) *under* $h \neq 0$

In this subsection, under the conditions $h = h_{A_0}$, and $h = h_{A_1}$, $h = h_{A_2}$, we will investigate exact solutions of (1.1) and discuss their properties. When $h \neq 0$, (2.6) can be reduced to

$$y = \pm\frac{\sqrt{h + (2nv/(n+1))\phi^{n+1} - (2n\sigma/(n+m))\phi^{n+m}}}{n\phi^{n-1}}. \tag{3.41}$$

Substituting (3.41) into the first expression of (2.3) yields

$$\int_{\phi_*}^{\phi} \frac{d\phi^n}{\sqrt{h + (2nv/(n+1))\phi^{n+1} - (2n\sigma/(n+m))\phi^{n+m}}} = \pm\xi, \tag{3.42}$$

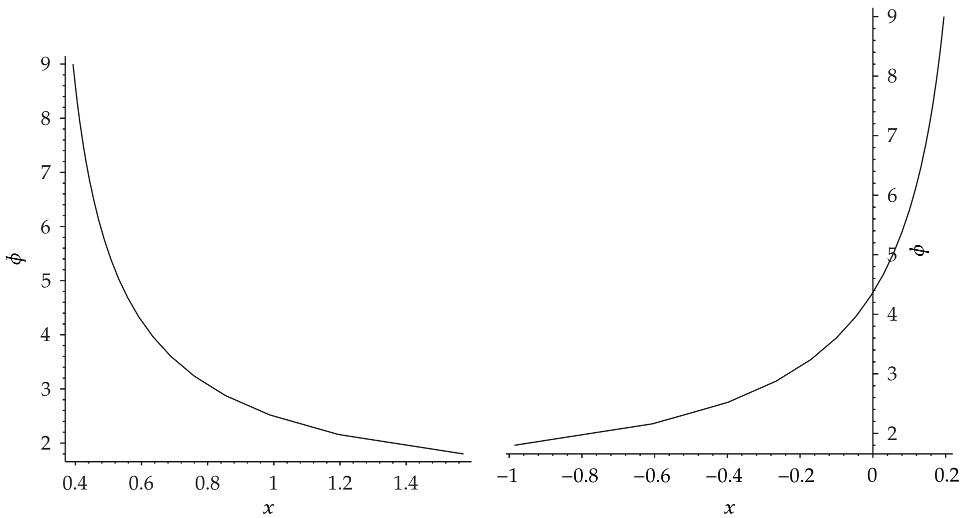
where ϕ_* is one of roots for equation $h + (2nv/(n+1))\phi^{n+1} - (2n\sigma/(n+m))\phi^{n+m} = 0$. However we cannot obtain any exact solutions by (3.42) when the degrees m and n are more great, because we cannot obtain coincidence relationship among different degrees n, $n+1$ and $n+m$. But, we can always obtain some exact solutions when the degree $m+n$ is not greater than four. For example, by using (3.42) directly, we can also obtain many exact solutions of $K(2,1)$ and $K(3,1)$ equations; see the next computation and discussion.

(i) If $m = n = 2$, then (3.41) can be reduced to

$$y = \pm\frac{\sqrt{h + (4v/3)\phi^3 - \sigma\phi^4}}{2\phi}. \tag{3.43}$$

Taking $h = h_{A_0}|_{m=n=2} = -v^4/6\sigma^3$ as Hamiltonian quantity, substituting (3.43) and $m = n = 2$ into the first expression of (2.5) yields

$$\frac{d\phi}{\sqrt{-(v^4/6\sigma^3) + (4v/3)\phi^3 - \sigma\phi^4}} = \pm d\tau. \tag{3.44}$$



(a) Asymptotically stable wave

(b) Asymptotically stable wave

Figure 5: Waveforms of two asymptotically stable solutions in (3.40) when $n = 4$, $\sigma = -1$, and $t = 1$.

Then $-(v^4/6\sigma^3) + (4v/3)\phi^3 - \sigma\phi^4 = 0$ has four roots, two real roots, and two complex roots as follows:

$$a, b = \frac{v}{\sigma}\left[\frac{1}{3} + \frac{\mu}{6} \pm \frac{1}{6}\sqrt{8 - 3\left(4 + 2\sqrt{2}\right)^{1/3} - 6\left(4 + 2\sqrt{2}\right)^{-1/3} + \frac{16}{\mu}}\,\right],$$
$$c, \overline{c} = \frac{v}{\sigma}\left[\frac{1}{3} - \frac{\mu}{6} \pm i\frac{1}{6}\sqrt{-8 + 3\left(4 + 2\sqrt{2}\right)^{1/3} + 6\left(4 + 2\sqrt{2}\right)^{-1/3} + \frac{16}{\mu}}\,\right], \tag{3.45}$$

with $\mu = \sqrt{4 + 3(4 + 2\sqrt{2})^{1/3} + 6(4 + 2\sqrt{2})^{-1/3}}$.

