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Research Article

Regularizing Model for the 2D MHD Equations with Zero Viscosity

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We consider the regularity of two dimensional incompressible magneto-hydrodynamics equations with zero viscosity. We provide an approximating system to the equations and prove global-in-time existence of classical solution to this approximating system. By using approximating system, a priori estimates for the equations can be justified.

1. Introduction

In this paper, we are concerned with regularity problem of solutions to the 2-dimensional incompressible magnetohydrodynamics (MHD) equations with zero viscosity

$$\begin{gathered}
u_t + (u \cdot \nabla)u + \nabla p = (b \cdot \nabla)b, \quad x \in \mathbb{R}^2,\ t > 0, \\
b_t - \Delta b + (u \cdot \nabla)b = (b \cdot \nabla)u, \quad x \in \mathbb{R}^2,\ t > 0, \\
\nabla \cdot u = \nabla \cdot b = 0, \quad x \in \mathbb{R}^2,\ t > 0,
\end{gathered} \tag{1.1}$$

where $u = (u_1, u_2)$, $b = (b_1, b_2)$, and p are fluid velocity vector field, magnetic field, and pressure function. The underlying idea of MHD is that dynamics of magnetic field induces the force on the fluid and, in turn, the motion of the conducting fluid affects the dynamics of magnetic field. MHD has many applications in electromagnetics, plasma theory, and cosmology. MHD equations describe the dynamics of the conducting fluids and thus, MHD equations are expressed as the combinations of the fluid equations and Maxwell system. We restrict our interest on the incompressible Euler equation (with Lorentz force of magnetic field) as the fluid equation. This is the special case that the fluid viscosity is quite smaller than the magnetic diffusivity. Recall that magnetic Prandtl number is approximately the ratio of viscosity and magnetic diffusivity. So (1.1) represents the zero magnetic Prandtl number case. In physics, liquid metal usually has the small magnetic Prandtl number. For the two-dimensional incompressible MHD equations with positive viscosity and magnetic diffusivity, it is well known that there exists a unique global classical solution for every initial data $(u_0, b_0) \in H^m$ with $m \geq 2$ and $\nabla \cdot u_0 = \nabla \cdot b_0 = 0$ (see [1, 2]). But if viscosity of the fluid or the magnetic diffusivity is zero, then the global regularity issue for 2D remains as an open problem. For the case that viscosity is positive and the magnetic diffusivity is zero, then there are some studies on the regularity or blow-up criterion. We briefly recall a few of them. In [3], the authors showed that if $\int_0^T \|\nabla \times \omega\|_{L^\infty} dt < \infty$, then the solution (u, b) remain smooth on $[0, T]$. In [4], the authors showed that if $\int_0^T \|\nabla \times b\|_{\mathrm{BMO}} dt < \infty$, then the solution (u, b) remain smooth on $[0, T]$. In [5], it was shown that if $\int_0^T \|b \times b\|_{\mathrm{BMO}} dt < \infty$, then the solution (u, b) remain smooth on $[0, T]$. For the case that the viscosity is zero and the magnetic diffusivity is positive, that is, for the system (1.1), we have better a prior estimates. In [6–8], the authors obtained the global existence of more regular weak solution, that is, we have $u \in L^\infty(0, T; H^1)$ and $b \in L^\infty(0, T; H^1) \cap L^2(0, T; H_0^2)$ for any $T > 0$.

But still the global existence of smooth solution is remained as a challenging open problem. Only some blow-up criterion for the system (1.1) is known. In [7], Cao and Wu showed that if for some $T > 0$,

$$\sup_{q \geq 2} \frac{1}{q} \int_0^T \|\nabla u\|_{L^q} dt < \infty, \tag{1.2}$$

then (u, b) remains smooth on $[0, T]$. Also Lei and Zhou [8] obtained the regularity criterion in terms of the $L^1(0, T; \mathrm{BMO})$ norm of $\nabla \times \omega$. In [6], Kozono studied the stability of the solution to (1.1) (see [9] also). And in [10], the authors studied the 2D incompressible MHD equations with horizontal dissipation and horizontal magnetic diffusion.

