

Research Article

Multiplicity of Solutions for Perturbed Nonhomogeneous Neumann Problem through Orlicz-Sobolev Spaces

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We investigate the existence of multiple solutions for a class of nonhomogeneous Neumann problem with a perturbed term. By using variational methods and three critical point theorems of B. Ricceri, we establish some new sufficient conditions under which such a problem possesses three solutions in an appropriate Orlicz-Sobolev space.

1. Introduction

Consider the following nonhomogeneous Neumann problem with a perturbed term:

$$\begin{aligned} -\operatorname{div}(\alpha(|\nabla u|)\nabla u) + \alpha(|u|)u &= \lambda f(x, u) + \mu g(x, u), \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} &= 0, \quad \text{on } \partial\Omega, \end{aligned} \tag{P_{\lambda, \mu}}$$

where Ω is a bounded domain in \mathbb{R}^N ($N \geq 3$) with smooth boundary $\partial\Omega$, ν is the outer normal to $\partial\Omega$, $f, g : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ are two Carathéodory functions, $\lambda > 0, \mu \geq 0$ are two parameters, and the function $\alpha : (0, \infty) \rightarrow \mathbb{R}$ is such that $\varphi(t) : \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$\varphi(t) = \begin{cases} \alpha(|t|)t, & t \neq 0, \\ 0, & t = 0, \end{cases} \tag{1.1}$$

is an odd, strictly increasing homeomorphism from \mathbb{R} to \mathbb{R} .

It is well known that these kinds of problems are important in applications in many fields, such as elasticity, fluid dynamics, and image processing (see [1–4]). Since the operator in the divergence form is nonhomogeneous, we introduce Orlicz-Sobolev space which is an appropriate setting for these problems. Such space originated with Nakano [5] and was developed by Musielak and Orlicz [6]. Many properties of Sobolev spaces have been extended to Orlicz-Sobolev space (see [7–10]). Several authors have widely studied the existence of solutions for the relevant problem by means of variational techniques, monotone operator methods, fixed point, and degree theory (see [11–15]). To the best of our knowledge, for the perturbed nonhomogeneous Neumann problem, there has so far been few papers concerning its multiple solutions. Motivated by the above facts, in this paper, we establish some new sufficient conditions under which such a problem possesses three weak solutions in Orlicz-Sobolev space.

This paper is organized as follows. In Section 2, some preliminaries are presented. In Section 3, we discuss the existence of three weak solutions for problem $(P_{\lambda,\mu})$.

2. Preliminaries

We start by recalling some basic facts about Orlicz-Sobolev space. Let φ be as in Introduction and $\Phi(t) : \mathbb{R} \rightarrow \mathbb{R}$,

$$\Phi(t) = \int_0^t \varphi(s) ds. \quad (2.1)$$

We observe that Φ is, a Young function, that is $\Phi(0) = 0$, Φ is convex and $\lim_{t \rightarrow \infty} \Phi(t) = +\infty$. Furthermore, since $\Phi(t) = 0$ if and only if $t = 0$, $\lim_{t \rightarrow 0} (\Phi(t)/t) = 0$, and $\lim_{t \rightarrow \infty} (\Phi(t)/t) = +\infty$, then Φ is called an N -function. The function Φ^* is called the complementary function of Φ and it satisfies

$$\Phi^*(t) = \sup\{st - \Phi(s); s \geq 0\}, \quad \forall t \geq 0. \quad (2.2)$$

Assume that Φ satisfies the following structural hypotheses

$$(\Phi_1) \quad 1 < \liminf_{t \rightarrow \infty} (t\varphi(t)/\Phi(t)) \leq p^0 := \sup_{t>0} (t\varphi(t)/\Phi(t)) < \infty;$$

$$(\Phi_2) \quad N < p_0 := \inf_{t>0} (t\varphi(t)/\Phi(t)) < \liminf_{t \rightarrow \infty} (\log(\Phi(t))/\log(t)).$$

