Hölder Continuity of Solutions to Parametric Generalized Vector Quasiequilibrium Problems

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By using a linear scalarization method, we establish sufficient conditions for the Hölder continuity of the solution mappings to a parametric generalized vector quasiequilibrium problem with set-valued mappings. These results extend the recent ones in the literature, e.g., Li et al. (2009), Li et al. (2011). Furthermore, two examples are given to illustrate the obtained result.

1. Introduction

The vector equilibrium problem has been attracting great interest because it provides a unified model for several important problems such as vector variational inequalities, vector complementarity problems, vector optimization problems, vector min-max inequality, and vector saddle point problems. Many different types of vector equilibrium problems have been intensively studied for the past years; see, for example, [1–3] and the references therein.

It is important to derive results for parametric vector equilibrium problems concerning the properties of the solution mapping when the problems data vary. Among many desirable properties of vector equilibrium problems, the stability analysis of solutions is an essential topic in vector optimization theory and applications. In general, stability may be understood as lower (upper) semicontinuity, continuity, Lipschitz and Hölder continuity and so on. Recently, semicontinuity, especially lower semicontinuity, of solution mappings to parametric vector variational inequalities and parametric vector equilibrium problems has been intensively studied in the literature; see [4–12]. On the other hand, Hölder continuity of solutions to parametric vector equilibrium problems has also been discussed recently; see [13–22], although there are less works in the literature devoted to this property than to semicontinuity. There have been many papers devoted to discussing the local uniqueness...

For general perturbed vector quasiequilibrium problems, it is well known that a solution mapping is, in general, a set-valued one, but not a single-valued one. Naturally, there is a need to study Hölder continuous properties of the set-valued solution mappings. Under the Hausdorff distance and the strong quasimonotonicity, Lee et al. [21] first showed that the set-valued solution mapping for a parametric vector variational inequality is Hölder continuous. Recently, by virtue of the strong quasimonotonicity, Ait Mansour and Aussel [22] discussed Hölder continuity of set-valued solution mappings for parametric generalized variational inequalities. Li et al. [23] introduced an assumption, which is weaker than the corresponding ones of [16, 18], and established the Hölder continuity of the set-valued solution mappings for two classes of parametric generalized vector quasiequilibrium problems in general metric spaces. Li et al. [24] extended the results of [23] to perturbed generalized vector quasiequilibrium problems. Later, S. J. Li and X. B. Li [25] use a scalarization technique to obtain the Hölder continuity of the set-valued solution mappings for a parametric vector equilibrium problem in general metric spaces.

Motivated by the work reported in [21, 23, 25], this paper aims at establishing sufficient conditions for Hölder continuity of the solution sets for a class of parametric generalized vector quasiequilibrium problem (PGVQEP, in short) with set-valued mapping, by using a linear scalarization method. The main results in this paper are different from corresponding results in [23, 24] and overcome the drawback, which requires the knowledge of detailed values of the solution mapping in a neighborhood of the point under consideration. Our main results also extend and improve the corresponding ones in [25].

The rest of the paper is organized as follows. In Section 2, we introduce the (PGVQEP) and define the solution and $\xi$-solution to the (PGVQEP). Then, we recall some notions and definitions which are needed in the sequel. In Section 3, we discuss Hölder continuity of the solution mapping for the (PGVQEP) and compare our main results with the corresponding ones in the recent literature. We also give two examples to illustrate that our main results are applicable.

### 2. Preliminaries

Throughout this paper, if not other specified, $\| \cdot \|$ and $d(\cdot, \cdot)$ denote the norm and metric in any metric space, respectively. Let $B(0, \delta)$ denote the closed ball with radius $\delta \geq 0$ and center
0 in any metric linear spaces. Let $X, \Lambda, M, Y$ be metric linear spaces. Let $Y^*$ be the topological dual space of $Y$. Let $C \subseteq Y$ be a pointed, closed, and convex cone with $\text{int } C \neq \emptyset$, where $\text{int } C$ denotes the interior of $C$. Let $C^* := \{ f \in Y^* : f(y) \geq 0, \text{ for all } y \in C \}$ be the dual cone of $C$. Since $\text{int } C \neq \emptyset$, the dual cone $C^*$ of $C$ has a weak* compact base. Letting $e \in \text{int } C$ be given, then $B^*_C := \{ \xi \in C^* : \| \xi \| = 1 \}$ is a weak* compact base of $C^*$.

Let $N(\lambda_0) \subset \Lambda$ and $N(\mu_0) \subset M$ be neighborhoods of considered points $\lambda_0$ and $\mu_0$, respectively. Let $K : X \times \Lambda \rightrightarrows X$ be a set-valued mapping, and let $F : X \times X \times M \rightrightarrows Y$ be a set-valued mapping. For each $\lambda \in N(\lambda_0)$ and $\mu \in N(\mu_0)$, consider the following parameterized generalized vector quasiequilibrium problem of finding $x_0 \in K(x_0, \lambda)$ such that

$$F(x_0, y, \mu) \subset Y \setminus \text{int } C, \quad \forall y \in K(x_0, \lambda). \quad \text{(PGVQEP)}$$

For each $\lambda \in N(\lambda_0)$ and $\mu \in N(\mu_0)$, let

$$E(\lambda) := \{ x \in X \mid x \in K(x, \lambda) \}. \quad (2.1)$$

Let $S(\lambda, \mu)$ be the solution set of (PGVQEP), that is,

$$S(\lambda, \mu) := \{ x \in E(\lambda) \mid F(x, y, \mu) \subset Y \setminus \text{int } C, \forall y \in K(x, \lambda) \}. \quad (2.2)$$

For each $\xi \in C^* \setminus \{0\}$, each $\lambda \in N(\lambda_0)$ and $\mu \in N(\mu_0)$, let $S_\xi(\lambda, \mu)$ denote the set of $\xi$-solution set to (PGVQEP), that is,

$$S_\xi(\lambda, \mu) := \left\{ x \in E(\lambda) : \inf_{z \in F(x, y, \mu)} f(z) \geq 0, \forall y \in K(x, \lambda) \right\}. \quad (2.3)$$

**Special Case**

(i) When $K(x, \lambda) = K(\lambda)$, that is, $K$ does not depend on $x$, the (PGVQEP) reduces to the parametric generalized vector equilibrium problem (PGVEP) considered by Li et al. [23].