In this paper, we consider approximating system of (1.1), which still preserves some properties of (1.1). We consider

$$\begin{gathered} \partial_t u + (u \cdot \nabla) u + \nabla p = (-\Delta)^{-\alpha}((b \cdot \nabla) b), \\ \partial_t b + (u \cdot \nabla) b - \Delta b = (b \cdot \nabla)\left((-\Delta)^{-\alpha} u\right), \\ \nabla \cdot u = \nabla \cdot b = 0. \end{gathered} \tag{1.3}$$

We study the regularity issue for (1.3). We can rewrite (1.3) into the equations of the vorticity $\omega = \partial_1 u_2 - \partial_2 u_1$ and the current density $j = \partial_1 b_2 - \partial_2 b_1$ as follows:

$$\begin{aligned} \omega_t + (u \cdot \nabla)\omega &= (-\Delta)^{-\alpha}\left((b \cdot \nabla) j\right), \quad x \in \mathbb{R}^2,\ t > 0, \\ j_t + (u \cdot \nabla) j - \Delta j &= (b \cdot \nabla)(-\Delta)^{-\alpha}\omega - \partial_2 b_1\left(1 + (-\Delta)^{-\alpha}\right)\partial_1 u_1 - \partial_2 b_2\left(1 + (-\Delta)^{-\alpha}\right)\partial_1 u_2 \\ &\quad + \partial_1 b_1\left(1 + (-\Delta)^{-\alpha}\right)\partial_2 u_1 + \partial_1 b_2\left(1 + (-\Delta)^{-\alpha}\right)\partial_2 u_2, \quad x \in \mathbb{R}^2,\ t > 0. \end{aligned} \tag{1.4}$$

We state our main results.

Theorem 1.1. *Assume* $(u_0, b_0) \in H^3$ *and* $\nabla \cdot u_0 = \nabla \cdot b_0 = 0$. *Then for any* $T > 0$ *and* $\alpha \in (0, 1/2)$, *there exists a unique solution* $(u, b) \in L^\infty(0, T; H^3) \times L^\infty(0, T; H^3) \cap L^2(0, T; H_0^4)$ *of* (1.3). *Furthermore, there exist constants C and D satisfying*

$$\|u\|_{L^\infty(0,T;H^3)} + \|b\|_{L^\infty(0,T;H^3)} + \|b\|_{L^2(0,T;H_0^4)} \le De^{CT}, \tag{1.5}$$

where C and D depend on $\|u_0\|_{H^3}$, $\|b_0\|_{H^3}$, *and* α.

2. Global-in-Time Existence of Smooth Solution

To prove Theorem 1.1, we present some regularity criterion for the solution to (1.3). The proof is standard and very similar to the criterion in [7, 8]. But for the readers' sake, we provide the sketch of the proof.

Proposition 2.1. *Assume the initial data* $(u_0, b_0) \in H^3$, $\nabla \cdot u_0 = 0$ *and* $\nabla \cdot b_0 = 0$. *Let* (u, b) *be the corresponding solution of* (1.3). *If, for some* $T > 0$,

$$\int_0^T \|\omega(t)\|_{L^\infty} dt < \infty, \tag{2.1}$$

then (u, b) *is regular on* $[0, T]$, *namely,* $(u, b) \in C([0, T]; H^3)$.

Proof. We provide brief sketch of proof. If we take ∇^3 operator on the fluid equations and magnetic field equations and take inner product with $\nabla^3 u$ and $\nabla^3 b$, respectively, then we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\left\|\nabla^3 u\right\|_{L^2}^2 + \left\|\nabla^3 b\right\|_{L^2}^2\right) + \left\|\nabla^4 b\right\|_{L^2}^2 \\ &\quad = -\int_{\mathbb{R}^2} \nabla^3((u\cdot\nabla)u)\nabla^3 u\,dx + \int_{\mathbb{R}^2} \nabla^3(-\Delta)^{-\alpha}((b\cdot\nabla)b)\nabla^3 u\,dx \\ &\qquad - \int_{\mathbb{R}^2} \nabla^3((u\cdot\nabla)b)\nabla^3 b\,dx + \int_{\mathbb{R}^2} \nabla^3\left((b\cdot\nabla)(-\Delta)^{-\alpha}u\right)\nabla^3 b\,dx. \end{aligned} \tag{2.2}$$

There are some cancellation properties, that is,

$$\begin{aligned} -\int_{\mathbb{R}^2} (u\cdot\nabla)\nabla^3 u\cdot\nabla^3 u\,dx &= 0, \\ \int_{\mathbb{R}^2} (-\Delta)^{-\alpha}\left((b\cdot\nabla)\nabla^3 b\right)\nabla^3 u\,dx &= \int_{\mathbb{R}^2} \left((b\cdot\nabla)\nabla^3 b\right)\nabla^3(-\Delta)^{-\alpha} u\,dx \\ &= -\int_{\mathbb{R}^2} \left((b\cdot\nabla)\nabla^3(-\Delta)^{-\alpha} u\right)\nabla^3 b\,dx. \end{aligned} \tag{2.3}$$

Using the previous cancellation, we have

$$\frac{1}{2}\frac{d}{dt}\left(\left\|\nabla^3 u\right\|_{L^2}^2+\left\|\nabla^3 b\right\|_{L^2}^2\right)+\left\|\nabla^4 b\right\|_{L^2}^2\leq C\|\nabla u\|_{L^\infty}\left(\left\|\nabla^3 u\right\|_{L^2}^2+\left\|\nabla^2 b\right\|_{L^2}^2\right). \tag{2.4}$$

Then recall the following Beale-Kato-Majda's logarithmic inequality [11],

$$\|\nabla u\|_{L^\infty}\leq C\|\omega\|_{L^\infty}(1+\ln(1+\|u\|_{H^3})). \tag{2.5}$$

We have the conclusion via Gronwall type inequality. □

We provide the Proof of Theorem 1.1.