Further, we also assume that the function

$$(\Phi_3) \quad [0, \infty) \ni t \rightarrow \Phi(\sqrt{t}) \text{ is convex.}$$

The Orlicz space $L_\Phi(\Omega)$ defined by Φ is the space of measurable functions $u : \Omega \rightarrow \mathbb{R}$ such that

$$\|u\|_{L_\Phi} := \sup \left\{ \int_\Omega u(x)v(x) dx; \int_\Omega \Phi^*(|v(x)|) dx \leq 1 \right\} < \infty. \quad (2.3)$$

Then $(L_\Phi(\Omega), \|\cdot\|_{L_\Phi})$ is a Banach space whose norm is equivalent to the Luxemburg norm

$$\|u\|_\Phi := \inf \left\{ k > 0; \int_\Omega \Phi\left(\frac{|u(x)|}{k}\right) dx \leq 1 \right\}. \quad (2.4)$$

We denote by $W^1L_\Phi(\Omega)$ the Orlicz-Sobolev space, defined by

$$W^1L_\Phi(\Omega) = \left\{ u \in L_\Phi; \frac{\partial u}{\partial x_i} \in L_\Phi, i = 1, 2, \dots, N \right\}. \quad (2.5)$$

This is a Banach space with respect to the norm

$$\|u\|_{1,\Phi} = \|\nabla u\|_\Phi + \|u\|_\Phi. \quad (2.6)$$

Lemma 2.1 (see [13]). *On $W^1L_\Phi(\Omega)$ the norms*

$$\begin{aligned} \|u\|_{1,\Phi} &= \|\nabla u\|_\Phi + \|u\|_\Phi, \\ \|u\|_{2,\Phi} &= \max\{\|\nabla u\|_\Phi, \|u\|_\Phi\}, \\ \|u\| &:= \inf \left\{ \mu > 0; \int_\Omega \left[\Phi\left(\frac{|u(x)|}{\mu}\right) + \Phi\left(\frac{|\nabla u(x)|}{\mu}\right) \right] dx \leq 1 \right\}, \end{aligned} \quad (2.7)$$

are equivalent. Moreover, for every $u \in W^1L_\Phi(\Omega)$, one has

$$\|u\| \leq 2\|u\|_{2,\Phi} \leq 2\|u\|_{1,\Phi} \leq 4\|u\|. \quad (2.8)$$

Lemma 2.2. *Let $u \in W^1L_\Phi(\Omega)$, then*

$$\begin{aligned} \|u\|^{p^0} &\leq \int_\Omega [\Phi(|u(x)|) + \Phi(|\nabla u(x)|)] dx \leq \|u\|^{p^0}, \quad \text{if } \|u\| > 1, \\ \|u\|^{p^0} &\leq \int_\Omega [\Phi(|u(x)|) + \Phi(|\nabla u(x)|)] dx \leq \|u\|^{p^0}, \quad \text{if } \|u\| < 1. \end{aligned} \quad (2.9)$$

Proof. For the proof of

$$\begin{aligned} \|u\|^{p^0} &\leq \int_\Omega [\Phi(|u(x)|) + \Phi(|\nabla u(x)|)] dx, \quad \text{if } \|u\| > 1, \\ \|u\|^{p^0} &\leq \int_\Omega [\Phi(|u(x)|) + \Phi(|\nabla u(x)|)] dx, \quad \text{if } \|u\| < 1, \end{aligned} \quad (2.10)$$

we can see Lemma 2.2 of the paper [13]. Since $p^0 \geq (t\varphi(t))/\Phi(t)$ for all $t \geq 0$, it follows that letting $\sigma > 1$, we have

$$\log(\Phi(\sigma t)) - \log(\Phi(t)) = \int_t^{\sigma t} \frac{\varphi(s)}{\Phi(s)} ds \leq \int_t^{\sigma t} \frac{p^0}{s} ds = \log(\sigma^{p^0}). \quad (2.11)$$

