

Research Article

The Asymptotic Solution of the Initial Boundary Value Problem to a Generalized Boussinesq Equation

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The L^2 space solution of an initial boundary problem for a generalized damped Boussinesq equation is constructed. Certain assumptions on the coefficients of the equation are found to show the existence and uniqueness of the global solution to the initial boundary problem. The explicit expression for the large time asymptotic solution is obtained.

1. Introduction

The classical Boussinesq equation can be expressed by

$$u_{tt} = -\alpha u_{xxxx} + u_{xx} + \beta (u^2)_{xx}, \quad (1.1)$$

where $u(x, t)$ is the elevation of the free surface of fluid, the subscripts denote partial derivatives, and the constant coefficients α and β depend on the depth of fluid and the characteristic speed of the long waves. The Boussinesq equation (1.1) was first derived by Boussinesq [1], in 1872, to describe propagation of long waves with small amplitude on the surface of shallow water. Since then, extensive research has been carried out to study the property and solutions of the equation and its associated initial boundary value problems. Clarkson [2] proposed a general approach for constructing exact solutions of (1.1). Hirota [3] deduced conservation laws and then examined the numerical solution of a Boussinesq equation. Yajima [4] investigated the nonlinear evolution of a linearly stable solution for (1.1). The exponentially decaying solution of the spherical Boussinesq equation has been

investigated by Nakamura [5]. Galkin et al. [6] developed rational solutions of the one-dimensional Boussinesq equation with both zero and nonzero boundary conditions at the infinity in the space from the known solutions of the Kadomtzev-Petviashvili equation. The structure of their solutions generalizes a family of rational solutions of the Korteweg-de Vries equation to the case of two-wave processes. For many other methods to investigate the Boussinesq system and other shallow water models, the reader is referred to [7–13] and the references therein.

Various generalizations of the classical Boussinesq equation have been proposed and studied from many aspects, in particular the well-posedness of the Cauchy problem for the following equation,

$$u_{tt} = -u_{xxxx} + u_{xx} + (f(u))_{xx}. \quad (1.2)$$

Bona and Sachs [14] showed that the special solitary wave solutions of (1.2) are nonlinearly stable for a range of wave speeds. The local and global well-posedness of the problem has been proved by transforming (1.2) into a system of nonlinear Schrödinger equations [15].

Varlamov [16] considered the initial-boundary condition for the following damped Boussinesq equation:

$$u_{tt} - 2bu_{txx} = -\alpha u_{xxxx} + u_{xx} + \beta(u^2)_{xx}, \quad (1.3)$$

where the second term on the left-hand side is responsible for dissipation, $\beta \in R^1$, and α and b are assumed to be positive constants and satisfy the assumption that $\alpha > b^2$. An initial-boundary value problem for (1.3) with small initial data is considered for the case of one space dimension. The classical solution of the problem is constructed, and the long-time asymptotics is obtained in explicit form. The asymptotics show the presence of both time and space oscillations and the exponential decay of the solution in time due to dissipations. Varlamov [17] has also used the eigenfunction expansion method to consider the long-time asymptotics for a damped Boussinesq equation in a ball which is similar to (1.3).

The aim of this paper is to study the initial-boundary value problem for the following generalized damped Boussinesq equation:

$$u_{tt} - au_{ttxx} - 2bu_{txx} + \frac{2b}{a}u_t = -cu_{xxxx} + u_{xx} - p^2u + \beta(u^2)_{xx}, \quad (1.4)$$

where a , b and c are positive constants and constants $p \geq 0$ and $\beta \in R^1$. We assume the cases $a + c > b^2$ and $p \geq \sqrt{2b}/a$. The classical solutions of the initial-boundary value problem for (1.4) will be constructed in the form of a Fourier series with coefficients in their own term represented as series in terms of a small parameter involved in the initial conditions. It is shown that the new solution of the initial-boundary value problem for (1.4) is well-posedness. In addition, the long time behavior of our solution also shows the presence of damped oscillations decaying exponentially in time as $t \rightarrow \infty$.

