Research Article

The Dirichlet Problem on the Upper Half-Space

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Received 18 April 2012; Accepted 19 September 2012

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A solution of the Dirichlet problem on the upper half-space is constructed by the generalized Dirichlet integral with a fast-growing continuous boundary function.

1. Introduction and Results

Let \( \mathbb{R}^n \) (\( n \geq 3 \)) denote the \( n \)-dimensional Euclidean space with points \( x = (x', x_n) \), where \( x' = (x_1, x_2, \ldots, x_{n-1}) \in \mathbb{R}^{n-1} \) and \( x_n \in \mathbb{R} \). The boundary and closure of an open set \( D \) of \( \mathbb{R}^n \) are denoted by \( \partial D \) and \( \overline{D} \), respectively. The upper half space is the set \( H = \{(x', x_n) \in \mathbb{R}^n : x_n > 0 \} \), whose boundary is \( \partial H \). We identify \( \mathbb{R}^n \) with \( \mathbb{R}^{n-1} \times \mathbb{R} \), and \( \mathbb{R}^{n-1} \) with \( \mathbb{R}^{n-1} \times \{0\} \), writing typical points \( x, y \in \mathbb{R}^n \) as \( x = (x', x_n) \), \( y = (y', y_n) \), where \( y' = (y_1, y_2, \ldots, y_{n-1}) \in \mathbb{R}^{n-1} \) and putting

\[
x \cdot y = \sum_{j=1}^{n} x_j y_j = x' \cdot y' + x_n y_n, \quad |x| = \sqrt{x \cdot x}, \quad |x'| = \sqrt{x' \cdot x'}.
\]  

(1.1)

Let \( B(r) \) denote the open ball with center at the origin and radius \( r \), and let \( \sigma \) denote \((n-1)\)-dimensional surface area measure. Let \([d] \) denote the integer part of the positive real number \( d \). In the sense of Lebesgue measure, \( dy' = dy_1 \cdots dy_{n-1} \) and \( dy = dy' dy_n \).

Given a continuous function \( f \) on \( \partial H \), we say that \( h \) is a solution of the (classical) Dirichlet problem on \( H \) with \( f \) if \( \Delta h = 0 \) in \( H \) and \( \lim_{x \in H, x \to z} h(x) = f(z') \) for every \( z' \in \partial H \).

The classical Poisson kernel for \( H \) is defined by \( P(x, x') = 2x_n \omega_n^{-1} |x - y'|^{-n} \), where \( \omega_n = 2\pi^{n/2}/\Gamma(n/2) \) is the area of the unit sphere in \( \mathbb{R}^n \).
To solve the Dirichlet problem on $H$, as in [1-6], we use the following modified Poisson kernel of order $m$ defined by

$$P_m(x, y') = \begin{cases} P(x, y') & \text{when } |y'| \leq 1, \\ P(x, y') - \frac{m-1}{\omega_n} \sum_{k=0}^{m-1} 2x_n|x|^{k} C_n^{(k/2)} \left( \frac{x \cdot y'}{|x||y'|} \right) & \text{when } |y'| > 1, \end{cases}$$

(1.2)

where $m$ is a nonnegative integer, and $C_k^{n/2}(t)$ is the ultraspherical (Gegenbauer) polynomials [7]. The expression arises from the generating function for Gegenbauer polynomials

$$(1 - 2tr + r^2)^{-n/2} = \sum_{k=0}^{\infty} C_k^{n/2}(t)r^k,$$  

(1.3)

where $|r| < 1$ and $|t| \leq 1$. The coefficient $C_k^{n/2}(t)$ is called the ultraspherical (Gegenbauer) polynomial of degree $k$ associated with $n/2$, and the function $C_k^{n/2}(t)$ is a polynomial of degree $k$ in $t$.

Put

$$U_m(f)(x) = \int_{\partial H} P_m(x, y') f(y') dy',$$  

(1.4)

where $f(y')$ is a continuous function on $\partial H$.

