Research Article

Fekete-Szegö Problems for Quasi-Subordination Classes

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An analytic function \( f \) is quasi-subordinate to an analytic function \( g \), in the open unit disk if there exist analytic functions \( \varphi \) and \( w \), with \( |\varphi/z| \leq 1 \), \( w(0) = 0 \) and \( |w(z)| < 1 \) such that

\[
f(z) = \varphi(z) g(w(z)).
\]

In particular, if the function \( g \) is univalent in \( D \), then \( f(z) < g(z) \) is equivalent to \( f(0) = g(0) \) and \( f(D) \subset g(D) \). For brief survey on the concept of subordination, see [1].

Ma and Minda [2] introduced the following class

\[
S^\ast(\phi) = \left\{ f \in A : \frac{zf'(z)}{f(z)} < \phi(z) \right\},
\]

1. Introduction and Motivation

Let \( A \) be the class of analytic function \( f \) in the open unit disk \( D = \{ z : |z| < 1 \} \) normalized by \( f(0) = 0 \) and \( f'(0) = 1 \) of the form

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n.
\]

For two analytic functions \( f \) and \( g \), the function \( f \) is subordinate to \( g \), written as follows:

\[
f(z) \prec g(z),
\]

if there exists an analytic function \( \varphi \), with \( \varphi(0) = 0 \) and \( |\varphi(z)| \leq 1 \) such that \( f(z) = \varphi(z) g(\varphi(z)) \).

In particular, if the function \( g \) is univalent in \( D \), then \( f(z) < g(z) \) is equivalent to \( f(0) = g(0) \) and \( f(D) \subset g(D) \). For brief survey on the concept of subordination, see [1].
where $\phi$ is an analytic function with positive real part in $\mathbb{D}$, $\phi(\mathbb{D})$ is symmetric with respect to the real axis and starlike with respect to $\phi(0) = 1$ and $\phi'(0) > 0$. A function $f \in S^*(\phi)$ is called Ma-Minda starlike (with respect to $\phi$). The class $C(\phi)$ is the class of functions $f \in A$ for which $1 + zf''(z)/f'(z) < \phi(z)$. The class $S^*(\phi)$ and $C(\phi)$ include several well-known subclasses of starlike and convex functions as special case.

In the year 1970, Robertson [3] introduced the concept of quasi-subordination. For two analytic functions $f$ and $g$, the function $f$ is quasi-subordinate to $g$, written as follows:

$$f(z) \ll_q g(z), \quad (1.3)$$

if there exist analytic functions $\varphi$ and $w$, with $|\varphi(z)| \leq 1$, $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = \varphi(z)g(w(z))$. Observe that when $\varphi(z) = 1$, then $f(z) = g(w(z))$, so that $f(z) \ll g(z)$ in $\mathbb{D}$. Also notice that if $w(z) = z$, then $f(z) = \varphi(z)g(z)$ and it is said that $f$ is majorized by $g$ and written $f(z) \ll g(z)$ in $\mathbb{D}$. Hence it is obvious that quasi-subordination is a generalization of subordination as well as majorization. See [4–6] for works related to quasi-subordination.

Throughout this paper it is assumed that $\phi$ is analytic in $\mathbb{D}$ with $\phi(0) = 1$. Motivated by [2, 3], we define the following classes.

**Definition 1.1.** Let the class $S^*_q(\phi)$ consists of functions $f \in A$ satisfying the quasi-subordination

$$zf''(z) f(z) - 1 \ll_q \phi(z) - 1. \quad (1.4)$$

**Example 1.2.** Since

$$zf''(z) f(z) - 1 = z(\phi(z) - 1) \ll_q \phi(z) - 1, \quad (1.5)$$

the function $f : \mathbb{D} \to \mathbb{C}$ defined by the following:

$$f(z) = z \exp \left( -z + \int_0^z \phi(\xi) d\xi \right) \quad (1.6)$$

belongs to the class $S^*_q(\phi)$.