2. Theorem of Existence, Uniqueness, and Asymptotics

In this paper, we consider the following initial-boundary value problem for the damped Boussinesq equation:

$$\begin{aligned}
 u_{tt} - au_{ttxx} - 2bu_{ttxx} + \frac{2b}{a}u_t &= -cu_{xxxx} + u_{xx} - p^2u + \beta(u^2)_{xx}, \\
 u(0, t) = u(\pi, t) &= 0, \quad t > 0, \\
 u_{xx}(0, t) = u_{xx}(\pi, t) &= 0, \quad t > 0, \\
 u(x, 0) = \varepsilon^2\varphi(x), \quad u_t(x, 0) &= \varepsilon^2\psi(x), \quad x \in (0, \pi),
 \end{aligned} \tag{2.1}$$

where a, b, c , and p are positive constants, $\beta \in R$, and ε is a small parameter.

Definition 2.1. The function $u(x) \in C^{2n}(0, \pi)$, $n \geq 1$, if $u(0) = u(\pi) = u''(0) = u''(\pi) = \dots = u^{(2n-2)}(0) = u^{(2n-2)}(\pi) = 0$ and $u^{(2n)}(x) \in L^2(0, \pi)$.

Definition 2.2. The function $u(x, t)$ defined on $[0, \pi] \times [0, +\infty)$ is said to be the classical solution of the problem defined by (2.1), if it has two continuous derivatives on $[0, \pi] \times [0, +\infty)$ and satisfies system (2.1).

The main results obtained are summarized in the following theorem.

Theorem 2.3. *If $a > 0$, $a + c > b^2$, $p \geq \sqrt{2b/a}$, $\varphi(x) \in C^8(0, \pi)$, $\psi(x) \in C^6(0, \pi)$, then there is a $\varepsilon_0 > 0$ such that, for $0 < \varepsilon \leq \varepsilon_0$, problem (2.1) has a unique classical solution of which is represented in the form of*

$$u(x, t) = \sum_{N=0}^{\infty} \varepsilon^{N+1} u^{(N)}(x, t), \tag{2.2}$$

where the function $u^{(N)}(x, t)$ is as that defined by expressions (3.24) in the proof below. This series and the series for the derivatives of $u(x, t)$ involved in (2.1) converge absolutely and uniformly with respect to $x \in [0, \pi]$, $t \geq 0$, $\varepsilon \in [0, \varepsilon_0]$. In addition, the solution of system (2.1) has the following long time behavior as $t \rightarrow +\infty$:

$$u(x, t) = e^{-(b/a)t} \left[(A \cos \sigma t + B \sin \sigma t) \sin x + O\left(e^{[-\eta b/a]t}\right) \right], \tag{2.3}$$

where $\sigma = \sqrt{ac + (a + c - b^2) + (1 + ap^2 - (b^2/a^2)) + p^2 - (b^2/a^2)/1 + a}$ is a positive constant, $0 < \eta < 1$, and the coefficients A and B are defined by (3.44).

3. Proof of Theorem 2.3

The proof of Theorem 2.3 includes three parts, namely, existence of a solution, uniqueness of the solution, and the asymptotics of the solution. The main techniques used in this paper are based on those presented in [16]. However, it should be emphasized that the method for proving uniqueness of the solution is different from that of [16].