Using the modified Poisson kernel $P_m(x, y')$, Yoshida (cf. [6, Theorem 1]) and Siegel and Talvila (cf. [5, Corollary 2.1]) gave classical solutions of the Dirichlet problem on $H$, respectively. Motivated by their results, we consider the Dirichlet problem for harmonic functions of infinite order (e.g., see [8, Definition 4.1, page 2, Line 12] for the definition of harmonic functions).

To do this, we define a nondecreasing and continuously differentiable function $\rho(r) \geq 1$ on the interval $[0, +\infty)$. We assume further that

$$\varepsilon_0 = \limsup_{r \to \infty} \frac{\rho'(r)r \log r}{\rho(r)} < 1.$$  

(1.5)

Let $F(\rho, \beta)$ be the set of continuous functions $f$ on $\partial H$ such that

$$\int_{\partial H} \frac{|f(y')|dy'}{1 + |y'|^{\rho(|y'|) + \beta - 1}} < \infty,$$  

(1.6)

where $\beta$ is a positive real number.

Now, we have the following.

**Theorem 1.1.** If $f \in F(\rho, \beta)$, then the integral $U_{[\rho(|y'|) + \beta]}(f)(x)$ is a solution of the Dirichlet problem on $H$ with $f$.

If one puts $[\rho(|y'|) + \beta] = m$ in Theorem 1.1, one immediately obtains the following (cf. [6, Theorem 1] and [5, Corollary 2.1]).
Corollary 1.2. If $f$ is a continuous function on $\partial H$ satisfying $\int_{\partial H} |f(y')(1+|y'|)^{-n-m} dy' < \infty$, then $U_m(f)(x)$ is a solution of the Dirichlet problem on $H$ with $f$.

Theorem 1.3. Let $u$ be harmonic in $H$ and continuous on $\overline{H}$. If $u \in F(\rho, \beta)$, then one has

$$u(x) = U_{[\rho(|y'|)+\beta]}(u)(x) + h(x), \quad (1.7)$$

for all $x \in \overline{H}$, where $h(x)$ is harmonic in $H$ and vanishes continuously on $\partial H$.

2. Proof of Theorem 1.1

We need to use the following inequality (see [5, page 3]):

$$|P_m(x, y')| \leq M|x|^m |y'|^{-n-m}, \quad (2.1)$$

for any $x \in H$ and $y' \in \partial H$ satisfying $|y'| \geq \max\{1, 2|x|\}$, where $M$ is a positive constant.

For any $\epsilon (0 < \epsilon < 1 - \epsilon_0)$, there exists a sufficiently large positive number $R$ such that $r > R$, and by (1.5), we have

$$\rho(r) < \rho(e)(\ln r)^{(\epsilon_0+\epsilon)}, \quad (2.2)$$

which yields that there exists a positive constant $M(r)$ dependent only on $r$ such that

$$k^{-\beta/2}(2r)^{\rho(k+1)+\beta+1} \leq M(r), \quad (2.3)$$

for any $k > k_r = [2r] + 1$.

For any $x \in H$ and $|x| \leq r$, we have by (1.6), (2.1), (2.3), $1/p + 1/q = 1$, and Hölder’s inequality

$$M \sum_{k=k_r}^{\infty} \int_{|y'\in\partial H; k\leq|y'|<k+1|} \frac{(2|x|)^{\rho(|y'|)+\beta+1}}{|y'|^{\rho(|y'|)+\beta+1}} |f(y')| dy'$$

$$\leq M \sum_{k=k_r}^{\infty} (2r)^{\rho(k+1)+\beta+1} \left( \int_{|y'\in\partial H; k\leq|y'|<k+1|} \frac{|f(y')|^{p}}{|y'|^{\rho(|y'|)+\beta+1}} dy' \right)^{1/p}$$

$$\times \left( \int_{|y'\in\partial H; k\leq|y'|<k+1|} |y'|^{-q(\rho(|y'|)+\beta+1-n(\rho(|y'|)+n-1)/p-\beta/2)} dy' \right)^{1/q} \quad (2.4)$$