(ii) If $F : X \times X \times M \rightarrow \mathbb{R}$, the (PGVQEP) collapses to the quasiequilibrium problem (QEP) considered by Anh and Khanh [26].

(iii) If $K(x, \lambda) = K(\lambda)$ and $F$ is a vector-valued mapping, that is, $F : X \times X \times M \rightarrow Y$, the (PGVQEP) reduce to the parametric Ky Fan inequality (PKI) considered by S. J. Li and X. B. Li [25].

Now we recall some basic definitions and their properties which are needed in this paper.

**Definition 2.1** (classical notion). A set-valued mapping $G : M \rightrightarrows X$ is said to be $\ell \cdot \alpha$-Hölder continuous at $\mu_0$ if there is a neighborhood $U(\mu_0)$ of $\mu_0$ such that, for all $\mu_1, \mu_2 \in U(\mu_0)$,

$$G(\mu_1) \subseteq G(\mu_2) + \ell B(0, d^\alpha(\mu_1, \mu_2)), \quad (2.4)$$

where $\ell \geq 0$ and $\alpha > 0$. 
Definition 2.2. A set-valued mapping $G: X \times \Lambda \rightrightarrows Y$ is said to be $(\ell_1 \cdot \alpha_1, \ell_2 \cdot \alpha_2)$-Hölder continuous at $(x_0, \lambda_0)$ if and only if there exists neighborhoods $N(x_0)$ of $x_0$ and $N(\mu_0)$ of $\mu_0$ such that, for all $x_1, x_2 \in N(x_0)$, for all $\lambda_1, \lambda_2 \in N(\lambda_0)$,
\[
G(x_1, \lambda_1) \subseteq G(x_2, \lambda_2) + (\ell_1 d^{\alpha_1}(x_1, x_2) + \ell_2 d^{\alpha_2}(\lambda_1, \lambda_2))B(0, 1),
\] (2.5)
where $\ell_1, \ell_2 \geq 0$ and $\alpha_1, \alpha_2 > 0$.

Definition 2.3 (see [25]). A set-valued mapping $G: M \rightrightarrows Y$ is said to be $(\ell \cdot \alpha)$-Hölder continuous with respect to $e \in \text{int} C$ at $\mu_0$ if and only if there exists neighborhoods $N(\mu_0)$ of $\mu_0$ such that, for all $\mu_1, \mu_2 \in N(\mu_0)$,
\[
G(\mu_1) \subseteq G(\mu_2) + \ell d^\alpha(\mu_1, \mu_2)[-e, e],
\] (2.6)
where $\ell \geq 0$, $\alpha > 0$ and $[-e, e] = \{x : x \in e - C, x \in -e + C\}$.

Definition 2.4. Let $F: X \times X \times \Lambda \rightrightarrows Y$ be a set-valued mapping with nonempty values; $F(x, \cdot, \mu)$ is called $C$-like convex on $A(\lambda)$ if and only if for any $x_1, x_2 \in X$ and any $t \in [0, 1]$, there exists $x_3 \in X$ such that
\[
tF(x, x_1, \lambda) + (1 - t)F(x, x_2, \lambda) \subseteq F(x, x_3, \lambda) + C.
\] (2.7)

Remark 2.5. If for each $\mu \in N(\mu_0)$ and each $x \in E(N(\lambda_0))$, $F(x, \cdot, \mu)$ is $C$-like convex on $E(N(\lambda_0))$, then $F(x, E(N(\lambda_0)), \mu) + C$ is a convex set.

3. Main Results

In this section, we mainly discuss the Hölder continuity of the solution mappings to (PGVQEP).

Lemma 3.1. Suppose that $N(\lambda_0), N(\mu_0)$ are the given neighborhoods of $\lambda_0, \mu_0$, respectively.

(a) If for each $x, y \in E(N(\lambda_0))$, $F(x, y, \cdot)$ is $m_1 \cdot \gamma_1$-Hölder continuous with respect to $e \in \text{int} C$ at $\mu_0 \in M$, then for any $\xi \in B^*_\nu$, the function $\varphi_\xi(x, y, \cdot) = \inf_{z \in F(x, y, \cdot)} \xi(z)$ is $m_1 \cdot \gamma_1$-Hölder continuous at $\mu_0$.

(b) If for each $x \in E(N(\lambda_0))$ and $\mu \in N(E(\mu_0))$, $F(x, \cdot, \mu)$ is $m_2 \cdot \gamma_2$-Hölder continuous with respect to $e \in \text{int} C$ on $E(N(\lambda_0))$, then for each $\xi \in B^*_\nu$, $\varphi_\xi(x, \cdot, \mu) = \inf_{z \in F(x, \cdot, \mu)} \xi(z)$ is also $m_2 \cdot \gamma_2$-Hölder continuous on $E(N(\lambda_0))$.