3.1. Existence of a Solution

We make an odd extension of $u(x, t)$ in x to the interval $[-\pi, 0]$ and represent u in the form of a complex Fourier series; namely,

$$u(x, t) = \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \hat{u}_n(t) e^{inx}, \quad \hat{u}_n(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(x, t) e^{-inx} dx, \quad (3.1)$$

where $\hat{u}_{-n}(t) = -\hat{u}_n(t)$ for $n \geq 1$. For the above equalities, we shall use the fact that $u(x, t)$ belongs to the space $L^2(-\pi, \pi)$ for each fixed $t > 0$, and we shall denote the corresponding norm by

$$\|u(t)\| = \|u(t)\|_{L^2(-\pi, \pi)} = \left(\int_{-\pi}^{\pi} |u(x, t)|^2 dx \right)^{1/2}. \quad (3.2)$$

From (3.1), we obtain, through a simple calculation, that

$$u(x, t) = 2i \sum_{n=1}^{\infty} \hat{u}_n(t) \sin nx, \quad \hat{u}_n(t) = \frac{1}{i\pi} \int_0^{\pi} u(x, t) \sin nx dx. \quad (3.3)$$

Noting the initial functions, we can write on $[-\pi, \pi]$ with

$$\hat{\varphi}_{-n} = \hat{\varphi}_n, \quad \hat{\psi}_{-n} = \hat{\psi}_n, \quad (3.4)$$

for $n \geq 1$ that

$$\varphi(x) = \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \hat{\varphi}_n e^{inx}, \quad \psi(x) = \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \hat{\psi}_n e^{inx}. \quad (3.5)$$

Then on the interval $[0, \pi]$, we obtain

$$\begin{aligned} \varphi(x) &= 2i \sum_{n=1}^{\infty} \hat{\varphi}_n \sin nx, & \psi(x) &= 2i \sum_{n=1}^{\infty} \hat{\psi}_n \sin nx, \\ \hat{\varphi}_n &= \frac{1}{i\pi} \int_0^{\pi} \varphi(x) \sin nx dx, & \hat{\psi}_n &= \frac{1}{i\pi} \int_0^{\pi} \psi(x) \sin nx dx. \end{aligned} \quad (3.6)$$

Integrating (3.6) by parts and using the smoothness assumption of the initial data, we get the following inequalities:

$$|\widehat{\varphi}_n| \leq C_1 n^{-8}, \quad |\widehat{\psi}_n| \leq C_1 n^{-6}, \quad n \geq 1, \quad (3.7)$$

where C_1 is a positive constant. Substituting (3.1) and (3.5) into (2.1), we have the following Cauchy problem for the function $\widehat{u}_n(t)$, $n \in \mathbb{Z}$:

$$\begin{aligned} (1 + an^2)\widehat{u}_n''(t) + 2\left(bn^2 + \frac{b}{a}\right)\widehat{u}_n'(t) + (cn^4 + n^2 + p^2)\widehat{u}_n(t) &= -\beta n^2 P(\widehat{u}_n(t)), \\ \widehat{u}_n(0) = \varepsilon^2 \widehat{\varphi}_n, \quad \widehat{u}_n'(0) &= \varepsilon^2 \widehat{\psi}_n, \end{aligned} \quad (3.8)$$

where

$$P(\widehat{u}_n(t)) = \sum_{\substack{g=-\infty \\ g \neq 0, n}}^{\infty} \widehat{u}_{n-g}(t)\widehat{u}_g(t), \quad \widehat{u}_{-n}(t) = -\widehat{u}_n(t) \quad \text{for } n \geq 1. \quad (3.9)$$

For $n = 1$, we have

$$P(\widehat{u}_1(t)) = 2 \sum_{g=1}^{\infty} \widehat{u}_{-g}(t)\widehat{u}_{1+g}(t) = -2 \sum_{g=1}^{\infty} \widehat{u}_g(t)\widehat{u}_{1+g}(t). \quad (3.10)$$

For $n \geq 2$, it has

$$P(\widehat{u}_n(t)) = \sum_{g=1}^{n-1} \widehat{u}_{n-g}(t)\widehat{u}_g(t) - 2 \sum_{g=1}^{\infty} \widehat{u}_g(t)\widehat{u}_{n+g}(t). \quad (3.11)$$

