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Research Article

A Perturbed Projection Algorithm with Inertial Technique for Split Feasibility Problem

Yazheng Dang,[1,2] Yan Gao,[1] and Yanli Han[1,2]

[1] *School of Management, University of Shanghai for Science and Technology, 516 Jungong Road, Shanghai 200093, China*
[2] *Department of Mathematics and Information Science, Henan Polytechnic University, Henan 454001, China*

Correspondence should be addressed to Yazheng Dang, jgdyz@163.com

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This paper deals with the split feasibility problem that requires to find a point closest to a closed convex set in one space such that its image under a linear transformation will be closest to another closed convex set in the image space. By combining perturbed strategy with inertial technique, we construct an inertial perturbed projection algorithm for solving the split feasibility problem. Under some suitable conditions, we show the asymptotic convergence. The results improve and extend the algorithms presented in Byrne (2002) and in Zhao and Yang (2005) and the related convergence theorem.

1. Introduction

Let $C \subset R^n$ and $Q \subset R^m$ be nonempty closed convex sets, and let A be an $m \times n$ real matrix. The split feasibility problem (SFP) is to find a point

$$x \in C \text{ such that } Ax \in Q. \tag{1.1}$$

This problem was first presented and analyzed by Censor and Elfving [1] and appeared in signal processing, image reconstruction [2], and so on. Many well-known iterative algorithms for solving (1.1) were established, see the papers [3–5]. Denoted by P_S, the orthogonal projection operator onto convex set S, that is

$$P_S(x) = \arg\min \|c - x\|, \quad c \in S, \tag{1.2}$$

where $\|\cdot\|$ indicates the 2-norm. The CQ algorithm proposed by Byrne in [6] has the following iterative process:

$$x^{k+1} = P_C\left(x^k + \gamma A^T (P_Q - I) A x^k\right), \quad k \geq 0, \tag{1.3}$$

where $\gamma \in (0, 2/L)$, L denotes the largest eigenvalue of the matrix $A^T A$, and I is the identity operator.

In some cases, it is difficult or even impossible to compute orthogonal projection; to avoid computing projection, Zhao and Yang in [7] proposed the perturbed projections algorithm for the SFP. This development was based on results of Santos and Scheimberg [8] who suggested replacing each nonempty closed convex set of the convex feasibility problem by a convergent sequence of supersets. If such supersets can be constructed with reasonable efforts and projecting onto them is simpler than projecting onto the original convex sets, then a perturbed algorithm is favorable. The concrete iterative process of perturbed CQ algorithm [7] is as follows:

$$x^{k+1} = (1 - \alpha_k) x^k + \alpha_k P_{C_k}\left(x^k + \gamma A^T \left(P_{Q_k} - I\right)\left(A x^k\right)\right), \tag{1.4}$$

where $\alpha_k \in [0, 1]$, $\gamma \in (0, 2/L)$, L, I defined as in algorithm (1.3), and $C_k \overset{M}{\to} C$ and $Q_k \overset{M}{\to} Q$ (see the definitions in the Section 2), while the perturbed projections algorithm sometime converges slowly by reason of using only the current point to get the next iterative point.

Many papers have studied the inertial-type extrapolation recently, see [9–12], which uses the term θ_k and the two previous iterative points x^{k-1}, x^k to get the next iterative point x^{k+1}. As an acceleration process, it can considerably improve the speed of convergence for the following causes: one is that the vector $x^k - x^{k-1}$ acts as an impulsion term, the other is that the parameter θ_k acts as a speed regulator.

To the best of our knowledge, no publications deal with perturbed projection algorithm and inertial process simultaneously. In this paper, we apply the inertial technique to the perturbed projection algorithm to get a perturbed inertial projection algorithm for the split feasibility problem. The results improve and extend the algorithms presented in [6] and in [7] and the related convergence theorem.

The paper is organized as follows. In Section 2, some preliminaries are given. The inertial perturbed algorithm and the corresponding convergence theorem for the split feasibility problem are presented in Section 3.

2. Preliminaries

Throughout the rest of the paper, I denotes the identity operator, $\mathrm{Fix}(T)$ denotes the fixed points of an operator T, that is, $\mathrm{Fix}(T) := \{x \mid x = T(x)\}$.