Proof. (a) By assumption, there exists a neighborhood $N(\mu_0)$ of $\mu_0$, such that for all $\mu_1, \mu_2 \in N(\mu_0)$, for all $x, y \in E(N(\lambda_0)) : x \neq y$,
\[
F(x, y, \mu_1) \subseteq F(x, y, \mu_2) + m_1 d^{\nu}(\mu_1, \mu_2)[-e, e].
\] (3.1)
So, for any $z_1 \in F(x, y, \mu_1)$, there exist $z_2 \in F(x, y, \mu_2)$ and $e_0 \in [-e, e]$ such that
\[
z_1 = z_2 + m_1 d^{\nu}(\mu_1, \mu_2)e_0.
\] (3.2)
Then, by the linearity of \( \xi \), we have
\[
\xi(z_1) - \xi(z_2) = m_1 d^n(\mu_1, \mu_2) \xi(e_0). 
\] (3.3)

It follows from \( \xi(e) = 1, e_0 \in [-e, e] \), and the structure of \([-e, e]\) that
\[
\xi(e_0) \geq -1. 
\] (3.4)

Therefore, (3.3) and (3.4) together yield that
\[
-m_1 d^n(\mu_1, \mu_2) \leq \xi(z_1) - \xi(z_2). 
\] (3.5)

Since \( z_1 \) is arbitrary and \( \xi(z_2) \geq \inf_{z \in F(x, y, \mu)} \xi(z) \), we have
\[
-m_1 d^n(\mu_1, \mu_2) \leq \inf_{z \in F(x, y, \mu_1)} \xi(z) - \inf_{z \in F(x, y, \mu_2)} \xi(z). 
\] (3.6)

Due to the symmetry between \( \mu_1 \) and \( \mu_2 \), the same estimate is also valid, that is,
\[
-m_1 d^n(\mu_1, \mu_2) \leq \inf_{z \in F(x, y, \mu_2)} \xi(z) - \inf_{z \in F(x, y, \mu_1)} \xi(z). 
\] (3.7)

Thus, it follows (3.6) and (3.7) that
\[
\left| \inf_{z \in F(x, y, \mu_1)} \xi(z) - \inf_{z \in F(x, y, \mu_2)} \xi(z) \right| = |\varphi_\xi(x, y, \mu_1) - \varphi_\xi(x, y, \mu_2)| \leq m_1 d^n(\mu_1, \mu_2) 
\] (3.8)

and the proof is completed.

(b) As the proof of (b) is similar to (a), we omit it. Then the proof is completed. \( \square \)

**Lemma 3.2.** If for each \( \mu \in N(\mu_0) \) and each \( x \in E(N(\lambda_0)), F(x, \cdot, \mu) \) is \( C \)-like convex on \( E(N(\lambda_0)) \), that is, \( F(x, E(N(\lambda_0)), \mu) + C \) is a convex set, then
\[
S(\lambda, \mu) = \bigcup_{\xi \in C \setminus 0} S_{\xi}(\lambda, \mu) = \bigcup_{\xi \in B_2^*} S_{\xi}(\lambda, \mu). 
\] (3.9)

**Proof.** In a similar way to the proof of Lemma 3.1 in [8], with suitable modifications, we can obtain the conclusion. \( \square \)

**Theorem 3.3.** Assume that for each \( \xi \in B_2^* \), the \( \xi \)-solution set for (PGVQEP) exists in a neighborhood \( N(\lambda_0) \times N(\mu_0) \) of the considered point \( (\lambda_0, \mu_0) \in \Lambda \times M \). Assume further that the following conditions hold.

(i) \( K(\cdot, \cdot) \) is \( (\ell_1 \cdot \alpha_1, \ell_2 \cdot \alpha_2) \)-Hölder continuous in \( E(N(\lambda_0)) \times N(\mu_0) \).

(ii) For each \( x, y \in E(N(\lambda_0)), F(x, y, \cdot) \) is \( m_1 \cdot \gamma_1 \)-Hölder continuous with respect to \( e \in \text{int} \ C \) at \( \mu_0 \in M \).
(iii) For each $x \in E(N(\lambda_0))$ and $\mu \in N(E(\mu_0))$, $F(x, \cdot , \mu)$ is $m_2 \cdot \gamma_2$-Hölder continuous with respect to $\ve \in \text{int} C$ on $E(N(\lambda_0))$.

(iv) for all $\xi \in B^*_\ve, \mu \in N(\mu_0)$, for all $x, y \in E(N(\lambda_0))$ : $x \neq y$, there exists two constants $h > 0$ and $\beta > 0$ such that

$$hd^\beta(x, y) \leq d\left(\inf_{z \in F(x,y, \mu)} \xi(z), \mathbb{R}_+\right) + d\left(\inf_{z \in F(y,x, \mu)} \xi(z), \mathbb{R}_+\right).$$

(v) $\alpha_1 \gamma_2 = \beta$ and $h > 2m_2 \ell_1^\beta$.

Then, for any $\xi \in B^*_\ve$, there exists open neighborhoods $N(\xi)$ of $\xi$, $N^*_\ve(\lambda_0)$ of $\lambda_0$ and $N^*_\ve(\mu_0)$ of $\mu_0$, such that the $\xi$-solution set $S_\xi(\cdot, \cdot)$ on $N(\xi) \times N^*_\ve(\lambda_0) \times N^*_\ve(\mu_0)$ satisfies the following Hölder condition: for all $\xi \in N(\xi)$, for all $(\lambda_1, \mu_1), (\lambda_2, \mu_2) \in N^*_\ve(\lambda_0) \times N^*_\ve(\mu_0)$,

$$d(x^\xi(\lambda_1, \mu_1), x^\xi(\lambda_2, \mu_2)) \leq \left(\frac{m_1}{h - 2m_2 \ell_1^\beta}\right)^{1/\beta} d^{1/\beta}(\mu_1, \mu_2) + \left(\frac{2m_2 \ell_1}{h - 2m_2 \ell_1^\beta}\right)^{1/\beta} d^{\alpha_1 \gamma_2 / \beta}(\lambda_1, \lambda_2),$$

where $x^\xi(\lambda_i, \mu_i) \in S_\xi(\lambda_i, \mu_i), i = 1, 2$.