(1) When $\sigma > 0$ and $a > \phi > b$, taking b as initial value, then integrating (3.44) yields

$$\int_b^\phi \frac{d\phi}{(a - \phi)(\phi - b)(\phi - c)(\phi - \overline{c})} = \pm\sqrt{\sigma}\int_0^\tau d\tau. \tag{3.46}$$

Solving the aforementioned integral equation yields

$$\phi = \frac{aB + bA}{A + B}\left[\frac{1 + \alpha_1 \mathrm{cn}\left(\sqrt{AB\sigma}\ \tau,\ k\right)}{1 + \alpha \mathrm{cn}\left(\sqrt{AB\sigma}\ \tau,\ k\right)}\right], \tag{3.47}$$

where $\alpha_1 = (bA - aB)/(aB + bA)$, $\alpha = (A - B)/(A + B)$ and $k = (1/2)\sqrt{((a-b)^2 - (A-B)^2)/AB}$ with $A = \sqrt{(a - ((c + \bar{c})/2))^2 - ((c - \bar{c})^2/4)}$ and $B = \sqrt{(b - ((c + \bar{c})/2))^2 - ((c - \bar{c})^2/4)}$ Substituting (3.47) and $n = 2$ into (2.4) yields

$$\xi = \frac{2(aB + bA)}{(A + B)\sqrt{AB\sigma}}\left[\frac{\alpha_1}{\alpha}u_1 + \frac{\alpha - \alpha_1}{\alpha(1 - \alpha^2)}\left(\Pi\left(\varphi, \frac{\alpha^2}{\alpha^2 - 1}, k\right) - \alpha f_1\right)\right], \tag{3.48}$$

where $u_1 = \mathrm{sn}^{-1}(\sqrt{AB\sigma}\tau, k) = F(\varphi, k)$, $\varphi = \mathrm{am}\ u_1 = \arcsin(\sqrt{AB\sigma}\tau)$, $\alpha^2 \neq 1$, the $\Pi(\varphi, \alpha^2/(\alpha^2 - 1), k)$ is an elliptic integral of the third kind, and the function f_1 satisfies the following three cases, respectively:

$$\begin{aligned}
f_1 &= \sqrt{\frac{1 - \alpha^2}{k^2 + k'^2\alpha^2}}\arctan\left[\sqrt{\frac{k^2 + k'^2\alpha^2}{1 - \alpha^2}}\mathrm{sd}(\sqrt{AB\sigma}\ \tau, k)\right], \quad \text{if} \frac{\alpha^2}{(\alpha^2 - 1)} < k^2,\\
&= \mathrm{sd}(\sqrt{AB\sigma}\tau, k), \quad \text{if} \frac{\alpha^2}{(\alpha^2 - 1)} = k^2,\\
&= \frac{1}{2}\sqrt{\frac{\alpha^2 - 1}{k^2 + k'^2\alpha^2}}\\
&\quad \times \ln\left[\frac{\sqrt{k^2 + k'^2\alpha^2}\mathrm{dn}(\sqrt{AB\sigma}\tau, k) + \sqrt{\alpha^2 - 1}\mathrm{sn}(\sqrt{AB\sigma}\tau, k)}{\sqrt{k^2 + k'^2\alpha^2}\mathrm{dn}(\sqrt{AB\sigma}\tau, k) - \sqrt{\alpha^2 - 1}\mathrm{sn}(\sqrt{AB\sigma}\tau, k)}\right], \quad \text{if} \ \frac{\alpha^2}{\alpha^2 - 1} > k^2.
\end{aligned}$$

In the previous three cases, $k'^2 = 1 - k^2$. Thus, by using (3.47) and (3.48), we obtain a parametric solution of Jacobian elliptic function for $K(2, 2)$ equation as follows:

$$\begin{aligned}
\phi &= \frac{aB + bA}{A + B}\left[\frac{1 + \alpha_1 \mathrm{cn}\left(\sqrt{AB\sigma}\ \tau,\ k\right)}{1 + \alpha \mathrm{cn}\left(\sqrt{AB\sigma}\ \tau,\ k\right)}\right],\\
\xi &= \frac{2(aB + bA)}{(A + B)\sqrt{AB\sigma}}\left[\frac{\alpha_1}{\alpha}u_1 + \frac{\alpha - \alpha_1}{\alpha(1 - \alpha^2)}\left(\Pi\left(\varphi, \frac{\alpha^2}{\alpha^2 - 1}, k\right) - \alpha f_1\right)\right].
\end{aligned} \tag{3.49}$$