An operator T is said to be nonexpansive (ne) if

$$\|T(x) - T(y)\| \leq \|x - y\|. \tag{2.1}$$

It is well known that the projection operator is nonexpansive.

Recall the following notions of the convergence and ρ-distance.

Definition 2.1. Let N be an operator on a Hilbert space H; let $N_k, k = 0, 1, 2, \ldots$ be a family of operators on a Hilbert space. $\{N_k\}$ is said to be convergent to N if $\|N_k(x) - N(x)\| \to 0$ as $k \to +\infty$ for all $x \in H$.

Definition 2.2. The ρ-distance ($\rho \geq 0$) for operators N_1 and N_2 on H is given by

$$D_\rho(N_1, N_2) = \sup_{\|x\| \leq \rho} \|N_1(x) - N_2(x)\|. \tag{2.2}$$

Now we introduce the Mosco-convergence for sequences of sets in a reflexive Banach space.

Definition 2.3 (see [13]). Let X be a reflexive Banach space and C and $\{C_k\}_{k \in N}$ (N is a set of natural numbers) a sequence of subsets of X. The sequence $\{C_k\}_{k \in N}$ is Mosco-convergent to C, denoted by $C_k \overset{M}{\to} C$, if

$$\begin{aligned}
&(1)\ \forall x \in C,\ \exists \left\{x^k\right\}_{k \in N} \text{ with } x^k \in C_k (k \in N) \text{ such that } x^k \xrightarrow{X_s} x, \\
&(2)\ \forall \{k_j\}_{j \in N},\ \forall \left\{x^j\right\}_{j \in N},\ x^j \in C_{k_j}(j \in N),\ x^j \xrightarrow{X_w} x \Longrightarrow x \in C,
\end{aligned} \tag{2.3}$$

where X_s and X_w denote the strong and weak topologies, respectively. In particular, if $\{C_k\}$ and C are in R^n, then $C_k \overset{M}{\to} C$ is equivalent to

$$\begin{aligned}
&(1)\ \forall x \in C,\ \exists \left\{x^k\right\}_{k \in N},\ x^k \in C_k (k \in N) \text{ such that } x^k \longrightarrow x, \\
&(2)\ \forall \{k_j\}_{j \in N},\ \forall \left\{x^j\right\}_{j \in N},\ x^j \in C_{k_j}(j \in N),\ x^j \longrightarrow x \Longrightarrow x \in C.
\end{aligned} \tag{2.4}$$

Using the notation NCCS(R^n) for the family of nonempty closed convex subsets of R^n, let C and C_k be sets in NCCS(R^n), for $k = 0, 1, \ldots$. It is easy to verify that if the sequence $\{C_k\}$ converges to C in the Mosco sense, then the operator sequence $\{P_{C_k}\}$ converges to P_C.

Definition 2.4. Let C_1 and C_2 be elements in NCCS(R^n). The ρ-distance is defined by

$$d_\rho(C_1, C_2) = \sup_{\|x\|_2 \leq \rho} \|P_{C_1}(x) - P_{C_2}(x)\|_2. \tag{2.5}$$

Let C and C_k be sets in NCCS(R^n), then $C_k \overset{M}{\to} C$ if and only if $d_\rho(C_k, C) \to 0$ for all $\rho \geq 0$.

The following lemmas will be used in convergence analysis later on.

Lemma 2.5 (see [14]). *Let $\{\delta_k\}$ and $\{\gamma_k\}$ be nonnegative sequences satisfying $\sum_k \delta_k < +\infty$ and $\gamma_{k+1} \leq \gamma_k + \delta_k$, $k = 0, 1, \ldots$. Then, $\{\gamma_k\}$ is a convergent sequence.*

Lemma 2.6 (see [10]). *Let $\phi_k \in [0,\infty)$, $k = 1,2,\dots$, and $\delta_k \in [0,\infty)$, $k = 1,2,\dots$ satisfying*

(1) $\phi_{k+1} - \phi_k \leq \theta_k(\phi_k - \phi_{k-1}) + \delta_k$,

(2) $\sum_k \delta_k < \infty$,

(3) $\theta_k \in [0,\theta]$, $k = 1,2,\dots$, *where* $\theta \in [0,1)$.