**Proof.** Let $(\lambda_1, \mu_1), (\lambda_2, \mu_2) \in N^*_\ve(\lambda_0) \times N^*_\ve(\mu_0)$ be arbitrarily given. For all $\xi \in B^*_\ve, x, y \in X$, and $\mu \in M$, we set $\varphi_\xi(x, y, \cdot) := \inf_{z \in F(x,y, \mu)} \xi(z)$ for the sake of convenient statement in the sequel. We prove that (3.11) holds by the following three steps.

**Step 1.** We first show that, for all $x^\xi(\lambda_1, \mu_1) \in S_\xi(\lambda_1, \mu_1)$, for all $x^\xi(\lambda_1, \mu_2) \in S_\xi(\lambda_1, \mu_2),$

$$d(x^\xi(\lambda_1, \mu_1), x^\xi(\lambda_1, \mu_2)) \leq \left(\frac{m_1}{h - 2m_2 \ell_1^\beta}\right)^{1/\beta} d^{1/\beta}(\mu_1, \mu_2).$$

Obviously, if $x^\xi(\lambda_1, \mu_1) = x^\xi(\lambda_1, \mu_2)$, we have that (3.12) holds. So we suppose $x^\xi(\lambda_1, \mu_1) \neq x^\xi(\lambda_1, \mu_2)$. Since $x^\xi(\lambda_1, \mu_1) \in K(x^\xi(\lambda_1, \mu_1), \lambda_1), x^\xi(\lambda_1, \mu_2) \in K(x^\xi(\lambda_1, \mu_2), \lambda_1)$, and by the Hölder continuity of $K(\cdot, \lambda_1)$, there exists $x_1 \in K(x^\xi(\lambda_1, \mu_1), \lambda_1)$ and $x_2 \in K(x^\xi(\lambda_1, \mu_2), \lambda_1)$ such that

$$d(x^\xi(\lambda_1, \mu_1), x_1) \leq \ell_1 d^{\alpha_1}(x^\xi(\lambda_1, \mu_1), x^\xi(\lambda_1, \mu_2)),$$

$$d(x^\xi(\lambda_1, \mu_2), x_1) \leq \ell_1 d^{\alpha_1}(x^\xi(\lambda_1, \mu_1), x^\xi(\lambda_1, \mu_2)).$$

Since $x^\xi(\lambda_1, \mu_1), x^\xi(\lambda_1, \mu_2)$ are $\xi$-solutions to (PGVQEP) at parameters $(\lambda_1, \mu_1), (\lambda_1, \mu_2)$, respectively, we obtain

$$\varphi_\xi(x^\xi(\lambda_1, \mu_1), x_1, \mu_1) \geq 0,$$

$$\varphi_\xi(x^\xi(\lambda_1, \mu_2), x_2, \mu_2) \geq 0.$$
By virtue of (iv), we get that

\[
hd^\beta\left(x^\ell_1(\lambda_1, \mu_1), x^\ell_1(\lambda_1, \mu_2)\right) \leq d\left(\phi_{\xi}(x^\ell_1(\lambda_1, \mu_2), x^\ell_1(\lambda_1, \mu_1), R_1)\right) + d\left(\phi_{\xi}(x^\ell_1(\lambda_1, \mu_1), x^\ell_1(\lambda_1, \mu_2), R_1)\right),
\]

which together with (3.14) yields that

\[
hd^\beta\left(x^\ell_1(\lambda_1, \mu_1), x^\ell_1(\lambda_1, \mu_2)\right) \leq \left|\phi_{\xi}(x^\ell_1(\lambda_1, \mu_2), x^\ell_1(\lambda_1, \mu_1), \mu_1) - \phi_{\xi}(x^\ell_1(\lambda_1, \mu_2), x_2, \mu_2)\right| \\
+ \left|\phi_{\xi}(x^\ell_1(\lambda_1, \mu_1), x^\ell_1(\lambda_1, \mu_2), \mu_1) - \phi_{\xi}(x^\ell_1(\lambda_1, \mu_1), x_1, \mu_1)\right| \\
\leq \left|\phi_{\xi}(x^\ell_1(\lambda_1, \mu_2), x^\ell_1(\lambda_1, \mu_1), \mu_1) - \phi_{\xi}(x^\ell_1(\lambda_1, \mu_2), x^\ell_1(\lambda_1, \mu_1), \mu_2)\right| \\
+ \left|\phi_{\xi}(x^\ell_1(\lambda_1, \mu_2), x^\ell_1(\lambda_1, \mu_1), \mu_2) - \phi_{\xi}(x^\ell_1(\lambda_1, \mu_2), x_2, \mu_2)\right| \\
+ \left|\phi_{\xi}(x^\ell_1(\lambda_1, \mu_1), x^\ell_1(\lambda_1, \mu_2), \mu_1) - \phi_{\xi}(x^\ell_1(\lambda_1, \mu_1), x_1, \mu_1)\right|.
\]