Letting $\Phi_n = \varepsilon \widehat{\varphi}_n$, $\Psi_n = \varepsilon \widehat{\psi}_n$ and solving the initial value problem (3.8), we get

$$\begin{aligned} \widehat{u}_n(t) &= \varepsilon e^{-(b/a)t} \left\{ \left[\cos(\sigma_n t) + \frac{b}{a} \cdot \frac{\sin(\sigma_n t)}{\sigma_n} \right] \Phi_n + \frac{\sin(\sigma_n t)}{\sigma_n} \Psi_n \right\} \\ &\quad - \frac{\beta n^2}{\sigma_n(1 + an^2)} \int_0^t \exp \left[-\frac{b}{a}(t - \tau) \right] \sin [\sigma_n(t - \tau)] P(\widehat{u}_n(\tau)) d\tau, \end{aligned} \quad (3.12)$$

where

$$\begin{aligned} \sigma_n &= \frac{\sqrt{acn^6 + (a + c - b^2)n^4 + (1 + ap^2 - (2b/a))n^2 + p^2 - (b^2/a^2)}}{1 + an^2}, \\ n \geq 1, \quad a + c &> b^2, \quad p \geq \frac{\sqrt{2}b}{a}. \end{aligned} \quad (3.13)$$

Now, in order to solve the integral equation, we use the perturbation theory. Firstly, we express $\hat{u}_n(t)$, $n \geq 1$, as a formal series in ε :

$$\hat{u}_n(t) = \sum_{N=0}^{\infty} \varepsilon^{N+1} \hat{\xi}_n^{(N)}(t). \quad (3.14)$$

Then, by substituting (3.14) into (3.12) and equating the coefficients of the like powers of ε , we get the following formulas, respectively, for $N = 0$ and $N \geq 1$:

$$\hat{\xi}_n^{(0)}(t) = \varepsilon e^{-(b/a)t} \left\{ \left[\cos(\sigma_n t) + \frac{b}{a} \cdot \frac{\sin(\sigma_n t)}{\sigma_n} \right] \Phi_n + \frac{\sin(\sigma_n t)}{\sigma_n} \Psi_n \right\}, \quad (3.15)$$

$$\hat{\xi}_n^{(N)}(t) = -\frac{\beta n^2}{\sigma_n(1 + an^2)} \int_0^t \exp \left[-\frac{b}{a}(t - \tau) \right] \times \sin[\sigma_n(t - \tau)] Q_N(\hat{\xi}_n^{(j)}(\tau)) d\tau, \quad N \geq 1, \quad (3.16)$$

where $n \geq 1$ and

$$Q_N(\hat{\xi}_n^{(j)}(\tau)) = \varepsilon_n \sum_{g=1}^{n-1} \sum_{j=1}^N \hat{\xi}_{n-g}^{(j-1)}(\tau) \hat{\xi}_g^{(N-j)}(\tau) - 2 \sum_{g=1}^{\infty} \sum_{j=1}^N \hat{\xi}_{n+g}^{(j-1)}(\tau) \hat{\xi}_g^{(N-j)}(\tau), \quad (3.17)$$

in which $\varepsilon_1 = 0$ and $\varepsilon_n = 1$ for $n \geq 2$.

Now we must prove that the formally constructed function (3.3), together with (3.12)–(3.14), does represent a solution of problem (2.1). To do this, we shall show that the series

$$\begin{aligned} u(x, t) &= \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} e^{inx} \sum_{N=0}^{\infty} \varepsilon^{N+1} \hat{\xi}_n^{(N)}(t) \\ &= 2i \sum_{n=1}^{\infty} \sin nx \sum_{N=0}^{\infty} \varepsilon^{N+1} \hat{\xi}_n^{(N)}(t) \end{aligned} \quad (3.18)$$

converges absolutely and uniformly. For this purpose, we firstly establish the following time estimates for $n \geq 1$, $t > 0$, $N \geq 0$:

$$\left| \hat{\xi}_n^{(N)}(t) \right| \leq C^N (N + 1)^{-2} n^{-6} e^{-(b/a)t}. \quad (3.19)$$

Here and in the sequel we denote by C any positive constant independent of N , n , ε , and t , but possibly depending on the coefficients of the equation and the initial functions.