Then, $\{\phi_k\}$ *is a converging sequence and* $\sum_k (\phi_{k+1} - \phi_k)_+ < \infty$, *where* $(t)_+ := \max\{t,0\}$ *for any* $t \in R$.

Lemma 2.7 (Opial [15]). *Let H be a Hilbert space, and let $\{x^k\}$ be a sequence in H such that there exists a nonempty set $S \subset H$ satisfying:*

(1) *for every* $x^* \in S$, $\lim_k \|x^k - x^*\|$ *exists,*

(2) *any weak cluster point of* $\{x^k\}$ *belongs to* S.

Then, there exists $\overline{x}^* \in S$ *such that* $\{x^k\}$ *weakly converges to* $\overline{x}^*$.

Lemma 2.8 (see [15]). *Let H be a Hilbert space, $T : H \to H$ a nonexpansive operator, and y a weak cluster point of a sequence $\{x^k\}$, and let $\|T(x^k) - x^k\| \to 0$. Then $y \in \mathrm{Fix}(T)$.*

3. The Inertial Perturbed Algorithm and the Asymptotic Convergence for the SPF

Let C and C_k be sets in NCCS(R^n), and let Q and Q_k be sets in NCCS(R^m), for $k = 0,1,\dots$, with $C_k \xrightarrow{M} C$ and $Q_k \xrightarrow{M} Q$. Then $d_\rho(C_k, C) \to 0$ and $d_\rho(Q_k, Q) \to 0$ for all $\rho > 0$. We denote

$$\begin{aligned} N &= P_C U, \qquad U = I - \gamma A^T(I - P_Q)A, \\ N_k &= P_{C_k} U_k, \qquad U_k = I - \gamma A^T(I - P_{Q_k})A. \end{aligned} \tag{3.1}$$

From Lemma 3.1 in [5], we know that x^* solves the SFP (1.1) if and only if $x^* \in \mathrm{Fix}(N)$.

It is well known that the operator $A^T(I - P_Q)A$ is λ-Lipschitz continuous with $\lambda = \rho(A^T A)$. The same is true for the operators $A^T(I - P_{Q_k})A$ for $k = 0,1,\dots$; it is easy to obtain the following conclusion.

Lemma 3.1 (see [7]). *Let C and C_k be sets in* NCCS(R^n), *and let Q and Q_k be sets in* NCCS(R^m), *for $k = 0,1,\dots$, with $C_k \xrightarrow{M} C$ and $Q_k \xrightarrow{M} Q$. Then, the operators N and N_k defined in* (3.1) *are nonexpansive operators for $\gamma \in (0, 2/\lambda)$. Moreover, the operator sequence $\{N_k\}$ converges to N.*

Now we give the perturbed inertial KM-type algorithm for SFP.

Algorithm 3.2. Given arbitrary elements in R^n for $k = 0,1,\dots$, let

$$\begin{aligned} x^{k+1} &= [(1 - \alpha_k)I + \alpha_k N_k]\left(y^k\right), \\ y^k &= x^k + \theta_k\left(x^k - x^{k-1}\right), \end{aligned} \tag{3.2}$$

where $N_k = P_{C_k}[(I - \gamma A^T(I - P_{Q_k})A)$, $\alpha_k \in (0,1)$, for any k and $\gamma \in (0, 2/\lambda)$ with $\lambda = \rho(A^T A)$, $\{\theta_k\} \subset [0, \theta]$, $\theta \in [0.1)$.

The following theorem is necessary for the convergence analysis of Algorithm 3.2.