**Definition 1.3.** Let the class $C_q(\phi)$ consists of functions $f \in A$ satisfying the quasi-subordination

$$zf''(z) f(z) - 1 \ll_q \phi(z) - 1. \quad (1.7)$$
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Example 1.4. The function \( f : \mathbb{D} \to \mathbb{C} \) defined by the following:

\[
f(z) = \int_0^z \exp \left( -\zeta + \int_0^\zeta \phi(\xi) d\xi \right) d\zeta
\]

belongs to the class \( C_q(\phi) \).

The classes \( S_q^*(\phi) \) and \( C_q(\phi) \) are analogous to the Ma-Minda starlike and convex classes defined in the form of quasi-subordination.

Definition 1.5. Let the class \( R_q(\phi) \) consist of functions \( f \in \mathcal{A} \) satisfying the quasi-subordination

\[
f'(z) - 1 < q \phi(z) - 1.
\]

Example 1.6. The function \( f : \mathbb{D} \to \mathbb{C} \) defined by the following:

\[
f(z) = z - \frac{z^2}{2} + \exp \left( \int_0^z \phi(\xi) d\xi \right)
\]

belongs to the class \( R_q(\phi) \).

It is known that a function \( f \in \mathcal{A} \) with \( \text{Re} \ f'(z) > 0 \) in \( D \) is univalent. The above class of functions defined in terms of the quasi-subordination is associated with the class of functions with positive real part.

Functions in the following classes, \( M_q(\alpha, \phi) \) and \( L_q(\alpha, \phi) \) are analogous to the \( \alpha \)-convex functions of Miller et al. [7] and \( \alpha \)-logarithmically convex functions introduced by Lewandowski et al. [8] (see also [9]), respectively.

Definition 1.7. Let the class \( M_q(\alpha, \phi) \), \( \alpha \geq 0 \) consist of functions \( f \in \mathcal{A} \) satisfying the quasi-subordination

\[
(1 - \alpha) \frac{zf'(z)}{f(z)} + \alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right) - 1 < q \phi(z) - 1.
\]

Example 1.8. The function \( f : \mathbb{D} \to \mathbb{C} \) defined by the following:

\[
(1 - \alpha) \frac{zf'(z)}{f(z)} + \alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right) - 1 = z(\phi(z) - 1)
\]

belongs to the class \( M_q(\phi) \).
**Definition 1.9.** Let the class $\mathcal{L}_q(\alpha, \phi)$, $(\alpha \geq 0)$ consist of functions $f \in \mathcal{A}$ satisfying the quasi-subordination

$$
\left( \frac{zf'(z)}{f(z)} \right)^\alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right)^{1-\alpha} - 1 \leq \phi(z) - 1.
$$

**Example 1.10.** The function $f : \mathbb{D} \to \mathbb{C}$ defined by the following:

$$
\left( \frac{zf'(z)}{f(z)} \right)^\alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right)^{1-\alpha} - 1 = \phi(z) - 1
$$

belongs to the class $\mathcal{L}_q(\phi)$.

It is well known (see [10]) that the $n$-th coefficient of a univalent function $f \in \mathcal{A}$ is bounded by $n$. The bounds for coefficient give information about various geometric properties of the function. Many authors have also investigated the bounds for the Fekete-Szegő coefficient for various classes [11–25]. In this paper, we obtain coefficient estimates for the functions in the above defined classes.

Let $\Omega$ be the class of analytic functions $w$, normalized by $w(0) = 0$, and satisfying the condition $|w(z)| < 1$. We need the following lemma to prove our results.

**Lemma 1.11 (see [26]).** If $w \in \Omega$, then for any complex number $t$

$$
|w_2 - tw_1^2| \leq \max\{1; |t|\}.
$$

The result is sharp for the functions $w(z) = z^2$ or $w(z) = z$.