Then, from Lemma 3.1, (3.13), we have

\[
hd^\beta\left(x^\ell_1(\lambda_1, \mu_1), x^\ell_1(\lambda_1, \mu_2)\right) \\
\leq m_1d^{\eta_1}(\mu_1, \mu_2) + m_2d^{\eta_1}(x^\ell_1(\lambda_1, \mu_2), x_1) + m_2d^{\eta_1}(x^\ell_1(\lambda_1, \mu_1), x_2) \\
\leq m_1d^{\eta_1}(\mu_1, \mu_2) + 2m_2d^{\eta_1}d^{\gamma_2}\left(x^\ell_1(\lambda_1, \mu_1), x^\ell_1(\lambda_1, \mu_2)\right).
\]

The assumption (v) yields that

\[
d\left(x^\ell_1(\lambda_1, \mu_1), x^\ell_1(\lambda_1, \mu_2)\right) \leq \left(\frac{m_1}{h - 2m_2\ell_1^{\gamma_2}}\right)^{1/\beta} d^{\eta_1/\beta}(\mu_1, \mu_2).
\]

Hence, we have that (3.12) holds.

**Step 2.** Now we show that, for all \(x^\ell(\lambda_1, \mu_2) \in S_\xi(\lambda_1, \mu_2)\), for all \(x^\ell(\lambda_2, \mu_2) \in S_\xi(\lambda_2, \mu_2)\),

\[
d\left(x^\ell(\lambda_1, \mu_2), x^\ell(\lambda_2, \mu_2)\right) \leq \left(\frac{2m_2\ell_2^{\gamma_2}}{h - 2m_2\ell_1^{\gamma_2}}\right)^{1/\beta} d^{\eta_1/\beta}(\lambda_1, \lambda_2).
\]
Obviously, we only need to prove that (3.19) holds when $x^i(\lambda_1, \mu_2) \neq x^i(\lambda_2, \mu_2)$. By virtue of assumption (i), there exists $x^i_1 \in K(x^i(\lambda_2, \mu_2), \lambda_1)$ and $x^i_2 \in K(x^i(\lambda_1, \mu_2), \lambda_2)$ such that
\begin{align}
    d\left( x^i(\lambda_2, \mu_2), x^i_1 \right) & \leq \ell_2 d^{m_1}(\lambda_1, \lambda_2), \\
    d\left( x^i(\lambda_1, \mu_2), x^i_2 \right) & \leq \ell_2 d^{m_2}(\lambda_1, \lambda_2). 
\end{align}

By the Hölder continuity of $K(\cdot, \cdot)$, there exists $x''_1 \in K(x^i(\lambda_1, \mu_2), \lambda_1)$ and $x''_2 \in K(x^i(\lambda_2, \mu_2), \lambda_2)$ such that
\begin{align}
    d\left( x^i_1, x''_1 \right) & \leq \ell_1 d^{m_1}(x^i(\lambda_1, \mu_2), x^i(\lambda_2, \mu_2)), \\
    d\left( x^i_2, x''_2 \right) & \leq \ell_1 d^{m_2}(x^i(\lambda_1, \mu_2), x^i(\lambda_2, \mu_2)).
\end{align}

From the definition of $\xi$-solution for (PGVQEP), we have
\begin{align}
    \phi_\xi\left( x^i(\lambda_1, \mu_2), x''_1, \mu_2 \right) & \geq 0, \\
    \phi_\xi\left( x^i(\lambda_2, \mu_2), x''_2, \mu_2 \right) & \geq 0.
\end{align}

From assumptions (ii)--(iv), (3.22), and Lemma 3.1, we have
\begin{align}
    hd^\theta\left( x^i(\lambda_1, \mu_2), x^i(\lambda_2, \mu_2) \right) & \\
    & \leq d\left( \phi_\xi\left( x^i(\lambda_1, \mu_2), x^i(\lambda_2, \mu_2), \mu_2 \right), R_i \right) + d\left( \phi_\xi\left( x^i(\lambda_1, \mu_2), x^i(\lambda_1, \mu_2), \mu_2 \right), R_i \right) \\
    & \leq d\left( \phi_\xi\left( x^i(\lambda_1, \mu_2), x^i(\lambda_2, \mu_2), \mu_2 \right), \phi_\xi\left( x^i(\lambda_1, \mu_2), x^i(\lambda_1, \mu_2), \mu_2 \right) \right) \\
    & \quad + d\left( \phi_\xi\left( x^i(\lambda_2, \mu_2), x^i(\lambda_1, \mu_2), \mu_2 \right), \phi_\xi\left( x^i(\lambda_2, \mu_2), x^i(\lambda_1, \mu_2), \mu_2 \right) \right) \\
    & \leq d\left( \phi_\xi\left( x^i(\lambda_1, \mu_2), x^i(\lambda_2, \mu_2), \mu_2 \right), \phi_\xi\left( x^i(\lambda_1, \mu_2), x^i(\lambda_1, \mu_2), \mu_2 \right) \right) \\
    & \quad + d\left( \phi_\xi\left( x^i(\lambda_1, \mu_2), x^i(\lambda_1, \mu_2), \mu_2 \right), \phi_\xi\left( x^i(\lambda_1, \mu_2), x^i(\lambda_1, \mu_2), \mu_2 \right) \right) \\
    & \leq m_2 d^{m_1}(x(\lambda_2, \mu_2), x_1^i) + m_2 d^{m_1}(x_1^i, x_1^i) + m_2 d^{m_1}(x(\lambda_1, \mu_2), x_2^i) + m_2 d^{m_1}(x_2^i, x_2^i).
\end{align}
By virtue of (3.20)–(3.21) and (3.23), we can get
\[ h d^{\beta}\left(x^\lambda(\lambda_1, \mu_2), x^\lambda(\lambda_2, \mu_2)\right) \leq m_2 e_2^{\beta} d^{\alpha_2}(\lambda_1, \lambda_2) + m_2 e_1^{\beta} d^{\alpha_1}(\lambda_1, \lambda_2) \]
\[ + m_2 e_2^{\beta} d^{\alpha_2}(\lambda_1, \lambda_2) + m_2 e_1^{\beta} d^{\alpha_1}(\lambda_1, \lambda_2). \]