Theorem 3.3. *Let N and N_k for $k = 0, 1, 2, \ldots$ be nonexpansive (ne) operators in finite-dimensional Hilbert space, with $N_k \to N$, and let $\{\alpha_k\}$ be a sequence in $(0,1)$ satisfying*

$$\sum_{k=0}^{+\infty} \alpha_k (1 - \alpha_k) = +\infty, \tag{3.3}$$

$$\sum_{k=0}^{+\infty} \alpha_k D_\rho(N_k, N) < +\infty \tag{3.4}$$

for all $\rho > 0$. Then, the sequence $\{x^k\}$ defined by the iterative step

$$x^{k+1} = (1 - \alpha_k) y^k + \alpha_k N_k\left(y^k\right), \tag{3.5}$$

$$y^k = x^k + \theta_k\left(x^k - x^{k-1}\right) \tag{3.6}$$

converges to a fixed point of N provided that we choose parameter θ_k satisfying

$$\{\theta_k\} \subset \left[0, \overline{\theta}_k\right], \quad \text{with } \overline{\theta}_k := \min\left\{\theta, \frac{1}{\max\left\{k^2\|x^k - x^{k-1}\|^2, k^2\|x^k - x^{k-1}\|\right\}}\right\}, \quad \theta \in [0,1), \tag{3.7}$$

whenever such fixed points exist.

Proof. We first prove that the sequence $\{x^k\}$ is bounded and $\{\|x^k - z\|\}$ is convergent for all $z \in \mathrm{Fix}(N)$, where $\mathrm{Fix}(N)$ denotes the set of the fixed points of the operator N, that is, $N(z) = z$. Since N and N_k are ne operators, we have

$$\begin{aligned}
\left\|x^{k+1} - z\right\| &= \left\|(1 - \alpha_k) y^k + \alpha_k N_k\left(y^k\right) - (1 - \alpha_k) z - \alpha_k N(z)\right\| \\
&\leq (1 - \alpha_k)\left\|y^k - z\right\| + \alpha_k\left\|N_k\left(y^k\right) - N(z)\right\| \\
&\leq (1 - \alpha_k)\left\|y^k - z\right\| + \alpha_k\left\|N_k\left(y^k\right) - N_k(z)\right\| + \alpha_k\|N_k(z) - N(z)\| \\
&\leq \left\|y^k - z\right\| + \alpha_k D_{\overline{\rho}}(N_k, N) \\
&= \left\|x^k + \theta_k\left(x^k - x^{k-1}\right) - z\right\| + \alpha_k D_{\overline{\rho}}(N_k, N) \\
&\leq \left\|x^k - z\right\| + \theta_k\left\|x^k - x^{k-1}\right\| + \alpha_k D_{\overline{\rho}}(N_k, N),
\end{aligned} \tag{3.8}$$

where $\overline{\rho} \geq \|z\|$. From the selection of parameter θ_k, we have

$$\theta_k \left\| x^k - x^{k-1} \right\| \leq \frac{\|x^k - x^{k-1}\|}{\max\left\{ k^2 \|x^k - x^{k-1}\|^2, k^2 \|x^k - x^{k-1}\| \right\}} \leq \frac{1}{k^2}. \tag{3.9}$$

It is easy to get

$$\sum_{k \geq 0} \left\| x^k - x^{k-1} \right\| < \infty. \tag{3.10}$$

By (3.4), (3.8)-(3.10), we obtain from Lemma 2.5 that the sequence $\{\|x^k - z\|\}$ is convergent and hence the sequence $\{x^k\}$ is bounded.

We next prove that $\liminf_{k \to \infty} \|y^k - N(y^k)\| = 0$. Let $e^k = N_k(y^k) - N(y^k)$ and notice the fact that

$$\left\| \lambda x + (1-\lambda) y \right\|^2 = \lambda \|x\|^2 + (1-\lambda) \left\| y \right\|^2 - \lambda(1-\lambda) \left\| x - y \right\|^2. \tag{3.11}$$