Thus, one has

$$\Phi(\sigma t) \leq \sigma^{p^0} \Phi(t), \quad t > 0, \sigma > 1. \quad (2.12)$$

Moreover, by the definition of the norm, we remark that

$$\int_{\Omega} \left[\Phi\left(\frac{|u(x)|}{\|u\|}\right) + \Phi\left(\frac{|\nabla u(x)|}{\|u\|}\right) \right] dx \leq 1. \quad (2.13)$$

Therefore, we have

$$\begin{aligned} \int_{\Omega} [\Phi(|u(x)|) + \Phi(|\nabla u(x)|)] dx &= \int_{\Omega} \left[\Phi\left(\|u\| \frac{|u(x)|}{\|u\|}\right) + \Phi\left(\|u\| \frac{|\nabla u(x)|}{\|u\|}\right) \right] dx \\ &\leq \|u\|^{p_0} \int_{\Omega} \left[\Phi\left(\frac{|u(x)|}{\|u\|}\right) + \Phi\left(\frac{|\nabla u(x)|}{\|u\|}\right) \right] dx \\ &\leq \|u\|^{p_0}, \end{aligned} \quad (2.14)$$

for all $\|u\| > 1$.

Similar techniques as those used in the proof of (2.12), we have

$$\Phi(t) \leq \tau^{p_0} \Phi\left(\frac{t}{\tau}\right), \quad t > 0, \quad 0 < \tau < 1. \quad (2.15)$$

Therefore, we can obtain

$$\begin{aligned} \int_{\Omega} [\Phi(|u(x)|) + \Phi(|\nabla u(x)|)] dx &\leq \|u\|^{p_0} \int_{\Omega} \left[\Phi\left(\frac{|u(x)|}{\|u\|}\right) + \Phi\left(\frac{|\nabla u(x)|}{\|u\|}\right) \right] dx \\ &\leq \|u\|^{p_0}, \end{aligned} \quad (2.16)$$

for all $\|u\| < 1$. □

Lemma 2.3 (see [13]). *Let $u \in W^1 L_{\Phi}(\Omega)$ and assume that*

$$\int_{\Omega} [\Phi(|u(x)|) + \Phi(|\nabla u(x)|)] dx \leq r \quad (2.17)$$

for some $0 < r < 1$, then one has $\|u\| < 1$.

Lemma 2.4 (see [13]). *If $p_0 > N$, then $W^1 L_{\Phi}(\Omega)$ is compactly embedded in $C^0(\overline{\Omega})$ and there exists a constant $c > 0$ such that*

$$\|u\|_{\infty} \leq c \|u\|_{1, \Phi}, \quad \forall u \in W^1 L_{\Phi}(\Omega), \quad (2.18)$$

where $\|u\|_{\infty} := \sup_{x \in \overline{\Omega}} |u(x)|$.

Now, one recall, a three critical theorem of B. Ricceri. If X is a real Banach space, denote by \mathcal{W}_X (see [16]) the class of all functionals $\Phi : X \rightarrow \mathbb{R}$ possessing the following property: if $\{u_n\}$ is a sequence in X converging weakly to u and $\liminf_{n \rightarrow \infty} \Phi(u_n) \leq \Phi(u)$, then $\{u_n\}$

has a subsequence converging strongly to u . For example, if X is uniformly convex and $g : [0, +\infty) \rightarrow \mathbb{R}$ is a continuous, strictly increasing function, then, by a classical results, the functional $u \rightarrow g(\|u\|)$ belongs to the class \mathcal{W}_X .