## 2. Main Results

Although Theorems 2.1 and 2.4 are contained in the corresponding results for the classes $\mathcal{M}_q(\alpha, \phi)$ and $\mathcal{L}_q(\alpha, \phi)$, they are stated and proved separately here because of the importance of the classes.

Throughout, let $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$, $\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots$, $\varphi(z) = c_0 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots$, $B_1 \in \mathbb{R}$ and $B_1 > 0$.

**Theorem 2.1.** If $f \in \mathcal{A}$ belongs to $\mathcal{S}_q(\phi)$, then

$$
|a_2| \leq B_1,
$$

$$
|a_3| \leq \frac{1}{2} \left( B_1 + \max \left\{ B_1, B_1^2 + |B_2| \right\} \right),
$$

and, for any complex number $\mu$,

$$
|a_3 - \mu a_2^2| \leq \frac{1}{2} \left( B_1 + \max \left\{ B_1, |1 - 2\mu| B_1^2 + |B_2| \right\} \right),
$$

(2.2)
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Proof. If \( f \in S^*_1(\phi) \), then there exist analytic functions \( \varphi \) and \( w \), with \( |\varphi(z)| \leq 1 \), \( w(0) = 0 \) and \( |w(z)| < 1 \) such that

\[
\frac{zf'(z)}{f(z)} - 1 = \varphi(z)(\phi(w(z)) - 1). \tag{2.3}
\]

Since

\[
\frac{zf'(z)}{f(z)} - 1 = a_2 z + \left( -a_2^2 + 2a_3 \right) z^2 + \cdots, \tag{2.4}
\]

\[
\phi(w(z)) - 1 = B_1 w_1 z + \left( B_1 w_2 + B_2 w_1^2 \right) z^2 + \cdots,
\]

\[
\varphi(z)(\phi(w(z)) - 1) = B_1 c_0 w_1 z + \left( B_1 c_1 w_1 + c_0 \left( B_1 w_2 + B_2 w_1^2 \right) \right) z^2 + \cdots, \tag{2.5}
\]

it follows from (2.3) that

\[
a_2 = B_1 c_0 w_1
\]

\[
a_3 = \frac{1}{2} \left( B_1 c_1 w_1 + B_1 c_0 w_2 + c_0 \left( B_2 + B_1^2 c_0 \right) w_1^2 \right). \tag{2.6}
\]

Since \( \varphi(z) \) is analytic and bounded in \( \mathbb{D} \), we have [27, page 172]

\[
|c_n| \leq 1 - |c_0|^2 \leq 1 \quad (n > 0). \tag{2.7}
\]

By using this fact and the well-known inequality, \( |w_1| \leq 1 \), we get

\[
|a_2| \leq B_1. \tag{2.8}
\]

Further,

\[
a_3 - \mu a_2^2 = \frac{1}{2} \left( B_1 c_1 w_1 + c_0 \left( B_1 w_2 + \left( B_2 + B_1^2 c_0 - 2 \mu B_1^2 c_0 \right) w_1^2 \right) \right). \tag{2.9}
\]

Then

\[
|a_3 - \mu a_2^2| \leq \frac{1}{2} \left( |B_1 c_1 w_1| + \left| B_1 c_0 \left( w_2 - \left( 2 \mu B_1 c_0 - B_1 c_0 - \frac{B_2}{B_1} \right) w_1^2 \right) \right| \right). \tag{2.10}
\]

Again applying \( |c_n| \leq 1 \) and \( |w_1| \leq 1 \), we have

\[
|a_3 - \mu a_2^2| \leq \frac{B_1}{2} \left( 1 + \left| w_2 - \left( (1 - 2 \mu) B_1 c_0 - \frac{B_2}{B_1} \right) w_1^2 \right| \right). \tag{2.11}
\]
Applying Lemma 1.11 to
\[ \left| w_2 - \left( (1 - 2\mu)B_1c_0 - \frac{B_2}{B_1} \right) w_1 \right| \] (2.12)
yields
\[ \left| a_3 - \mu a_2^2 \right| \leq \frac{B_1}{2} \left( 1 + \max \left\{ 1, \left| (1 - 2\mu)B_1c_0 - \frac{B_2}{B_1} \right| \right\} \right). \] (2.13)