Then, we have

$$\begin{aligned}
\left\| x^{k+1} - z \right\|^2 &= \left\| (1-\alpha_k)\left(y^k - z\right) + \alpha_k\left(N_k\left(y^k\right) - N\left(y^k\right)\right) + \alpha_k\left(N\left(y^k\right) - N(z)\right) \right\|^2 \\
&= \left\| (1-\alpha_k)\left(y^k - z + \alpha_k e^k\right) + \alpha_k\left(N\left(y^k\right) - N(z) + \alpha_k e^k\right) \right\|^2 \\
&= (1-\alpha_k)\left\| y^k - z + \alpha_k e^k \right\|^2 + \alpha_k \left\| N\left(y^k\right) - N(z) + \alpha_k e^k \right\|^2 \\
&\quad - \alpha_k(1-\alpha_k)\left\| y^k - N\left(y^k\right) \right\|^2 \\
&\leq (1-\alpha_k)\left[\left\| y^k - z \right\|^2 + 2\alpha_k \left\| y^k - z \right\| \left\| e^k \right\| + {\alpha_k}^2 \left\| e^k \right\|^2 \right] \\
&\quad + \alpha_k \left[\left\| N\left(y^k\right) - N(z) \right\|^2 + 2\alpha_k \left\| N\left(y^k\right) - N(z) \right\| \left\| e^k \right\| + {\alpha_k}^2 \left\| e^k \right\|^2 \right] \\
&\quad - \alpha_k(1-\alpha_k)\left\| y^k - N\left(y^k\right) \right\|^2 \\
&\leq \left\| y^k - z \right\|^2 + 2\alpha_k \left\| y^k - z \right\| \left\| e^k \right\| + {\alpha_k}^2 \left\| e^k \right\|^2 - \alpha_k(1-\alpha_k)\left\| y^k - N\left(y^k\right) \right\|^2.
\end{aligned} \tag{3.12}$$

From (3.6), we obtain

$$\begin{aligned}
\left\| y^k - z \right\|^2 &= \left\| \left(x^k - z\right) - \theta_k\left(x^{k-1} - x^k\right) \right\|^2 \\
&= \left\| x^k - z \right\|^2 - 2\theta_k \left\langle x^k - z, x^{k-1} - x^k \right\rangle + \theta_k^2 \left\| x^{k-1} - x^k \right\|^2.
\end{aligned} \tag{3.13}$$

By (3.12) and observing that $\theta_k^2 \leq \theta_k$ (since $\theta_k \in [0,1]$), we have

$$\begin{aligned}\left\|x^{k+1}-z\right\|^2 \leq \left\|x^k-z\right\|^2 - 2\theta_k\left\langle x^k-z, x^{k-1}-x^k\right\rangle + \theta_k\left\|x^{k-1}-x^k\right\|^2 \\ + 2\alpha_k\left\|y^k-z\right\|\left\|e^k\right\| + \alpha_k^2\left\|e^k\right\|^2 - \alpha_k(1-\alpha_k)\left\|y^k-N\left(y^k\right)\right\|^2.\end{aligned} \tag{3.14}$$

Combining $\langle a,b\rangle = -(1/2)\|a-b\|^2 + (1/2)\|a\|^2 + (1/2)\|b\|^2$ with (3.14), we get

$$\begin{aligned}\left\|x^{k+1}-z\right\|^2 - \left\|x^k-z\right\|^2 &\leq \theta_k\left(\left\|x^k-z\right\|^2 - \left\|x^{k-1}-z\right\|^2\right) + 2\theta_k\left\|x^{k-1}-x^k\right\|^2 \\ &\quad + 2\alpha_k\left\|y^k-z\right\|\left\|e^k\right\| + \alpha_k^2\left\|e^k\right\|^2 - \alpha_k(1-\alpha_k)\left\|y^k-N\left(y^k\right)\right\|^2 \\ &\leq \theta_k\left(\left\|x^k-z\right\|^2 - \left\|x^{k-1}-z\right\|^2\right) + 2\theta_k\left\|x^{k-1}-x^k\right\|^2 \\ &\quad + 2\alpha_k\left(\left\|x^k-z\right\| + \theta_k\left\|x^{k-1}-x^k\right\|\right)\left\|e^k\right\| + \alpha_k^2\left\|e^k\right\|^2 \\ &\quad - \alpha_k(1-\alpha_k)\left\|y^k-N\left(y^k\right)\right\|^2.\end{aligned} \tag{3.15}$$

We have known that the sequence $\{x^k\}$ is bounded and $\{\|x^k-z\|\}$ is convergent; hence, there exist $\rho \geq \overline{\rho} > 0$ and $G > 0$ such that $\|x^k\| \leq \rho$ and $\|x^k-z\| \leq G$ for all k.