Proof of Theorem 1.1. For simplicity of the exposition, the calculations are presented on 0 smooth solutions. All the calculations can be justified by using continuation method of the local solutions. If we multiply both sides of the first and second equations of (1.4) by ω and j, respectively, and integrate over $\mathbb{R}^2$, then we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\omega\|_{L^2}^2 &\leq \int_{\mathbb{R}^2}(b\cdot\nabla)j\cdot(-\Delta)^{-\alpha}\omega\,dx,\\
\frac{1}{2}\frac{d}{dt}\|j\|_{L^2}^2+\|\nabla j\|_{L^2}^2 &\leq \int_{\mathbb{R}^2}(b\cdot\nabla)(-\Delta)^{-\alpha}\omega\cdot j\,dx+C\int_{\mathbb{R}^2}|\nabla u||\nabla B||j|dx\\
&\leq \int_{\mathbb{R}^2}(b\cdot\nabla)(-\Delta)^{-\alpha}\omega\cdot j\,dx+C\|\nabla u\|_{L^2}\|j\|_{L^4}^2.
\end{aligned} \tag{2.6}$$

Since we have

$$\int_{\mathbb{R}^2}(b\cdot\nabla)(-\Delta)^{-\alpha}\omega\cdot j\,dx=-\int_{\mathbb{R}^2}(b\cdot\nabla)(-\Delta)^{-\alpha}j\cdot\omega\,dx, \tag{2.7}$$

we obtain the following by adding the above inequalities:

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\left(\|\omega\|_{L^2}^2+\|j\|_{L^2}^2\right)+\|\nabla j\|_{L^2}^2 &\leq C\|\nabla u\|_{L^2}\|j\|_{L^4}^2\\
&\leq C\|\omega\|_{L^2}\|j\|_{L^2}\|\nabla j\|_{L^2}\\
&\leq C\|\omega\|_{L^2}^2\|j\|_{L^2}^2+\frac{1}{2}\|\nabla j\|_{L^2}^2.
\end{aligned} \tag{2.8}$$

By using Gronwall's inequality, we have

$$\begin{aligned}
&\sup_{0\leq t\leq T}\left(\|\omega(t)\|_{L^2}^2+\|j(t)\|_{L^2}^2\right)+\int_0^T\|\nabla j(t)\|_{L^2}^2dt\\
&\quad\leq\left(\|\omega_0\|_{L^2}^2+\|j_0\|_{L^2}^2\right)\exp\left(C\int_0^T\|j\|_{L^2}^2dt\right).
\end{aligned} \tag{2.9}$$

Then we obtain $\omega\in L^\infty(0,T;L^2)$ and $j\in L^\infty(0,T;L^2)\cap L^2(0,T;H^1)$.

By standard $L^{p,q}$ estimate of the Stokes system or heat equation (see [12, 13]), we have

$$\int_0^t \|\nabla j\|_{L^p}^q dt \le C\int_0^t \left\|(b\cdot\nabla)\left((-\Delta)^{-\alpha}u\right)\right\|_{L^p}^q dt + C\|j_0\|_{H^2}, \tag{2.10}$$

for any $p, q \in (1,\infty)$. We also recall the following inequality:

$$\left\|(b\cdot\nabla)\left((-\Delta)^{-\alpha}u\right)\right\|_{L^p} \le \|b\|_{L^{2p}}\|u\|_{W_0^{1-2\alpha,2p}}. \tag{2.11}$$

Since $j \in L^\infty(0,T;L^2)\cap L^2(0,T;H^1)$, $b \in L^\infty(0,T;L^{2p})$ for $p<\infty$. Then we have

$$\int_0^t \|\nabla j\|_{L^p}^q dt \le C\int_0^t \|u\|_{W_0^{1-2\alpha,2p}}^q dt + C. \tag{2.12}$$

If we consider a usual trajectory map $X(x,t)$ such that $(dX/dt)(x,t) = u(X(x,t),t)$ and $X(x,0)=x$, then we can rewrite the first equation of (1.4) as

$$\frac{d}{dt}\omega(X(x,t),t) = (-\Delta)^{-\alpha}\nabla\cdot\left(bj(X(x,t),t)\right). \tag{2.13}$$

Thus we have

$$\sup_{0\le t\le T}\|\omega\|_{L^r} \le \|\omega_0\|_{L^r} + \int_0^t \|bj\|_{W^{1-2\alpha,r}} dt, \tag{2.14}$$

for any $r \in [1,\infty]$.