Therefore, it follows from (v) that
\[ d\left(x^\lambda(\lambda_1, \mu_2), x^\lambda(\lambda_2, \mu_2)\right) \leq \left(\frac{2m_2 e_2^{\beta}}{h-2m_2 e_1^{\beta}}\right)^{1/\beta} d^{\alpha_2/\beta}(\lambda_1, \lambda_2) \]  
(3.25)

and the conclusion (3.19) holds.

Step 3. Finally, by the arbitrariness of \(x^\lambda(\lambda_1, \mu_1) \in S_\xi(\lambda_1, \mu_1), x^\lambda(\lambda_1, \mu_2) \in S_\xi(\lambda_1, \mu_2), x^\lambda(\lambda_2, \mu_2) \in S_\xi(\lambda_2, \mu_2),\) (3.12) and (3.19), we can easily get that
\[ d\left(x^\lambda(\lambda_1, \mu_1), x^\lambda(\lambda_2, \mu_2)\right) \leq d\left(x^\lambda(\lambda_1, \mu_1), x^\lambda(\lambda_1, \mu_2)\right) + d\left(x^\lambda(\lambda_1, \mu_2), x^\lambda(\lambda_2, \mu_2)\right) \]
\[ \leq \left(\frac{m_1}{h-2m_2 e_1^{\beta}}\right)^{1/\beta} d^{\gamma/\beta}(\mu_1, \mu_2) + \left(\frac{2m_2 e_2^{\beta}}{h-2m_2 e_1^{\beta}}\right)^{1/\beta} d^{\alpha_2/\beta}(\lambda_1, \lambda_2) \]
(3.26)

and the conclusion (3.11) holds. This completes the proof.

Remark 3.4. Theorem 3.3 generalizes Lemma 3.3 in S. J. Li and X. B. Li [25] from vector-valued version to set valued version. Moreover, the assumption \((H_4)\) of Lemma 3.3 in [25] is removed.

Now, we give an example to illustrate that Theorem 3.3 is applicable under the case that the mapping \(F\) is set valued.

Example 3.5. Let \(X = Y = R, A = M = [0, 1], C = R_+\) and \(e = 3/2 \in \text{int } C.\) Let \(K: X \times M \rightrightarrows Y\) be defined by
\[ K(x, \lambda) = \left[\frac{\lambda^2 + x}{16}, 1\right] \]
(3.27)

and \(F: X \times X \times M \rightrightarrows Y\) a set-valued mapping defined by
\[ F(x, y, \lambda) = \left[(1 + \lambda)\left(x + \frac{1}{2}\right)(y - x), 28 - 2x^{3/2}\right]. \]
(3.28)

Then, \(E(\lambda) = [\lambda^2/15, 1].\) Consider that \(\lambda_0 = 0.5\) and \(N(\lambda_0) = A.\) Direct computation shows that \(E(\Lambda) = E(N(\lambda_0)) = [0, 1].\)
It can be checked that $K(\cdot, \cdot)$ is $((1/16) \cdot 1, (3/2) \cdot 1)$-Hölder continuous in $E(N(\lambda_0)) \times N(\mu_0)$; for all $x, y \in E(N(\lambda_0))$, $F(x, y, \cdot)$ is $6\sqrt{2}$-1-Hölder continuous with respect to $e = 3/2 \in \text{int } C$ at $\lambda_0 \in M$; for each $x \in E(N(\lambda_0))$ and $\lambda \in N(E(\lambda_0))$, $F(x, \cdot, \lambda)$ is $3.1$-Hölder continuous with respect to $e \in \text{int } C$ on $E(N(\lambda_0))$. Here $\xi_1 = 1/16, \alpha_1 = 1, \xi_2 = 3/2, \alpha_2 = 1, m_1 = 6\sqrt{2}, \gamma_1 = 1, m_2 = 3, \gamma_2 = 1$. Take $\beta = 1$ and $h = 1/2$, for any $\xi \in B^*_e$ and for all $x, y \in E(N(\lambda_0)) : x \neq y$, we have

$$hd^\beta(x, y) \leq d\left(\inf_{z \in F(x,y,\mu)} \xi(z), R_+\right) + d\left(\inf_{z \in F(y,x,\mu)} \xi(z), R_+\right),$$

and also have $\alpha_1 \gamma_2 = \beta$ and $h > 2m_2\ell_1^{\gamma_1} = 3/8$. Hence, all assumptions of Theorem 3.3 hold, and thus it is valid.