(2) When $\sigma < 0$ and $b < a < \phi < \infty$, taking a as initial value, integrating (3.44) yields

$$\int_a^{\phi} \frac{d\phi}{(\phi - a)(\phi - b)(\phi - c)(\phi - \bar{c})} = \pm\sqrt{-\sigma}\int_0^{\tau} d\tau. \tag{3.50}$$

Solving the aforementioned integral equation yields

$$\phi = \frac{aB - bA}{B - A}\left[\frac{1 + \tilde{\alpha}_1 \mathrm{cn}\left(\sqrt{-AB\sigma}\ \tau, \tilde{k}\right)}{1 + \tilde{\alpha}\mathrm{cn}\left(\sqrt{-AB\sigma}\ \tau, \tilde{k}\right)}\right], \tag{3.51}$$

where $\tilde{\alpha}_1 = (aB + bA)/(aB - bA)$, $\tilde{\alpha} = (A + B)/(B - A)$, $\tilde{k} = (1/2)\sqrt{(A + B)^2\ (a - b)^2/AB}$, and A and B are given in case (1). Substituting (3.51) and $n = 2$ into (2.4) yields

$$\begin{aligned} \xi &= \frac{aB - bA}{(B - A)\sqrt{-AB\sigma}} \\ &\quad \times \left[\frac{\tilde{\alpha}_1}{\tilde{\alpha}}\tilde{u}_1 + \frac{\tilde{\alpha} - \tilde{\alpha}_1}{\tilde{\alpha}(1 - \tilde{\alpha}^2)}\left(\Pi\left(\tilde{\varphi}, \frac{\tilde{\alpha}^2}{\tilde{\alpha}^2 - 1}, \tilde{k}\right) - \tilde{\alpha}\tilde{f}_1\right)\right], \end{aligned} \tag{3.52}$$

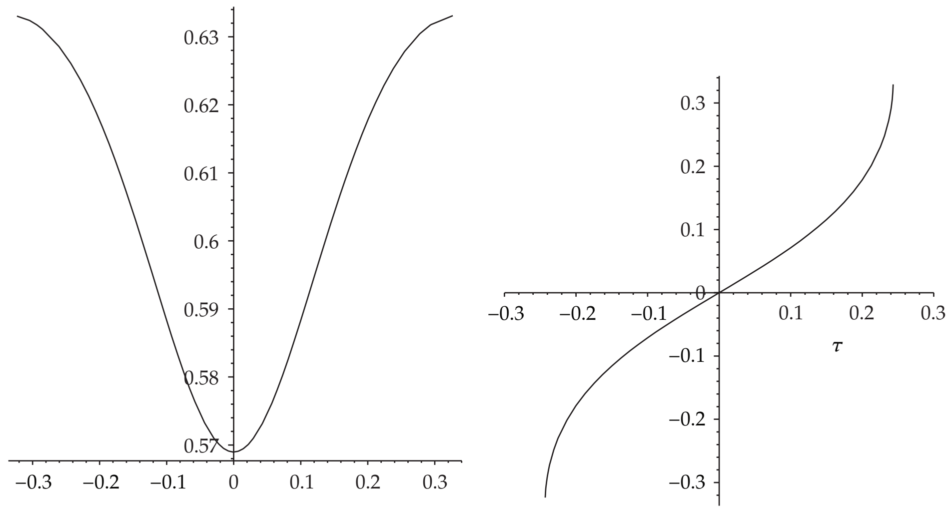
where $\tilde{u}_1 = \mathrm{sn}^{-1}(\sqrt{-AB\sigma}\tau, \tilde{k}) = F(\tilde{\varphi}, k)$, $\tilde{\varphi} = \mathrm{am}\ \tilde{u}_1 = \arcsin(\sqrt{-AB\sigma}\tau)$, $\tilde{\alpha}^2 \neq 1$, $\Pi(\tilde{\varphi}, \tilde{\alpha}^2/(\tilde{\alpha}^2 - 1), \tilde{k})$ is an elliptic integral of the third kind, and the function $\tilde{f}_1$ satisfies the following three cases, respectively:

$$\begin{aligned} \tilde{f}_1 &= \sqrt{\frac{1 - \tilde{\alpha}^2}{\tilde{k}^2 + \tilde{k}'^2\tilde{\alpha}^2}}\arctan\left[\sqrt{\frac{\tilde{k}^2 + \tilde{k}'^2\tilde{\alpha}^2}{1 - \tilde{\alpha}^2}}\mathrm{sd}(\sqrt{-AB\sigma}\ \tau, \tilde{k})\right], \quad \text{if} \frac{\tilde{\alpha}^2}{(\tilde{\alpha}^2 - 1)} < \tilde{k}^2, \\ &= \mathrm{sd}(\sqrt{-AB\sigma}\tau, \tilde{k}), \quad \text{if} \frac{\tilde{\alpha}^2}{(\tilde{\alpha}^2 - 1)} = \tilde{k}^2, \\ &= \frac{1}{2}\sqrt{\frac{\tilde{\alpha}^2 - 1}{\tilde{k}^2 + \tilde{k}'^2\tilde{\alpha}^2}} \\ &\quad \times \ln\left[\frac{\sqrt{\tilde{k}^2 + \tilde{k}'^2\tilde{\alpha}^2}\mathrm{dn}(\sqrt{-AB\sigma}\tau, \tilde{k}) + \sqrt{\tilde{\alpha}^2 - 1}\mathrm{sn}(\sqrt{-AB\sigma}\tau, \tilde{k})}{\sqrt{\tilde{k}^2 + \tilde{k}'^2\alpha^2}\mathrm{dn}(\sqrt{-AB\sigma}\tau, \tilde{k}) - \sqrt{\tilde{\alpha}^2 - 1}\mathrm{sn}(\sqrt{-AB\sigma}\tau, \tilde{k})}\right], \quad \text{if} \frac{\tilde{\alpha}^2}{\tilde{\alpha}^2 - 1} > \tilde{k}^2. \end{aligned}$$

In the previous three cases, $\tilde{k}'^2 = 1 - \tilde{k}^2$. Thus, by using (3.51) and (3.52), we obtain another parametric solution of Jacobian elliptic function for $K(2, 2)$ equation as follows:

$$\begin{aligned} u = \phi &= \frac{aB - bA}{B - A}\left[\frac{1 + \tilde{\alpha}_1\mathrm{cn}\left(\sqrt{-AB\sigma}\ \tau,\ \tilde{k}\right)}{1 + \tilde{\alpha}\mathrm{cn}\left(\sqrt{-AB\sigma}\ \tau,\ \tilde{k}\right)}\right], \\ \xi &= \frac{aB - bA}{(B - A)\sqrt{-AB\sigma}}\left[\frac{\tilde{\alpha}_1}{\tilde{\alpha}}\tilde{u}_1 + \frac{\tilde{\alpha} - \tilde{\alpha}_1}{\tilde{\alpha}(1 - \tilde{\alpha}^2)}\left(\Pi\left(\tilde{\varphi}, \frac{\tilde{\alpha}^2}{\tilde{\alpha}^2 - 1}, \tilde{k}\right) - \tilde{\alpha}\tilde{f}_1\right)\right]. \end{aligned} \tag{3.53}$$

In addition, when $h < -v^4/6\sigma^3$, $h + (4v/3)\phi^3 - \sigma\phi^4 = 0$ has four complex roots; in this case, we cannot obtain any useful results for $K(2, 2)$ equation. When $h > -v^4/6\sigma^3$, the case is very



(a) Peculiar compacton wave (b) Bounded region of independent variable ξ

Figure 6: Peculiar compacton wave and its bounded region of independent variable ξ.

similar to (3.52); that is, the equation $h + (4v/3)\phi^3 - \sigma\phi^4 = 0$ has two real roots and two complex roots. So we omit the discussions for these parts of results.

In order to describe the dynamic properties of the traveling wave solutions (3.49) and (3.53) intuitively, as an example, we draw profile figure of solution (3.53) by using the software *Maple*, when $v = 4$, and $\sigma = -2$, see Figure 6(a).

Figure 6(a) shows a shape of peculiar compacton wave; its independent variable ξ is bounded region (i.e., $|\xi| < \alpha_1 + 1$); see Figure 6(b). From Figure 6(a), we find that its shape is very similar to that of the solitary wave, but it is not solitary wave because when $|\xi| \geq \alpha_1 + 1$, $u \equiv 0$. So, this is a new compacton.

(ii) Under $m = 2$, $n = 1$, taking $h = h_{A_0}\big|_{m=2,n=1} = -v^3/3\sigma^2$ as Hamiltonian quantity, (3.42) can be reduced to

$$\int_{\phi_*}^{\phi} \frac{d\phi}{\sqrt{-(v^3/3\sigma^2) + v\phi^2 - (2\sigma/3)\phi^3}} = \pm\xi, \tag{3.54}$$