We shall use the induction on the number N . For $N = 0$, we have from (3.7) and (3.15) that

$$\begin{aligned} \left| \hat{\xi}_n^{(0)}(t) \right| &\leq \varepsilon e^{-(b/a)t} \left[\left(1 + \frac{b}{a\sigma_n} \right) \left| \hat{\Phi}_n \right| + \frac{1}{\sigma_n} \left| \Psi_n \right| \right] \\ &\leq \varepsilon n^{-6} e^{-(b/a)t}. \end{aligned} \quad (3.20)$$

Assuming that (3.19) is valid for all $\widehat{\xi}_n^{(s)}(t)$ with $0 \leq s \leq N - 1$, we shall prove that (3.19) also holds for $s = N$. According to [16], for any integer $n \geq 1$, $g \geq 1$, and $g \neq n$, we have

$$\begin{aligned} |n - g|^{-6} g^{-6} &\leq 2^6 n^{-6} [g^{-6} + |n - g|^{-6}], \\ j^{-2} (N + 1 - j)^{-2} &\leq 2^2 (N + 1)^{-2} [j^{-2} + (N + 1 - j)^{-2}]. \end{aligned} \tag{3.21}$$

From (3.16) and noting that

$$\frac{bn^2}{1 + an^2} \geq \frac{b}{a} \quad \text{for } n \neq 0, \tag{3.22}$$

we have

$$\begin{aligned} \left| \widehat{\xi}_n^{(N)}(t) \right| &\leq C |\beta| (N + 1)^{-2} n^{-6} \sum_{g=1}^{\infty} (g^{-6} + |n - g|^{-6}) \\ &\quad \times \sum_{j=1}^{\infty} C^{j-1} C^{N-j} [(N + 1 - j)^{-2} + j^{-2}] |S_N(n, t)|, |S_N(n, t)| \\ &\leq C e^{-(b/a)t} \int_0^t e^{((b/a)-(2b/a)\tau)} d\tau \\ &\leq C e^{-(b/a)t} \times \left| \frac{e^{((b/a)-(2b/a)t)} - 1}{(b/a) - (2b/a)} \right| \\ &\leq C e^{-(b/a)t}. \end{aligned} \tag{3.23}$$

Therefore, we know that inequality (3.19) holds.

For the derivation of (3.15), we recall (3.3) and (3.14) with $\widehat{\xi}_n^{(N)}$ defined by (3.15) and (3.16) and interchange the order of summations in the series; namely,

$$u(x, t) = 2i \sum_{n=1}^{\infty} \widehat{u}_n(t) \sin nx = 2i \sum_{n=1}^{\infty} \sin nx \sum_{N=0}^{\infty} \varepsilon^{N+1} \widehat{\xi}_n^{(N)}(t) = \sum_{N=0}^{\infty} \varepsilon^{N+1} u^{(N)}(x, t), \tag{3.24}$$

where

$$u^{(N)}(x, t) = 2i \sum_{n=1}^{\infty} \widehat{\xi}_n^{(N)} \sin nx. \tag{3.25}$$