**Theorem 3.6.** Assume that for each $\xi \in B^*_e$, the $\xi$-solution set for (PGVQEP) exists in a neighborhood $N(\lambda_0) \times N(\mu_0)$ of the considered point $(\lambda_0, \mu_0) \in \Lambda \times M$. Assume further that the following conditions hold:

(i) $K(\cdot, \cdot)$ is $(\ell_1 \cdot \alpha_1, \ell_2 \cdot \alpha_2)$-Hölder continuous in $E(N(\lambda_0)) \times N(\mu_0)$;

(ii) for each $x, y \in E(N(\lambda_0))$, $F(x, y, \cdot)$ is $m_1 \cdot \gamma_1$-Hölder continuous with respect to $e \in \text{int } C$ at $\mu_0 \in M$;

(iii) for each $x \in E(N(\lambda_0))$ and $\mu \in N(E(\mu_0))$, $F(x, \cdot, \mu)$ is $m_2 \cdot \gamma_2$-Hölder continuous with respect to $e \in \text{int } C$ on $E(N(\lambda_0))$;

(iv) for all $\xi \in B^*_e, \mu \in N(\mu_0)$, for all $x, y \in E(N(\lambda_0))(x \neq y)$, there exist two constants $h > 0$ and $\beta > 0$ such that

$$hd^\beta(x, y) \leq d\left(\inf_{z \in F(x,y,\mu)} \xi(z), R_+\right) + d\left(\inf_{z \in F(y,x,\mu)} \xi(z), R_+\right),$$

(v) for all $x \in E(N(\lambda_0))$, for all $\mu \in N(\mu_0)$, $F(x, \cdot, \mu)$ is $C$-like convex on $E(N(\lambda_0))$;

(vi) $\alpha_1 \gamma_2 = \beta$ and $h > 2m_2\ell_1^{\gamma_1}$.

Then there exist neighborhoods $\tilde{N}(\lambda_0)$ of $\lambda_0$ and $\tilde{N}(\mu_0)$ of $\mu_0$, such that the solution set $S(\cdot, \cdot)$ on $\tilde{N}(\lambda_0) \times \tilde{N}(\mu_0)$ is nonempty and satisfies the following Hölder continuous condition, for all $(\lambda_1, \mu_1), (\lambda_2, \mu_2) \in \tilde{N}(\lambda_0) \times \tilde{N}(\mu_0)$:

$$S(\lambda_1, \mu_1) \subset S(\lambda_2, \mu_2) + \left(\left(\frac{m_1}{h - 2m_2\ell_1^{\gamma_1}}\right)\frac{1}{\beta} d^{1/\beta}(\mu_1, \mu_2) + \left(\frac{2m_2\ell_2^{\gamma_2}}{h - 2m_2\ell_1^{\gamma_1}}\right)\frac{1}{\beta} d^{2\gamma_2/\beta}(\lambda_1, \lambda_2)\right)B(0, 1).$$
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Proof. Since the system of $\{N'(\hat{\xi})\}_{\hat{\xi} \in B^*_e}$, which are given by Theorem 3.3, is an open covering of the weak* compact set $B^*_e$, there exist a finite number of points $(\hat{\xi}_i)$ ($i = 1, 2, \ldots, n$) from $B^*_e$ such that

$$B^*_e \subset \bigcup_{i=1}^{n} N'(\hat{\xi}_i). \quad (3.32)$$

Hence, let $\widetilde{N}(\lambda_0) = \bigcap_{\hat{\xi}_i}^{n} N'(\hat{\xi}_i)(\lambda_0)$ and $\widetilde{N}(\mu_0) = \bigcap_{\hat{\xi}_i}^{n} N'(\hat{\xi}_i)(\mu_0)$. Then $\widetilde{N}(\lambda_0)$ and $\widetilde{N}(\mu_0)$ are desired neighborhoods of $\lambda_0$ and $\mu_0$, respectively. Indeed, let $(\lambda, \mu) \in \widetilde{N}(\lambda_0) \times \widetilde{N}(\mu_0)$ be given arbitrarily. For any $\hat{\xi} \in B^*_e$, by virtue of (3.32), there exists $i_0 \in \{1, 2, \ldots, n\}$ such that $\hat{\xi} \in N'(\hat{\xi}_{i_0})$. From the construction of the neighborhoods $\widetilde{N}(\lambda_0)$ and $\widetilde{N}(\mu_0)$, one has

$$(\lambda, \mu) \in N'_{\hat{\xi}_{i_0}}(\lambda_0) \times N'_{\hat{\xi}_{i_0}}(\mu_0). \quad (3.33)$$

Then, from the assumption of existence for $\hat{\xi}$-solution set and Lemma 3.2, $S(\lambda, \mu) = \bigcup_{\hat{\xi} \in B^*_e} S_{\hat{\xi}}(\lambda, \mu)$ is nonempty.

Now, we show that (3.31) holds. Indeed, taking any $(\lambda_1, \mu_1), (\lambda_2, \mu_2) \in \widetilde{N}(\lambda_0) \times \widetilde{N}(\mu_0)$, we need to show that for any $x_1 \in S(\lambda_1, \mu_1)$, there exists $x_2 \in S(\lambda_2, \mu_2)$ satisfying

$$d(x_1, x_2) \leq \left( \frac{m_1}{h - 2m_2\ell_1^{\beta}} \right)^{1/\beta} d^{\gamma/\beta}(\mu_1, \mu_2) + \left( \frac{2m_2\ell_2^{\beta}}{h - 2m_2\ell_1^{\beta}} \right)^{1/\beta} d^{\alpha/\beta}(\lambda_1, \lambda_2). \quad (3.34)$$

Since $x_1 \in S(\lambda_1, \mu_1) = \bigcup_{\hat{\xi} \in B^*_e} S_{\hat{\xi}}(\lambda_1, \mu_1)$, there exists $\hat{\xi} \in B^*_e$ such that

$$x_1 = x_{\hat{\xi}}(\lambda_1, \mu_1) \in S_{\hat{\xi}}(\lambda_1, \mu_1). \quad (3.35)$$