Lemma 2.5 (see [16]). *Let X be a separable and reflexive real Banach space; let $I : X \rightarrow \mathbb{R}$ be a coercive, sequentially weakly lower semicontinuous C^1 functional, belonging to \mathcal{W}_X , bounded on each bounded subset of X and whose derivative admits a continuous inverse on X^* ; $J : X \rightarrow \mathbb{R}$ a C^1 functional with compact derivative. Assume that I has a strict local minimum u_0 with $I(u_0) = J(u_0) = 0$. Finally, setting*

$$\alpha' = \max \left\{ 0, \limsup_{\|u\| \rightarrow +\infty} \frac{J(u)}{I(u)}, \limsup_{u \rightarrow u_0} \frac{J(u)}{I(u)} \right\}, \quad (2.19)$$

$$\beta' = \sup_{u \in I^{-1}(0, +\infty)} \frac{J(u)}{I(u)},$$

assume that $\alpha' < \beta'$. Then for each compact interval $[a, b] \subset (1/\beta', 1/\alpha')$ (with the conventions $(1/0) = +\infty, (1/+\infty) = 0$), there exists $B > 0$ with the following property: for every $\lambda \in [a, b]$ and every C^1 functional $\Psi : X \rightarrow \mathbb{R}$ with compact derivative, there exists $\delta > 0$ such that, for each $\mu \in [0, \delta]$, the equation

$$I'(x) = \lambda J'(x) + \mu \Psi'(x) \quad (2.20)$$

has at least three solutions in X whose norms are less than B .

Lemma 2.6 (see [17]). *Let X be a reflexive real Banach space; $S \subset \mathbb{R}$ an interval, let $I : X \rightarrow \mathbb{R}$ be a sequentially weakly lower semicontinuous C^1 functional, bounded on each bounded subset of X and whose derivative admits a continuous inverse on X^* ; $J : X \rightarrow \mathbb{R}$ a C^1 functional with compact derivative. Assume that*

$$\lim_{\|u\| \rightarrow \infty} [I(u) - \lambda J(u)] = +\infty \quad (2.21)$$

for all $\lambda \in S$, and that there exists $\rho \in \mathbb{R}$ such that

$$\sup_{\lambda \in S} \inf_{u \in X} [I(u) + \lambda(\rho - J(u))] < \inf_{u \in X} \sup_{\lambda \in S} [I(u) + \lambda(\rho - J(u))]. \quad (2.22)$$

Then there exist a nonempty open set $A \subset S$ and a positive number B , with the following property: for every $\lambda \in A$ and every C^1 functional $\Psi : X \rightarrow \mathbb{R}$ with compact derivative, there exists $\delta > 0$ such that, for each $\mu \in [0, \delta]$, the equation

$$I'(x) = \lambda J'(x) + \mu \Psi'(x) \quad (2.23)$$

has at least three solutions in X whose norms are less than B .

Lemma 2.7 (see [18]). *Let X be a nonempty set and I, J two real functions on X . Assume that there are $r > 0$ and $x_0, x_1 \in X$ such that*

$$I(x_0) = J(x_0) = 0, \quad I(x_1) > r, \quad \sup_{x \in I^{-1}([-\infty, r])} J(x) < r \frac{J(x_1)}{I(x_1)}. \quad (2.24)$$

Then for each ρ satisfying

$$\sup_{x \in I^{-1}([-\infty, r])} J(x) < \rho < r \frac{J(x_1)}{I(x_1)}, \quad (2.25)$$

one has

$$\sup_{\lambda \geq 0} \inf_{u \in X} [I(u) + \lambda(\rho - J(u))] < \inf_{u \in X} \sup_{\lambda \geq 0} [I(u) + \lambda(\rho - J(u))]. \quad (2.26)$$

3. Proof of the Main Results

Set $\gamma = \inf\{(\int_{\Omega} (\Phi(|\nabla u(x)|) + \Phi(|u(x)|)) dx) / (\int_{\Omega} F(x, u(x)) dx) : u \in X, \int_{\Omega} F(x, u(x)) dx > 0\}$.