Observe that
\[ \left| (1 - 2\mu)B_1c_0 - \frac{B_2}{B_1} \right| \leq B_1|c_0||1 - 2\mu| + \left\| \frac{B_2}{B_1} \right\|, \] (2.14)
and hence we can conclude that
\[ \left| a_3 - \mu a_2^2 \right| \leq \frac{B_1}{2} \left( 1 + \max \left\{ 1, B_1|1 - 2\mu| + \left\| \frac{B_2}{B_1} \right\| \right\} \right). \] (2.15)

For \( \mu = 0 \), the above will reduce to the estimate of \( |a_3| \).

**Remark 2.2.** For \( \varphi(z) \equiv 1 \), Theorem 2.1 gives a particular case of the estimates in [13, Theorem 1] for \( p = 1 \) and [14, Theorem 2.1] for \( k = 1 \).

**Theorem 2.3.** If \( f \in A \) satisfies
\[ \frac{zf'(z)}{f(z)} - 1 \ll \phi(z) - 1, \] (2.16)
then the following inequalities hold:
\[ |a_2| \leq B_1, \] \[ |a_3| \leq \frac{1}{2} \left( B_1 + B_1^2 + |B_2| \right), \] (2.17)
and, for any complex number \( \mu \),
\[ \left| a_3 - \mu a_2^2 \right| \leq \frac{1}{2} \left( B_1 + |1 - 2\mu|B_1^2 + |B_2| \right). \] (2.18)

**Proof.** The result follows by taking \( w(z) = z \) in the proof of Theorem 2.1.
Theorem 2.4. If \( f \in \mathcal{A} \) belongs to \( C_q(\phi) \), then

\[
|a_2| \leq \frac{B_1}{2},
\]

and

\[
|a_3| \leq \frac{1}{6} \left( B_1 + \max \left\{ B_1, B_1^2 + |B_2| \right\} \right),
\]

and, for any complex number \( \mu \),

\[
|a_3 - \mu a_2^2| \leq \frac{1}{6} \left( B_1 + \max \left\{ B_1, \left| B_2 \right| \right\} \right).
\]

Proof. Observe that when \( zf' \in S_q^{*} \), equality (2.3) becomes

\[
\frac{z(zf'(z))'}{zf'(z)} - 1 = \phi(z) (\phi(w(z)) - 1),
\]

or equally

\[
\frac{zf''(z)}{f'(z)} \prec \phi(w(z)) - 1,
\]

and the converse can be verified easily. By the Alexander relation, that is \( f \in C_q \) if and only if \( zf' \in S_q^{*} \) we can obtain the required estimates.

Theorem 2.5. If \( f \in \mathcal{A} \) satisfies

\[
\frac{zf''(z)}{f'(z)} \ll \phi(z) - 1,
\]

then the following inequalities hold:

\[
|a_2| \leq \frac{B_1}{2},
\]

\[
|a_3| \leq \frac{1}{6} \left( B_1 + B_1^2 + |B_2| \right),
\]

and, for any complex number \( \mu \),

\[
|a_3 - \mu a_2^2| \leq \frac{1}{6} \left( B_1 + \left| B_2 \right| \right).
\]
Theorem 2.6. If \( f \in \mathcal{A} \) belongs to \( \mathcal{R}_q(\phi) \), then

\[
|a_2| \leq \frac{B_1}{2}, \\
|a_3| \leq \frac{1}{3}(B_1 + \max \{B_1, |B_2|\}),
\]

and, for any complex number \( \mu \),

\[
|a_3 - \mu a_2^2| \leq \frac{1}{3} \left( B_1 + \max \left\{ B_1, \frac{3}{4} |\mu|B_1^2 + |B_2| \right\} \right).
\]