It follows from (3.32) that there exists $i_0 \in \{1, 2, \ldots, n\}$ such that $\hat{\xi} \in N'(\hat{\xi}_{i_0})$. Thus, by the construction of the neighborhoods $\widetilde{N}(\lambda_0)$ and $\widetilde{N}(\mu_0)$, we have

$$(\lambda_1, \mu_1), (\lambda_2, \mu_2) \in N_{\hat{\xi}_{i_0}}(\lambda_0) \times N_{\hat{\xi}_{i_0}}(\mu_0). \quad (3.36)$$

Obviously, thanks to Theorem 3.3, we have

$$d\left(x_{\hat{\xi}}(\lambda_1, \mu_1), x_{\hat{\xi}}(\lambda_2, \mu_2)\right) \leq \left( \frac{m_1}{h - 2m_2\ell_1^{\beta}} \right)^{1/\beta} d^{\gamma/\beta}(\mu_1, \mu_2) + \left( \frac{2m_2\ell_2^{\beta}}{h - 2m_2\ell_1^{\beta}} \right)^{1/\beta} d^{\alpha/\beta}(\lambda_1, \lambda_2). \quad (3.37)$$

Let $x_2 = x_{\hat{\xi}}(\lambda_2, \mu_2)$. Then, (3.34) holds, and the proof is complete.

Remark 3.7. Theorem 3.6 generalizes, and improves the corresponding results of S. J. Li and X. B. Li [25] in the following three aspects.
(i) The vector-valued mapping \( F(x, y, \mu) \) is extended to set-valued, and the parametric vector equilibrium problem is extended to the parametric vector quasiequilibrium problem.

(ii) The assumption \((H_4)\) of Theorem 3.1 in [25] is removed.

(iii) The C-convexity of \( F(x, \cdot, \mu) \) (see Theorem 3.1 in [25]) is extended to C-convexlike-ness.

In addition, it is easy to see that the assumption (iv) of Theorem 3.6 is different form the assumption \((H_1)\) of Theorem 3.1 in S. J. Li and X. B. Li [25].

Moreover, we also can see that the obtained result extends the ones of [23]. Now, we give the following example to illustrate the case.

**Example 3.8.** Let \( X = Y = \mathbb{R}, \Lambda = M = [0, 1], C = \mathbb{R}_+, \) and \( e = \sqrt{2}/2 \in \text{int} C. \) Let \( K : X \times M \rightrightarrows Y \) be defined by \( K(x, \lambda) = [\lambda^2, 1], \) and let \( F : X \times X \times M \rightrightarrows Y \) be a set-valued mapping defined by

\[
F(x, y, \lambda) = \left[ \left( \frac{3}{4} + 2\lambda \right) (y + 3)(x - y), 20 - \left| x \right|^{1/2} \right].
\]

(3.38)

Consider that \( \lambda_0 = 0.5 \) and \( N(\lambda_0) = \Lambda. \) Then, \( E(\lambda) = [\lambda^2, 1] \) and \( E(\Lambda) = E(N(\lambda_0)) = [0, 1]. \)

Obviously, \( K(\cdot, \cdot) \) is \((0.1, \sqrt{2} \cdot 1)\)-Hölder continuous in \( E(N(\lambda_0)) \times N(\mu_0); \) for all \( x, y \in E(N(\lambda_0)), F(x, y, \cdot) \) is \( 7\sqrt{2} \cdot 1\)-Hölder continuous with respect to \( e = \sqrt{2}/2 \in \text{int} C \) at \( \lambda_0 \in M; \) for each \( x \in E(N(\lambda_0)) \) and \( \lambda \in E(N(\lambda_0)), F(x, \cdot, \lambda) \) is \( 9\sqrt{5} \cdot \)1-Hölder continuous with respect to \( e \in \text{int} C \) on \( E(N(\lambda_0)). \) Here \( \ell_0 = 0, \alpha_1 = 1, \ell_1 = \sqrt{2}, \alpha_2 = 1, m_1 = 7\sqrt{2}, \gamma_1 = 1, m_2 = 9\sqrt{5}, \gamma_2 = 1. \) Take \( \beta = 1 \) and \( h = 3/4, \) for any \( \xi \in B^*_2 \) and for all \( x, y \in E(N(\lambda_0))(x \neq y), \) we have

\[
hd^\beta(x, y) \leq d\left( \inf_{z \in F(x, y, \mu)} \xi(z), \mathbb{R}_+ \right) + d\left( \inf_{z \in F(y, x, \mu)} \xi(z), \mathbb{R}_+ \right)
\]

(3.39)

and also have \( \alpha_1 \gamma_2 = \beta \) and \( h = 3/4 > 2m_2\ell_1^{1\beta}. \) Therefore, all assumptions of Theorem 3.3 hold, and thus it is applicable.

However, the assumption (ii) of Theorem 3.1 (or (ii') of Theorem 4.1) in [23] does not hold. In fact, for any \( \lambda \in \Lambda, \) for any \( h > 0 \) and \( \beta > 0, \) there exists \( y_0 = 0 \in E(N(\lambda_0)) \setminus S(\lambda, \mu) \) such that

\[
F(y_0, \bar{x}, \lambda) + hB(0, d^\beta(\bar{x}, y_0)) = \left[ \left( \frac{3}{4} + 2\lambda \right) (\bar{x} + 3)\bar{x}, 20 - \left| \bar{x} \right|^{1/2} \right] + hB(0, d^\beta(0, \bar{x}))
\]

(3.40)

for all \( \bar{x} \in S(\lambda, \mu). \) Thus, Theorems 3.1 and 4.1 in Li et al. [23] are not applicable.

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