Theorem 3.1. *Let Φ be a function satisfying the structural hypotheses $(\Phi_1-\Phi_3)$ and the following conditions hold*

$$(H_1) \max\{\limsup_{\xi \rightarrow 0} (\sup_{x \in \Omega} F(x, \xi) / |\xi|^{p_0}), \limsup_{|\xi| \rightarrow \infty} (\sup_{x \in \Omega} F(x, \xi) / |\xi|^{p_0})\} \leq 0,$$

$$(H_2) \sup_{u \in X} \int_{\Omega} F(x, u(x)) dx > 0.$$

Then, for each compact interval $[a, b] \subset (\gamma, \infty)$, there exists $B > 0$ with the following property: for every $\lambda \in [a, b]$ and g , there exists $\delta > 0$ such that, for each $\mu \in [0, \delta]$, the problem $(P_{\lambda, \mu})$ has at least three weak solutions whose norms in X are less than B .

Proof of Theorem 3.1. In order to apply Lemma 2.5, we let

$$\begin{aligned} I(u) &= \int_{\Omega} [\Phi(|\nabla u(x)|) + \Phi(|u(x)|)] dx, \\ J(u) &= \int_{\Omega} F(x, u(x)) dx, \quad \Psi(u) = \int_{\Omega} G(x, u(x)) dx. \end{aligned} \quad (3.1)$$

We divide our proof into two steps as follows.

Step 1. We show that some fundamental assumptions are satisfied.

$X := W^1 L_{\Phi}(\Omega)$. Obviously, X is a separable and reflexive real Banach space (see [13]). By Lemma 2.2, it is easy to see that $I(u)$ is a coercive, bounded on each bounded subset of X . On the other hand, $I, J, \Psi \in C^1(X, \mathbb{R})$ with the derivatives given by

$$\begin{aligned} \langle I'(u), v \rangle &= \int_{\Omega} [\alpha(|\nabla u(x)|) \nabla u(x) \cdot \nabla v(x) + \alpha(|u(x)|) u(x) v(x)] dx, \\ \langle J'(u), v \rangle &= \int_{\Omega} f(x, u(x)) v(x) dx, \quad \langle \Psi'(u), v \rangle = \int_{\Omega} g(x, u(x)) v(x) dx, \end{aligned} \quad (3.2)$$

for any $u, v \in X$. Hence, the critical points of the functional $I - \lambda J - \mu \Psi$ are exactly the weak solutions for problem $(P_{\lambda, \mu})$. Moreover, owing that Φ is convex, it follows that I is convex. Hence, one has that I is sequentially weakly lower semicontinuous. The fact X is compactly embedded into $C^0(\overline{\Omega})$ implies that operators J', Ψ' is compact. As the proof of Lemma 3.2 in [15], we know that I' has a continuous inverse.

Moreover, if $\{u_n\}$ is a sequence in X converging weakly to u and $\liminf_{n \rightarrow \infty} I(u_n) \leq I(u)$, remark that I is sequentially weakly lower semicontinuous, one has

$$I(u) \leq \liminf_{n \rightarrow \infty} I(u_n) \leq I(u). \tag{3.3}$$

Then, up to a subsequence, we deduce that $I(u_n) \rightarrow I(u) = d$. Taking into account that $\{(u_n + u)/2\}$ converges weakly to u and I is sequentially weakly lower semicontinuous, we have

$$d = I(u) \leq \liminf_{n \rightarrow \infty} I\left(\frac{u_n + u}{2}\right). \tag{3.4}$$

We assume by contradiction that u_n does not converge to u in X . Hence, there exist $\varepsilon_0 > 0$ and a subsequence $\{u_{n_m}\}$ of (u_n) such that

$$\left\| \frac{u_{n_m} - u}{2} \right\| > \varepsilon_0, \quad \forall m. \tag{3.5}$$

Then there exists $\varepsilon_1 > 0$ such that

$$I\left(\frac{u_{n_m} - u}{2}\right) > \varepsilon_1, \quad \forall m. \tag{3.6}$$