The interchange of the order of summations is allowable as the series is absolutely and uniformly convergent for $x \in [0, \pi]$, $t \geq 0$ and $\varepsilon \in [0, \varepsilon_0]$. Differentiating (3.15) and (3.16), we get, for $k = 1, 2$, that

$$\begin{aligned} \partial_t^k \widehat{\xi}_n^{(0)}(t) &= \sum_{l=0}^k c_k^l (-1)^l \left(\frac{b}{a}\right)^l e^{-(b/a)t} \partial_k^{k-l} \left\{ \left[\cos(\sigma_n t) + \frac{b \sin(\sigma_n t)}{a \sigma_n} \right] \Phi_n + \frac{\sin(\sigma_n t)}{\sigma_n} \Psi_n \right\}, \\ \partial_t^k \widehat{\xi}_n^{(N)}(t) &= -\frac{\beta n^2}{(1 + an^2)\sigma_n} \int_0^t g_k(n, t - \tau) Q_N(\widehat{\xi}_n^{(j)}(\tau)) d\tau + R_k(n, t), \end{aligned} \quad (3.26)$$

where

$$g_k(n, t) = \sum_{l=0}^k c_k^l (-1)^l \left(\frac{b}{a}\right)^l e^{-(b/a)t} \sigma_n^{k-l} \sin \left[\sigma_n t + \frac{(k-l)\pi}{2} \right], \quad (3.27)$$

$Q_N(\widehat{\xi}_n^{(j)}(\tau))$ is defined by (3.16), c_k^l ($l = 0, 1, \dots, k$) are binomial coefficients, and $R_k(n, t)$ ($k = 1, 2$) are obtained by differentiating the integral in (3.16) and are as follows:

$$R_1(n, t) = 0, \quad R_2(n, t) = -\beta n^2 Q_N(\widehat{\xi}_n^{(j)}(t)). \quad (3.28)$$

Hence, using (3.19), we deduce that for, $n \geq 1$, $N \geq 0$, $t > 0$, and $k \geq 0, 1, 2$,

$$\begin{aligned} \left| \partial_t^k \widehat{\xi}_n^{(N)}(t) \right| &\leq C^N (N+1)^{-2} n^{2k-6} e^{-(b/a)t}, \\ \left| \partial_t^k \widehat{u}_n(t) \right| &\leq C n^{2k-6} e^{-(b/a)t}. \end{aligned} \quad (3.29)$$

Using these estimates and calculating the necessary derivatives of (3.29), we can prove straightforwardly that (3.29) represents the classical solution of the initial-boundary value problem defined by system (2.1).

3.2. Uniqueness of the Solution

For proving the uniqueness of the constructed solution, we shall assume that there exist two classical solutions $u^{(1)}(x, t)$ and $u^{(2)}(x, t)$ to system (2.1) and then deduce that $u^{(1)}(x, t)$ must be equal to $u^{(2)}(x, t)$.

Making an odd extension of the two solutions to the segment $(-\pi, 0]$, we notice that both of them belong to the $L^2(-\pi, \pi)$ space, and according to Definition 2.2, we have for each fixed time $t > 0$ that

$$\max_{x \in [-\pi, \pi]} |u^{(1)}(x, t)| \leq C_t, \quad \max_{x \in [-\pi, \pi]} |u^{(2)}(x, t)| < C_t, \tag{3.30}$$

where C_t is a constant depending on t . Let $w(x, t) = u^{(1)}(x, t) - u^{(2)}(x, t)$, and make an even extension of $w(x, t)$ to $\dots(-3\pi, -2\pi), (-2\pi, -\pi), (\pi, 2\pi), (2\pi, 3\pi), \dots$. Then, we have from (2.1) that

$$\begin{aligned} w_{tt} - aw_{ttxx} - 2bw_{txx} + \frac{2b}{a}w_t &= -cw_{xxxx} + w_{xx} - p^2w + \beta \left[w(x, t) \left(u^{(1)}(x, t) + u^{(2)}(x, t) \right) \right]_{xx}, \\ w(x, 0) &= w_t(x, 0) = 0. \end{aligned} \tag{3.31}$$