From the calculus inequality, we have

$$\|bj\|_{W^{1-2\alpha,r}} \le C\left(\|b\|_{L^{2r}}\|j\|_{W^{1-2\alpha,2r}} + \|b\|_{W^{1-2\alpha,2r}}\|j\|_{L^{2r}}\right). \tag{2.15}$$

For $r \in [2,\infty)$ and $p \ge 2r/(1+\alpha r)$, by Sobolev inequality, we have

$$\|j\|_{W^{1-2\alpha,2r}} \le C\|j\|_{W_0^{1,p}} + C\|j\|_{L^2}. \tag{2.16}$$

For the case $r=\infty$, we choose p such that $\alpha p > 1$, then we have

$$\|j\|_{W^{1-2\alpha,\infty}} \le C\|j\|_{W_0^{1,p}} + C\|j\|_{L^2}. \tag{2.17}$$

Hence for any $r < \infty$ and $p \geq 2r/(1+\alpha r)$, we have

$$\begin{aligned}\sup_{0\leq t\leq T}\|\omega\|_{L^r} &\leq \|\omega_0\|_{L^r} + C\left(\int_0^T \|b\|_{L^{2r}}\|j\|_{W^{1-2\alpha,2r}} + \|b\|_{W^{1-2\alpha,2r}}\|j\|_{L^{2r}}dt\right)\\ &\leq \|\omega_0\|_{L^r} + C\left(\int_0^T \|j\|_{W_0^{1,p}}dt + C\right).\end{aligned} \tag{2.18}$$

Since $j \in L^\infty(0,T;L^2) \cap L^2(0,T;H_0^1)$, we have the finiteness of $\|b\|_{L^\infty(0,T;L^{2r})} + \|b\|_{L^2(0,T;W^{1-2\alpha,r})} + \|j\|_{L^2(0,T;H_0^1)}$ in the above inequality.

If we use (2.12), then for any $q \in (1,\infty)$, we have

$$\sup_{0\leq t\leq T}\|\omega\|_{L^r} \leq C + CT^{(q-1)/q}\left(\int_0^T \|u\|^q_{W_0^{1-2\alpha,2p}}dt\right)^{1/q}. \tag{2.19}$$

If we choose p such that $4r/(1+\alpha r) \leq 2p \leq r/(1-\alpha r)$(we can choose such p if $r \geq 3/5\alpha$), then for any $q \in (1,\infty)$, we have

$$\sup_{0\leq t\leq T}\|\omega\|_{L^r} \leq C + CT^{(q-1)/q}\left(\int_0^T \|\omega\|^q_{L^r}dt\right)^{1/q}. \tag{2.20}$$

By using Gronwall's inequality, we have

$$\sup_{0\leq t\leq T}\|\omega\|_{L^r} \leq C, \tag{2.21}$$

for any $r \in [3/5\alpha,\infty)$. In turn, it gives the bound of $\|\nabla j\|_{L^q(0,t;L^p)}$ for all $p,q \in (1,\infty)$.

For $r = \infty$ and $p > 1/\alpha$, we have

$$\begin{aligned}\sup_{0\leq t\leq T}\|\omega\|_{L^\infty} &\leq \|\omega_0\|_{L^\infty} + C\left(\int_0^T \|b\|_{L^\infty}\|j\|_{W^{1-2\alpha,\infty}} + \|b\|_{W^{1-2\alpha,\infty}}\|j\|_{L^\infty}dt\right)\\ &\leq \|\omega_0\|_{L^r} + C\left(\int_0^T \|j\|_{W_0^{1,p}}dt + C\right).\end{aligned} \tag{2.22}$$

We already have the finiteness of $\|\nabla j\|_{L^1(0,T;L^p)}$; this gives our conclusion by Proposition 2.1. □

Remark 2.2. With the similar arguments in the proof of Theorem 1.1 and this approximating system, we can prove rigorously that the solution (u,b) to (1.1) satisfies $u \in L^\infty(0,T;W^{1,p})$ and $b \in L^\infty(0,T;W^{1,p}) \cap L^q(0,T;W_0^{2,p})$ for any $T > 0$ and $(p,q) \in [2,\infty) \times (1,\infty)$ as the remark in [8].

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