On the other hand,

$$\frac{1}{2}I(u) + \frac{1}{2}I(u_{n_m}) - I\left(\frac{u_{n_m} + u}{2}\right) \geq I\left(\frac{u_{n_m} - u}{2}\right) > \varepsilon_1, \tag{3.7}$$

(see [19]). Letting $m \rightarrow \infty$ in the above inequality we obtain

$$d - \varepsilon_1 \geq \limsup_{m \rightarrow \infty} I\left(\frac{u_{n_m} + u}{2}\right) \tag{3.8}$$

and that is a contradiction with (3.4). It follows that u_n converges strongly to u and $I \in \mathcal{W}_X$. In addition, $I(0) = J(0) = 0$.

Step 2. We show that $\alpha' = 0, \beta' > 0$.

In view of (H_1) , for all $\varepsilon > 0$, there exists $\tau_1 > 0$ such that

$$F(x, \xi) \leq \varepsilon |\xi|^{p_0}, \tag{3.9}$$

for any $|\xi| \in [0, \tau_1]$. For $\|u\| < \min\{1, (\tau_1/2c)\}$, we have

$$\begin{aligned} |u(x)| &\leq \|u\|_\infty \leq c\|u\|_{1,\Phi} \leq 2c\|u\| \leq \tau_1, \\ \limsup_{u \rightarrow 0} \frac{J(u)}{I(u)} &\leq \limsup_{u \rightarrow 0} \frac{\varepsilon \int_\Omega |u|^{p^0} dx}{\|u\|^{p^0}} \leq \limsup_{u \rightarrow 0} \frac{\varepsilon |\Omega| (2c\|u\|)^{p^0}}{\|u\|^{p^0}} \leq 0. \end{aligned} \quad (3.10)$$

By (H_1) , for all $\varepsilon > 0$, there exists $0 < \tau_1 < \tau_2$ such that

$$F(x, \xi) \leq \varepsilon |\xi|^{p_0}, \quad (3.11)$$

for any $|\xi| > \tau_2$. Further, for each $\|u\| > 1$, we have

$$\begin{aligned} \frac{J(u)}{I(u)} &\leq \frac{\int_{\Omega(|u| \leq \tau_2)} F(x, u(x)) dx}{\|u\|^{p_0}} + \frac{\int_{\Omega(|u| > \tau_2)} F(x, u(x)) dx}{\|u\|^{p_0}} \\ &\leq \frac{\int_{\Omega(|u| \leq \tau_2)} F(x, u(x)) dx}{\|u\|^{p_0}} + \frac{\varepsilon |\Omega| (2c\|u\|)^{p_0}}{\|u\|^{p_0}}. \end{aligned} \quad (3.12)$$

So we get

$$\limsup_{\|u\| \rightarrow \infty} \frac{J(u)}{I(u)} \leq 0. \quad (3.13)$$

Then, with the notation of Lemma 2.5, we have $\alpha' = 0$. By assumption (H_2) , we have $\beta' > 0$. Thus, all the hypotheses of Lemma 2.5 are satisfied. Clearly, $\gamma = 1/\beta'$. Finally, by Lemma 2.5, we can obtain the Theorem 3.1. \square

Example 3.2. Let $p > N + 1$. Define

$$\varphi(t) = \frac{|t|^{p-2}t}{\log(1 + |t|)}, \quad t \neq 0, \quad (3.14)$$

and $\varphi(0) = 0$. By [13], one has

$$p_0 = p - 1 < p^0 = p. \quad (3.15)$$

Let $F(x, t) = |t|^{p+1} - |t|^{p+2}$. Since $F \leq 0$ if $|t|$ is large enough and $F > 0$ if $|t|$ is small enough, moreover, it is easy to see $\limsup_{\xi \rightarrow 0} (\sup_{x \in \Omega} F(x, \xi)/|\xi|^{p^0}) = 0$, the conditions of Theorem 3.1 can be satisfied.