Thus

$$\left\|e^k\right\| = \left\|N_k\left(y^k\right) - N\left(y^k\right)\right\| \leq D_\rho(N_k, N). \tag{3.16}$$

Moreover, one has that

$$\begin{aligned}\left\|x^{k+1}-z\right\|^2 - \left\|x^k-z\right\|^2 \leq \theta_k\left(\left\|x^k-z\right\|^2 - \left\|x^{k-1}-z\right\|^2\right) + 2\theta_k\left\|x^{k-1}-x^k\right\|^2 \\ + 2\alpha_k G D_\rho(N_k,N) + 2\alpha_k\theta_k\left\|x^{k-1}-x^k\right\|D_\rho(N_k,N) \\ + \alpha_k^2 D_\rho(N_k,N)^2 - \alpha_k(1-\alpha_k)\left\|y^k-N\left(y^k\right)\right\|^2.\end{aligned} \tag{3.17}$$

Denoting

$$\sigma_k = 2\theta_k\left\|x^{k-1}-x^k\right\|^2 + 2\alpha_k G D_\rho(N_k,N) + 2\alpha_k\theta_k\left\|x^{k-1}-x^k\right\|D_\rho(N_k,N) + \alpha_k^2 D_\rho(N_k,N)^2, \tag{3.18}$$

we get

$$\left\|x^{k+1}-z\right\|^2 - \left\|x^k-z\right\|^2 \leq \theta_k\left(\left\|x^k-z\right\|^2 - \left\|x^{k-1}-z\right\|^2\right) + \sigma_k. \tag{3.19}$$

Similarly, from the selection of parameter θ_k, we have

$$\theta_k \left\| x^k - x^{k-1} \right\|^2 \leq \frac{\| x^k - x^{k-1} \|^2}{\max\left\{ k^2 \| x^k - x^{k-1} \|^2, k^2 \| x^k - x^{k-1} \| \right\}} \leq \frac{1}{k^2}. \tag{3.20}$$

It is easy to get

$$\sum_{k \geq 0} \left\| x^k - x^{k-1} \right\|^2 < \infty. \tag{3.21}$$

Both (3.10) and (3.21) manifest

$$\sum_{k \geq 0} \alpha_k \theta_k \left\| x^{k-1} - x^k \right\| D_\rho(N_k, N) < \infty. \tag{3.22}$$

Then, from (3.4), (3.21), and (3.22), we get

$$\sum_{k=0}^{+\infty} \sigma_k < +\infty. \tag{3.23}$$

According to Lemma 2.6, we obtain $\sum_{k \geq 0} [\| x^k - z \|^2 - \| x^{k-1} - z \|^2]_+ < \infty$.

From (3.17), it follows that

$$\sum_{k=0}^{+\infty} \alpha_k (1 - \alpha_k) \left\| y^k - N\left(y^k \right) \right\|^2 < +\infty. \tag{3.24}$$

Because of (3.3) $\sum_{k=0}^{+\infty} \alpha_k (1 - \alpha_k) = +\infty$, we conclude that $\liminf_{k \to \infty} \| y^k - N(y^k) \| = 0$.

Finally, we prove that $\{x^k\}$ converges to a fixed point of N. From the above computation, we know that the sequence $\{y^k\}$ is also bounded; hence there exist x^* and a subsequence of $\{y^k\}$ (denoted $\{y^{k_l}\}$) such that

$$\begin{gathered} \lim_{l \to \infty} \left\| y^{k_l} - x^* \right\| = 0, \\ \lim_{l \to \infty} \left\| y^{k_l} - N\left(y^{k_l} \right) \right\| = \liminf_{k \to \infty} \left\| y^k - N\left(y^k \right) \right\| = 0. \end{gathered} \tag{3.25}$$