where ϕ_* is one of roots for the equation $-(v^3/3\sigma^2) + v\phi^2 - (2\sigma/3)\phi^3 = 0$. Clearly, this equation has three real roots, one single root $-v/2\sigma$ and two double roots v/σ, v/σ. If $\sigma > 0$, then the function $\sqrt{-(v^3/3\sigma^2) + v\phi^2 - (2\sigma/3)\phi^3} = \sqrt{(2\sigma/3)}|\phi - (v/\sigma)|\sqrt{-(v/2\sigma) - \phi}$; if $\sigma < 0$, then the function $\sqrt{(-v^3/3\sigma^2) + v\phi^2 - (2\sigma/3)\phi^3} = \sqrt{-(2\sigma/3)}|\phi - (v/\sigma)|\sqrt{(v/2\sigma) + \phi}$. In these two conditions, taking $\phi_* = -(v/2\sigma)$ as initial value and completing the (3.54), we obtain a periodic solution and a solitary wave solutions for $K(2,1)$ as follows:

$$\begin{aligned} u(x,t) &= \phi(\xi) = -\left[\frac{v}{2\sigma} + \frac{3v}{2\sigma}\tan^2\left(\frac{1}{2}\sqrt{v}\xi\right)\right], \quad v > 0, \\ u(x,t) &= \phi(\xi) = -\left[\frac{v}{2\sigma} - \frac{3v}{2\sigma}\tanh^2\left(\frac{1}{2}\sqrt{-v}\xi\right)\right], \quad v < 0. \end{aligned} \tag{3.55}$$

Similarly, taking $\phi_* = v/\sigma$ as initial value, we obtain two periodic solutions for $K(2,1)$ as follows:

$$u(x,t) = \phi(\xi) = -\frac{v}{2\sigma} - \frac{3v}{2\sigma}\tan^2\left(\frac{\pi}{4} \pm \frac{1}{2}\sqrt{v}\xi\right), \quad v > 0. \tag{3.56}$$

(iii) Under $m = 2$, $n = 1$, taking arbitrary constant h as Hamiltonian quantity, (3.42) can be reduced to

$$\int_{\phi_*}^{\phi} \frac{d\phi}{\sqrt{-(2\sigma/3)(\phi^3 + p\phi^2 + q)}} = \pm\xi, \tag{3.57}$$

where $p = -3v/2\sigma$, $q = -3h/2\sigma$. Write $\Delta = (q^2/4) + (p^3/27) = (9v^4/64\sigma^4) - ((v^3 + 6h\sigma^2)^3/1728\sigma^9)$. It is easy to know that $\Delta = 0$ as $h = h_{A_0}|_{m=2,n=1} = -v^3/3\sigma^2$; this case is same as case (ii). So, we only discuss the case $\Delta < 0$ in the next.

When h, σ, and v satisfy $\Delta < 0$, $\phi^3 + p\phi^2 + q = 0$ has three real roots z_1, z_2, and z_3 such as $\sqrt{v/2\sigma}\cos(\theta/3)$, $\sqrt{v/2\sigma}\cos(\theta/3 + 2\pi/3)$, and $\sqrt{v/2\sigma}\cos(\theta/3 + 4\pi/3)$ with $\theta = \arccos[(3h/2\sigma)\sqrt{2\sigma^3/v^3}]$ and $v/\sigma > 0$. Under these conditions, taking the z_1, z_2, and z_3 as initial values replacing ϕ_*, respectively, (3.57) can be reduced to the following three integral equations:

$$\begin{aligned}
&\int_{z_1}^{\phi} \frac{d\phi}{(\phi - z_1)(\phi - z_2)(\phi - z_3)} = \pm\sqrt{-\frac{2\sigma}{3}}\xi \quad (\sigma < 0, z_3 < z_2 < z_1 < \phi < \infty),\\
&\int_{z_2}^{\phi} \frac{d\phi}{(z_1 - \phi)(\phi - z_2)(\phi - z_3)} = \pm\sqrt{\frac{2\sigma}{3}}\xi \quad (\sigma > 0,\ z_3 < z_2 < \phi < z_1),\\
&\int_{z_3}^{\phi} \frac{d\phi}{(z_1 - \phi)(\phi - z_2)(\phi - z_3)} = \pm\sqrt{\frac{2\sigma}{3}}\xi \quad (\sigma > 0,\ z_3 < \phi < z_2 < z_1).
\end{aligned} \tag{3.58}$$

Integrating the (3.58), then solving them, respectively, we obtain three periodic solutions of elliptic function type for $K(2,1)$ as follows:

$$u(x - vt) = \phi(\xi) = \frac{z_1 - z_2\mathrm{sn}^2(\omega_1\xi, k_1)}{\mathrm{cn}^2(\omega_1\xi, k_1)}, \tag{3.59}$$

$$u(x - vt) = \phi(\xi) = \frac{z_2 - z_3k_2^2\mathrm{sn}^2(\omega_2\xi, k_2)}{\mathrm{dn}^2(\omega_2\xi, k_2)}, \tag{3.60}$$

$$u(x - vt) = \phi(\xi) = z_3 + (z_2 - z_3)\mathrm{sn}^2(\omega_2\xi, k_1), \tag{3.61}$$

where $\omega_1 = (1/2)\sqrt{-(2\sigma/3)(z_1 - z_3)}$, $k_1 = \sqrt{(z_2 - z_3)/(z_1 - z_3)}$, $\omega_2 = (1/2)\sqrt{(2\sigma/3)(z_1 - z_3)}$, and $k_2 = \sqrt{(z_1 - z_2)/(z_1 - z_3)}$.