Taking the Fourier transform of w in the interval $(-\infty, +\infty)$; namely,

$$\widehat{w}(\xi, t) = \int_{-\infty}^{+\infty} e^{-ix\xi} w(x, t) dx, \tag{3.32}$$

we have

$$\left(1 + a\xi^2\right)\widehat{w}''(\xi, t) + 2b\left(\xi^2 + \frac{b}{a}\right)\widehat{w}(\xi, t) + \left(c\xi^4 + \xi^2 + p^2\right)\widehat{w}(\xi, t) = -\beta\xi^2\widehat{f}(\xi, t), \tag{3.33}$$

where

$$f(x, t) = w\left(u^{(1)} + u^{(2)}\right)(x, t). \tag{3.34}$$

From (3.33), we have

$$\widehat{w}(\xi, t) = -\frac{\beta\xi^2}{\sigma_\xi(1 + a\xi^2)} \int_0^t \exp\left[-\frac{b}{a}(t - \tau)\right] \sin(\sigma_\xi(t - \tau)) \widehat{f}(\xi, \tau) d\tau, \tag{3.35}$$

where

$$\sigma_\xi = \frac{\sqrt{ac\xi^6 + (a + c - b)\xi^4 + (1 + ap^2 - (2b^2/a))\xi^2 + p^2 - (b^2/a^2)}}{1 + a\xi^2}. \tag{3.36}$$

Thus, it has

$$\begin{aligned}
 |\widehat{w}(\xi, t)| &\leq C \int_0^t \exp\left[-\frac{b}{a}(t-\tau)\right] \widehat{f}(\xi, \tau) d\tau \\
 &\leq C \left[\int_0^t \exp\left[-\frac{b}{a}(t-\tau)\right] d\tau \right]^{1/2} \left[\int_0^t |\widehat{f}(\xi, \tau)|^2 d\tau \right]^{1/2} \\
 &\leq C \left[\int_0^t |\widehat{f}(\xi, \tau)|^2 d\tau \right]^{1/2}.
 \end{aligned} \tag{3.37}$$

It follows from (3.37) and the Parseval inequality that

$$\begin{aligned}
 \int_{-\infty}^{+\infty} |\widehat{w}(\xi, t)|^2 d\xi &\leq C \int_{-\infty}^{+\infty} \int_0^t |\widehat{f}(\xi, \tau)|^2 d\tau d\xi \\
 &\leq C \int_0^t \|\widehat{f}(\xi, \tau)\|_{L^2}^2 d\tau \\
 &\leq C \int_0^t \|w(x, \tau) (u^{(1)}(x, \tau) + u^{(2)}(x, \tau))\|_{L^2}^2 d\tau \\
 &\leq C \int_0^t C_\tau \|w(x, \tau)\|_{L^2}^2 d\tau.
 \end{aligned} \tag{3.38}$$

Using the Gronwall inequality, we obtain

$$w(x, t) = 0 \quad (\text{in } L^2), \tag{3.39}$$

which implies that $u^{(1)}(x, t) \equiv u^{(2)}(x, t)$ and thus the solution is unique.

3.3. Long Time Asymptotics

To find the long-time behavior of the constructed solution, we firstly determine a subtle asymptotic estimate of $\widehat{u}_1(t)$ which will contribute to the major term and then estimate the remaining terms $\sum_{n=2}^{\infty} \widehat{u}_n(t) \sin nx$.

Since

$$\widehat{u}_1(t) = \sum_{N=0}^{\infty} \varepsilon^{N+1} \widehat{\xi}_1^{(N)}(t), \tag{3.40}$$

we substitute it into the integral equations (3.15) and (3.16) to obtain

$$\begin{aligned} \widehat{\xi}_1^{(0)}(t) &= e^{-(b/a)t} \left[A^{(0)} \cos(\sigma t) + B^{(0)} \sin(\sigma t) \right], \\ \widehat{\xi}_1^{(N)}(t) &= e^{-(b/a)t} \left\{ \left[A^{(N)} + R_A^{(N)}(t) \right] \cos(\sigma t) + \left[B^{(N)} + R_B^{(N)}(t) \right] \sin(\sigma t) \right\}, \end{aligned} \tag{3.41}$$