From Lemmas 2.7 and 2.8, we have $x^* \in \mathrm{Fix}(N)$. It is easy to obtain that $\lim_{l \to \infty} \| x^{k_l} - x^* \| = 0$, because $\| y^k - x^k \| = \theta_k \| x^k - x^{k-1} \| \to 0$ by (3.10). Since $\{\| x^k - x^* \|\}$ is convergent, it follows that $\lim_{k \to \infty} \| x^k - x^* \| = \lim_{l \to \infty} \| x^{k_l} - x^* \| = 0$. The proof is completed. □

Remark 1. Since the current value of $\| x^k - x^{k-1} \|$ is known when choosing the parameter θ_k, then θ_k is well defined in Theorem 3.3. In fact, from the process of proof for the Theorem 3.3, we can get the following assert: the convergence result of Theorem 3.3 always holds provided that we select $\theta_k \in [0, \theta]$, $\theta \in [0, 1)$, for all $k \geq 0$ with

$$\sum_{k\geq 0} \left\| x^k - x^{k-1} \right\| < \infty,$$
$$\sum_{k\geq 0} \left\| x^k - x^{k-1} \right\|^2 < \infty. \tag{3.26}$$

Now let us return to the convergence analysis of Algorithm 3.2.

Theorem 3.4. *Let the hypotheses in Lemma 3.1 be satisfied. Then the sequence* $\{x^k\}$ *generated by* (3.2) *converges to a fixed point of* N *if*

$$\sum_{k=0}^{+\infty} \alpha_k (1 - \alpha_k) = +\infty, \qquad \sum_{k=0}^{+\infty} \alpha_k \left(d_{\overline{\rho}}(C_k, C) + 2\lambda^{-1/2} d_{\overline{\rho}}(Q_k, Q) \right) < +\infty, \quad \overline{\rho} > 0, \tag{3.27}$$

where parameter θ_k *is satisfying* (3.10) *and* (3.21).

Proof. For any $y \in R^n$, $\|y\| \leq \rho$, by reason of the nonexpansive properties of projection, we have

$$\begin{aligned}
\|N_k(y) - N(y)\| &= \left\| P_{C_k}\left(y - \gamma A^T (I - P_{Q_k}) A y \right) - P_C\left(y - \gamma A^T (I - P_Q) A y \right) \right\| \\
&\leq \left\| P_{C_k}\left(y - \gamma A^T (I - P_{Q_k}) A y \right) - P_{C_k}\left(y - \gamma A^T (I - P_Q) A y \right) \right\| \\
&\quad + \left\| P_{C_k}\left(y - \gamma A^T (I - P_Q) A y \right) - P_C\left(y - \gamma A^T (I - P_Q) A y \right) \right\| \\
&\leq \gamma \left\| A^T (I - P_{Q_k}) A y - A^T (I - P_Q) A y \right\| \\
&\quad + \left\| P_{C_k}\left(y - \gamma A^T (I - P_Q) A y \right) - P_C\left(y - \gamma A^T (I - P_Q) A y \right) \right\| \\
&\leq \gamma \lambda^{1/2} \| P_{Q_k}(Ay) - P_Q(Ay) \| \\
&\quad + \left\| P_{C_k}\left(y - \gamma A^T (I - P_Q) A y \right) - P_C\left(y - \gamma A^T (I - P_Q) A y \right) \right\| \\
&\leq 2\lambda^{-1/2} \| P_{Q_k}(Ay) - P_Q(Ay) \| \\
&\quad + \left\| P_{C_k}\left(y - \gamma A^T (I - P_Q) A y \right) - P_C\left(y - \gamma A^T (I - P_Q) A y \right) \right\|.
\end{aligned} \tag{3.28}$$

Obviously,

$$D_\rho(N_k, N) \leq d_{\overline{\rho}}(C_k, C) + 2\lambda^{-1/2} d_{\overline{\rho}}(Q_k, Q), \tag{3.29}$$

where $\overline{\rho} \geq \max\{\|Ay\|, \|y - \gamma A^T (I - P_Q) A y\|\}$.

Since $\sum_{k=0}^{+\infty} \alpha_k (d_{\overline{\rho}}(C_k, C) + 2\lambda^{-1/2} d_{\overline{\rho}}(Q_k, Q)) < +\infty$ for any given $\overline{\rho} > 0$, the result of this theorem can be obtained using Theorem 3.3. □

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