Remark 3.3. Since F in [13] is p_0 -sublinear, the results of [13] do not fit to the problem treated in the previous Example 3.2 even if $\mu = 0$, that is, there is no perturbed nonlinear term. In addition, for nonhomogeneous Neumann problem with a perturbed term, we can have the following result when F is p_0 -sublinear, which extends the results of [13].

Theorem 3.4. *Let Φ be a function satisfying the structural hypotheses (Φ_1) – (Φ_3) and the following conditions.*

(H₃) There exist two constants γ, δ with $\gamma < 2c$ such that $\Phi(\delta) > (\gamma^{p_0} / (2c)^{p_0} |\Omega|)$ and

$$\frac{\int_{\Omega} \max_{|\xi| \leq \gamma} F(x, \xi) dx}{\gamma^{p_0}} < \frac{\int_{\Omega} F(x, \delta) dx}{(2c)^{p_0} |\Omega| \Phi(\delta)}, \quad (3.16)$$

where $|\Omega|$ denotes the Lebesgue measure of the set Ω .

(H₄) There exist $h(x), k(x) \in L^1(\Omega; \mathbb{R}^+)$ and $0 < s < p_0$ such that $|F(x, t)| \leq h(x) + k(x)|t|^s$ for every $(x, t) \in \Omega \times \mathbb{R}$.

Then, there exist a nonempty open set $A \subset [0, \infty)$ and a positive number B , for each $\lambda \in A$ and for every g , there exists $\delta > 0$ such that, for each $\mu \in [0, \delta]$, the problem $(P_{\lambda, \mu})$ has at least three weak solutions whose norms in X are less than B .

Proof of Theorem 3.4. Let us consider I, J, Ψ as the proof of Theorem 3.1. For any $\lambda > 0$, $u \in X$, by (H_4) we have

$$\begin{aligned} I(u) - \lambda J(u) &\geq \int_{\Omega} [\Phi(|u(x)|) + \Phi(|\nabla u(x)|)] dx - \lambda \|u\|_{\infty}^s \int_{\Omega} k(x) dx - \lambda \int_{\Omega} h(x) dx \\ &\geq \|u\|^{p_0} - \lambda (2c)^s \|u\|^s \int_{\Omega} k(x) dx - \lambda \int_{\Omega} h(x) dx, \end{aligned} \quad (3.17)$$

for $\|u\| > 1$. Since $0 < s < p_0$, one has $\lim_{\|u\| \rightarrow \infty} [I(u) - \lambda J(u)] = +\infty$ for all $\lambda \geq 0$.

Let $r := (\gamma^{p_0} / (2c)^{p_0})$, $x_1 = \delta$. For $r > 0$, $I(u) \leq r$, we have

$$|u(x)| \leq \|u\|_{\infty} \leq c \|u\|_{1, \Phi} \leq 2c \|u\| \leq \gamma, \quad \forall x \in \Omega. \quad (3.18)$$

Hence, one has

$$\frac{\sup_{u \in I^{-1}([-\infty, r])} J(u)}{r} \leq \frac{(2c)^{p_0} \int_{\Omega} \max_{|\xi| \leq \gamma} F(x, \xi) dx}{\gamma^{p_0}}. \quad (3.19)$$

From (H_3) , it follows that

$$\frac{\sup_{I(u) \leq r} J(u)}{r} < \frac{J(\delta)}{I(\delta)}. \quad (3.20)$$

Since all the assumptions of Lemmas 2.7 and 2.6 are satisfied, then, there is a nonempty open set $A \subset [0, \infty)$ and a positive number B , for each $\lambda \in A$ and for every g there exists $\delta > 0$ such that, for each $\mu \in [0, \delta]$, the problem has at least three weak solutions whose norms in X are less than B . \square

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