where

$$\begin{aligned} A^{(0)} &= \varepsilon \widehat{\varphi}_1, \\ B^{(0)} &= \frac{\varepsilon}{\sigma} \left(\frac{b}{1+a} \widehat{\varphi}_1 + \widehat{\psi}_1 \right), \\ \sigma &= \frac{\sqrt{ac + 1 + a + c - b^2}}{1+a}, \\ A^{(N)} &= \frac{\beta}{\sigma(1+a)} \int_0^t e^{(b/a)\tau} \sin(\sigma\tau) S_{u_n}(\tau) d\tau, \\ B^{(N)} &= -\frac{\beta}{\sigma(1+a)} \int_0^t e^{(b/a)\tau} \cos(\sigma\tau) S_{u_n}(\tau) d\tau, \\ R_A^{(N)} &= \frac{\beta}{\sigma(1+a)} \int_0^{+\infty} e^{(b/a)\tau} \sin(\sigma\tau) S_{u_n}(\tau) d\tau, \\ R_B^{(N)} &= -\frac{\beta}{\sigma(1+a)} \int_0^{+\infty} e^{(b/a)\tau} \cos(\sigma\tau) S_{u_n}(\tau) d\tau, \\ S_{u_n}(t) &= \sum_{g=1}^{\infty} \sum_{j=1}^N \widehat{\xi}_{1+g}^{(j-1)}(t) \widehat{\xi}_g^{(N-j)}(t), \quad N \geq 1, \end{aligned} \tag{3.42}$$

and the function $\widehat{\xi}^{(j)}(t)$, $j = 0, 1, \dots, N - 1$ are defined by (3.15)-(3.16). Taking $n \geq 2$, $N \geq 1$ and using a method similar to that used in [16], it follows from (3.19) and (3.24) that there exists a positive number $0 < \eta < 1$ such that

$$\left| R_A^{(N)}(t) \right| \leq c e^{-(\eta b t/a)}, \quad \left| R_B^{(N)}(t) \right| \leq c e^{-(\eta b t/a)}. \tag{3.43}$$

Hence, we have proved that, as $t \rightarrow +\infty$,

$$\begin{aligned} 2i\widehat{u}_1(t) &= e^{-(b/a)t} [A \cos(\sigma t) + B \sin(\sigma t)] + O\left(e^{-(1+\eta)b/a)t}\right), \\ A &= 2i \sum_{N=0}^{\infty} \varepsilon^{N+1} A^{(N)}, \quad B = 2i \sum_{N=0}^{\infty} \varepsilon^{N+1} B^{(N)}, \end{aligned} \tag{3.44}$$

where $0 < \eta < 1$, $A^{(N)}$ and $B^{(N)}$ are defined by (3.41), and the series above converge absolutely and uniformly for $\varepsilon \in [0, \varepsilon_0]$.

Now, we can represent the solution by

$$\begin{aligned} u(x, t) &= 2i\widehat{u}_1(t) + R_u(x, t), \\ R_u(x, t) &= 2i \sum_{n=2}^{\infty} \sin nx \sum_{N=0}^{\infty} \widehat{\xi}_n^{(N)}(t). \end{aligned} \quad (3.45)$$

Using (3.24), we deduce that

$$|R_u(x, t)| \leq \exp\left[\frac{-\eta b}{a}t\right] \sum_{n=0}^{\infty} c^N \varepsilon^{N+1} (N+1)^{-2} \sum_{N=2}^{\infty} n^{-6} \leq c \exp\left[\frac{-\eta b}{a}t\right], \quad (3.46)$$

$$u(x, t) = e^{-(b/a)t} \left[(A \cos \sigma t + B \sin \sigma t) \sin x + O\left(e^{-[(\eta b/a)t]}\right) \right], \quad 0 < \eta < 1. \quad (3.47)$$

It follows from (3.47) that the inequality (2.3) holds.

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