(iv) When $m = 3, n = 1$, taking the constant $h = h_{A_1} = h_{A_2}|_{m=3,n=1} = -v/2\sigma$ as Hamiltonian quantity, (3.42) can be reduced to

$$\int_{\phi_*}^{\phi} \frac{d\phi}{(v/\sigma) - \phi^2} = \pm\sqrt{-\frac{\sigma}{2}}\xi \quad (\sigma < 0,\ v < 0). \tag{3.62}$$

Clearly, $(v/\sigma) - \phi^2 = 0$ has two real roots $\sqrt{v/\sigma}$ and $-\sqrt{v/\sigma}$. Taking $\phi_* = (\sqrt{v/\sigma} + (-\sqrt{v/\sigma}))/2 = 0$ as initial value, solving (3.62), we obtain a kink wave solution and an antikink wave solution for $K(3,1)$ as follows:

$$u(x - vt) = \phi(\xi) = \pm\sqrt{\frac{v}{\sigma}} \tanh\left(\sqrt{-\frac{v}{2}}\xi\right), \tag{3.63}$$

where $v < 0$ shows that the waves defined by (3.63) are reverse traveling waves.

(v) Under $m = 3, n = 1$, taking arbitrary constant h as Hamiltonian quantity and $h \neq -(v^2/2\sigma)$, (3.42) can be reduced to

$$\int_{\phi_*}^{\phi} \frac{d\phi}{\sqrt{\phi^4 - (2v/\sigma)\phi^2 - (2h/\sigma)}} = \pm\sqrt{-\frac{\sigma}{2}}\xi \quad (\sigma < 0,\ v < 0), \tag{3.64}$$

or

$$\int_{\phi_*}^{\phi} \frac{d\phi}{\sqrt{-(\phi^4 - (2v/\sigma)\phi^2 - (2h/\sigma))}} = \pm\sqrt{\frac{\sigma}{2}}\xi \quad (\sigma > 0,\ v > 0). \tag{3.65}$$

Clearly, $\phi^4 - (2v/\sigma)\phi^2 - (2h/\sigma) = 0$ has four real roots $r_{1,2,3,4} = \pm\sqrt{v/\sigma \pm \sqrt{v^2/\sigma^2 + 2h/\sigma}}$ if $\sigma < 0$, $v < 0$, and $0 < h < -(v^2/2\sigma)$ or $\sigma > 0$, $v > 0$, and $-(v^2/2\sigma) < h < 0$; it has two real roots $s_{1,2} = \pm\sqrt{v/\sigma \pm \sqrt{v^2/\sigma^2 + 2h/\sigma}}$ and two complex roots $s, \overline{s} = \pm i\sqrt{|v/\sigma - \sqrt{v^2/\sigma^2 + 2h/\sigma}|}$ if $\sigma < 0$, $v < 0$, and $h < 0$ or $\sigma > 0$, $v > 0$, and $h > 0$; it has not any real roots if $\sigma < 0$, $v < 0$, and $h > -v^2/2\sigma$ or $\sigma > 0$, $v > 0$, and $h < -v^2/2\sigma$.

(1) Under the conditions $\sigma < 0$, $v < 0$, and $0 < h < -v^2/2\sigma$ or $\sigma > 0$, $v > 0$, and $-v^2/2\sigma < h < 0$, taking $\phi_* = r_1$ as an initial value, (3.64) and (3.65) can be reduced to

$$\begin{aligned} \int_{r_1}^{\phi} \frac{d\phi}{\sqrt{(\phi - r_1)(\phi - r_2)(\phi - r_3)(\phi - r_4)}} &= \pm\sqrt{-\frac{\sigma}{2}}\xi, \\ \int_{\phi}^{r_1} \frac{d\phi}{\sqrt{(r_1 - \phi)(\phi - r_2)(\phi - r_3)(\phi - r_4)}} &= \pm\sqrt{\frac{\sigma}{2}}\xi, \end{aligned} \tag{3.66}$$