$$\leq M \sum_{k=k_r}^{\infty} (2r)^{\rho(k+1)+\beta+1} \frac{k^{\beta/2}}{k} \int_{|y'\in\partial H; k\leq|y'|<k+1|} \frac{|f(y')|}{|y'|^{\rho(|y'|)+\beta+1}} dy'$$

$$\leq 2MM(r) \int_{|y'\in\partial H; |y'| \geq k_r|} \frac{|f(y')|}{1 + |y'|^{\rho(|y'|)+\beta/2}} dy' < \infty.$$
Thus, $U_{[p(|y'|)+\rho]}(f)(x)$ is finite for any $x \in H$. Since $P_{[p(|y'|)+\rho]}(x, y')$ is a harmonic function of $x \in H$ for any fixed $y' \in \partial H$, $U_{[p(|y'|)+\rho]}(f)(x)$ is also a harmonic function of $x \in H$.

To verify the boundary behavior of $U_{[p(|y'|)+\rho]}(f)(x)$, we fix a boundary point $z' \in \partial H$, choose a large $t > |z'| + 1$, and write

$$U_{[p(|y'|)+\rho]}(f)(x) = X(x) - Y(x) + Z(x),$$  \hspace{1cm} (2.5)

where

$$X(x) = \int_{\{y' \in \partial H : |y'| \leq |x|\}} P(x, y') f(y') dy',$$

$$Y(x) = \sum_{k=0}^{[p(|y'|)+\rho]-1} \frac{2x_n|x|^k}{\omega_n} \int_{\{y' \in \partial H : 1 < |y'| \leq |x|\}} \frac{1}{|y'|^{n+k}} C_k^{n/2} \left( \frac{x' \cdot y'}{|x||y'|} \right) f(y') dy',$$

$$Z(x) = \int_{\{y' \in \partial H : |y'| > t\}} P_{[p(|y'|)+\rho]}(x, y') f(y') dy'.$$

Notice that $X(x)$ is the Poisson integral of $f(y') \chi_{B(t)}(y')$, where $\chi_{B(t)}$ is the characteristic function of the ball $B(t)$. So it tends to $f(z')$ as $x \to z'$. Since $Y(x)$ are polynomial times $x_n$ and $Z(x) = O(x_n)$, both of them tend to zero as $x \to z'$. Thus, the function $U_{[p(|y'|)+\rho]}(f)(x)$ can be continuously extended to $\overline{H}$ such that $U_{[p(|y'|)+\rho]}(f)(z') = f(z')$, for any $z' \in \partial H$. Theorem 1.1 is proved.

3. Proof of Theorem 1.3

Consider that the function $u(x) - U_{[p(|y'|)+\rho]}(u)(x)$, which is harmonic in $H$, can be continuously extended to $\overline{H}$ and vanishes on $\partial H$.

The Schwarz reflection principle [9, page 68] applied to $u(x) - U_{[p(|y'|)+\rho]}(u)(x)$ shows that there exists a harmonic function $h(x)$ in $H$ such that $h(x^*) = -h(x) = -(u(x) - U_{[p(|y'|)+\rho]}(u)(x))$ for $x \in \overline{H}$, where $\ast$ denotes reflection in $\partial H$ just as $x^* = (x^*, -x_n)$.

Thus, $u(x) = h(x) + U_{[p(|y'|)+\rho]}(u)(x)$ for all $x \in \overline{H}$, where $h(x)$ is a harmonic function on $H$ vanishing continuously on $\partial H$. We complete the proof of Theorem 1.3.

Acknowledgments

The authors wish to express their appreciation to Professor Guantie Deng for some very useful conversations related to this problem. They are grateful to the referee for her or his careful reading and helpful suggestions which led to an improvement of their original manuscript. This work is supported by The National Natural Science Foundation of China under Grant 11271045 and Specialized Research Fund for the Doctoral Program of Higher Education under Grant 20100003110004.
Abstract and Applied Analysis

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