where $r_1 > r_2 > r_3 > r_4$. Solving the integral equations (3.66), we obtain two periodic solutions of Jacobian elliptic function for $K(3,1)$ equation as follows:

$$u(x-vt) = \phi(\xi) = \frac{r_1(r_2-r_4) - r_2(r_1-r_4)\mathrm{sn}^2\left(\Omega_1\xi, \tilde{k}_1\right)}{r_2-r_4-(r_1-r_4)\mathrm{sn}^2\left(\Omega_1\xi, \tilde{k}_1\right)} \quad (\phi < r_1), \tag{3.67}$$

where $\Omega_1 = (1/2)\sqrt{-(\sigma/2)(r_1-r_3)(r_2-r_4)}$, $\tilde{k}_1 = \sqrt{(r_2-r_3)(r_1-r_4)/(r_1-r_3)(r_2-r_4)}$,

$$u(x-vt) = \phi(\xi) = \frac{r_1(r_2-r_4) + r_4(r_1-r_2)\mathrm{sn}^2\left(\Omega_2\xi, \tilde{k}_2\right)}{r_2-r_4-(r_1-r_2)\mathrm{sn}^2\left(\Omega_2\xi, \tilde{k}_2\right)} \quad (r_2 < \phi < r_1), \tag{3.68}$$

where $\Omega_2 = (1/2)\sqrt{(\sigma/2)(r_1-r_3)(r_2-r_4)}$, and $\tilde{k}_2 = \sqrt{(r_1-r_2)(r_3-r_4)/(r_1-r_3)(r_2-r_4)}$. The case for taking $\phi_* = r_2, r_3, r_4$ as initial values can be similarly discussed; here we omit these discussions because these results are very similar to the solutions (3.67) and (3.68).

(2) Under the conditions $\sigma < 0$, $v < 0$, and $h < 0$ or $\sigma > 0$, $v > 0$, and $h > 0$, respectively taking $\phi_* = s_1, s_2$ as initial value, (3.64) and (3.65) can be reduced to

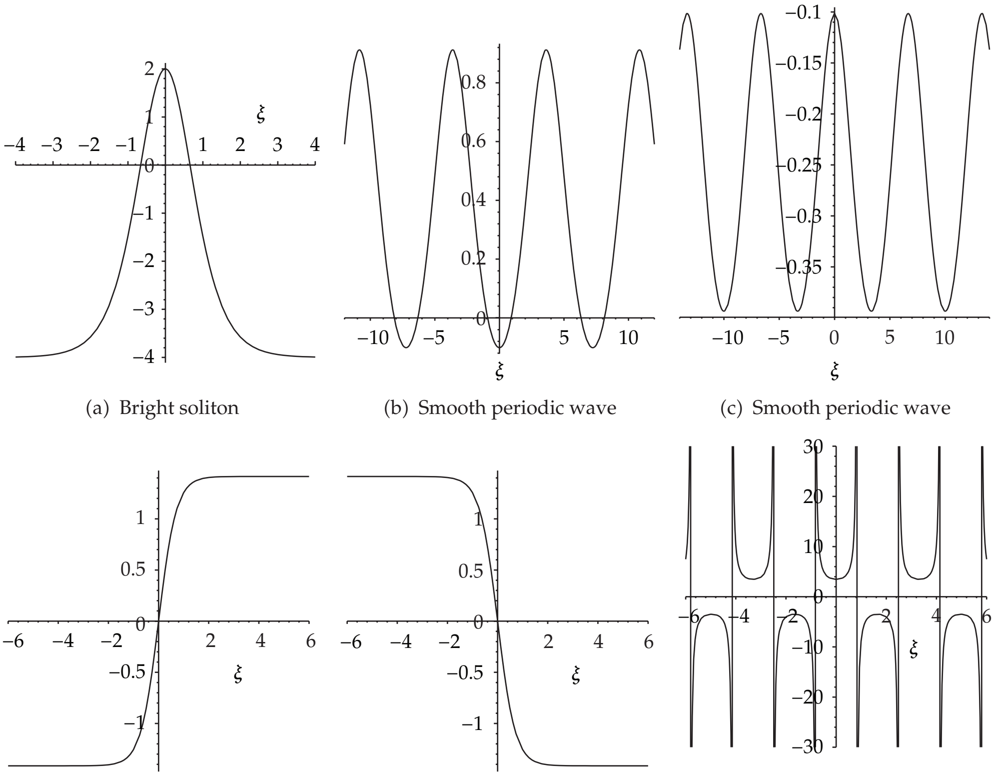
$$\begin{aligned} &\int_{s_1}^{\phi} \frac{d\phi}{\sqrt{(\phi-s_1)(\phi-s_2)(\phi-s)(\phi-\overline{s})}} = \pm\sqrt{-\frac{\sigma}{2}}\xi, \\ &\int_{s_2}^{\phi} \frac{d\phi}{\sqrt{(s_1-\phi)(\phi-s_2)(\phi-s)(\phi-\overline{s})}} = \pm\sqrt{\frac{\sigma}{2}}\xi. \end{aligned} \tag{3.69}$$

Solving the aforementioned two integral equations, we obtain two periodic solutions of Jacobian elliptic function for $K(3,1)$ equation as follows:

$$\begin{aligned} u(x-vt) = \phi(\xi) &= \frac{s_1\tilde{B} - s_2\tilde{A} + \left(s_1\tilde{B} + s_2\tilde{A}\right)\mathrm{cn}\left((1/\tilde{g})\ \sqrt{(-\sigma/2)}\xi, \tilde{k}_3\right)}{\tilde{B} - \tilde{A} + \left(\tilde{A} + \tilde{B}\right)\mathrm{cn}\left((1/\tilde{g})\ \sqrt{(-\sigma/2)}\xi, \tilde{k}_3\right)}, \\ u(x-vt) = \phi(\xi) &= \frac{s_1\tilde{B} + s_2\tilde{A} + \left(s_2\tilde{A} - s_1\tilde{B}\right)\mathrm{cn}\left((1/\tilde{g})\sqrt{(\sigma/2)}\xi, \tilde{k}_4\right)}{\tilde{B} + \tilde{A} + \left(\tilde{A} - \tilde{B}\right)\mathrm{cn}\left((1/\tilde{g})\sqrt{(\sigma/2)}\xi, \tilde{k}_4\right)}, \end{aligned} \tag{3.70}$$

where $\tilde{g} = (1/\sqrt{\tilde{A}\tilde{B}})$, $\tilde{k}_3 = \sqrt{((\tilde{A}+\tilde{B})^2 - (s_1-s_2)^2)/4\tilde{A}\tilde{B}}$, $\tilde{k}_4 = \sqrt{((s_1-s_2)^2 - (\tilde{A}-\tilde{B})^2)/4\tilde{A}\tilde{B}}$ with $\tilde{A} = \sqrt{(s_1-\tilde{b}_1)^2 + \tilde{a}_1^2}$, $\tilde{B} = \sqrt{(s_2-\tilde{b}_1)^2 + \tilde{a}_1^2}$, $\tilde{a}_1^2 = -(s-\overline{s})^2/4 = |v/\sigma - \sqrt{v^2/\sigma^2 + 2h/\sigma}|$, $\tilde{b}_1 = (s+\overline{s})/2 = 0$, and s_1 and s_2 are given previously.

Among these aforementioned solutions, (3.59) shows a shape of solitary wave for given parameters $v = 4$, and $\sigma = 1$ which is shown in Figure 7(a). Equation (3.60) shows a shape of smooth periodic wave for given parameters $v = 2$, $\sigma = 1$, and $h = 4$ which is shown in Figure 7(b). Also (3.61) shows a shape of smooth periodic wave for given parameters



(a) Bright soliton (b) Smooth periodic wave (c) Smooth periodic wave

(d) Antikink wave (e) Kink wave (f) Singular periodic wave

Figure 7: The graphs of six kinds of waveforms for solutions (3.59), (3.60), (3.61), (3.63), and (3.68).

$v = 2$, $\sigma = 1$, and $h = 0.4$ which is shown in Figure 7(c). Equation (3.63) shows two shapes of kink wave and antikink wave for given parameters $v = -4$, and $\sigma = -2$ which are shown in Figures 7(d)–7(e). Equation (3.68) shows a shape of singular periodic wave for given parameters $v = -10$, $\sigma = -1$, and $h = 48$ which is shown in Figure 7(f).

4. Conclusion

In this work, by using the integral bifurcation method, we study the nonlinear $K(m, n)$ equation for all possible values of m and n. Some travelling wave solutions such as normal compactons, peculiar compacton, smooth solitary waves, smooth periodic waves, periodic blowup waves, singular periodic waves, compacton-like periodic waves, asymptotically stable waves, and kink and antikink waves are obtained. In order to show their dynamic properties intuitively, the solutions of $K(n, n)$, $K(2n - 1, n)$, $K(3n - 2, n)$, $K(4n - 3, n)$, and $K(m, 1)$ equations are chosen to illustrate with the concrete features; using software *Maple*, we display their profiles by graphs; see Figures 1–7. These phenomena of traveling waves are different from those in existing literatures and they are very interesting. Although we do not know how they are relevant to the real physical or engineering problem for the moment, these interesting phenomena will attract us to study them further in the future works.

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