Proof. For \( f \in \mathcal{R}_q(\phi) \), we know that by Definition 1.5 there exist analytic functions \( \varphi \) and \( w \), with \( |\varphi(z)| \leq 1 \), \( w(0) = 0 \) and \( |w(z)| < 1 \) such that

\[
f'(z) - 1 = \varphi(z) (\phi(w(z)) - 1).
\]

Since

\[
f'(z) - 1 = 2a_2 z + 3a_3 z^2 + \cdots,
\]

it follows from (2.28) and (2.5) that

\[
a_2 = \frac{1}{2} B_1 c_0 w_1,
\]

\[
a_3 = \frac{1}{3} \left( B_1 c_1 w_1 + c_0 \left( B_1 w_2 + B_2 w_2^2 \right) \right).
\]

Following the same argument as in Theorem 2.1, where \( |c_0| \leq 1 \) and \( |c_1| \leq 1 \), we can deduce that

\[
|a_2| \leq \frac{B_1}{2},
\]

\[
|a_3 - \mu a_2^2| \leq \frac{B_1}{3} \left( 1 + \left| w_2 - \left( \frac{3B_1 c_0}{4} \mu - \frac{B_2}{B_1} \right) w_1^2 \right| \right).
\]

Applying Lemma 1.11, we get

\[
|a_3 - \mu a_2^2| \leq \frac{B_1}{3} \left( 1 + \max \left\{ 1, \left| \frac{3B_1 c_0}{4} \mu - \frac{B_2}{B_1} \right| \right\} \right).
\]
Since
\[
\left| \frac{3B_1c_0}{4} - \frac{B_2}{B_1} \right| \leq \frac{3B_1}{4} |\mu| |c_0| + \left| \frac{B_2}{B_1} \right|,
\]
(2.33)
and $|c_0| \leq 1$ we can conclude the hypothesis.

\[\Box\]

**Theorem 2.7.** If $f \in \mathcal{A}$ satisfies
\[
f'(z) - 1 \ll \phi(z) - 1,
\]
then the following inequalities hold:
\[
|a_2| \leq \frac{B_1}{2},
\]
(2.35)
\[
|a_3| \leq \frac{1}{3}(B_1 + |B_2|),
\]
and, for any complex number $\mu$,
\[
|a_3 - \mu a_2^2| \leq \frac{1}{3} \left( B_1 + \frac{3}{4} |\mu| B_1^2 + |B_2| \right).
\]
(2.36)

Let the class $\mathcal{R}_\rho^\phi(\phi)$ consist of functions $f \in \mathcal{A}$ satisfying the quasi-subordination
\[
\frac{1}{\rho} (f'(z) - 1) \ll \phi(z) - 1,
\]
(2.37)
where $\rho \in \mathbb{C} \setminus \{0\}$. The following corollary gives the results for $f \in \mathcal{R}_\rho^\phi(\phi)$.

**Corollary 2.8.** Let $\rho \in \mathbb{C} \setminus \{0\}$. If $f \in \mathcal{A}$ belongs to $\mathcal{R}_\rho^\phi(\phi)$, then
\[
|a_2| \leq \frac{\rho}{2} B_1,
\]
(2.38)
\[
|a_3| \leq \frac{\rho}{3} \left( B_1 + \max\{|B_1|, |B_2|\} \right),
\]
and, for any complex number $\mu$,
\[
|a_3 - \mu a_2^2| \leq \frac{\rho}{3} \left( B_1 + \max\left\{ B_1, \frac{3}{4} |\mu| B_1^2 + |B_2| \right\} \right).
\]
(2.39)
Remark 2.9. (1) For \( \varphi(z) \equiv 1 \), Corollary 2.8 gives a particular case of the estimates in [13, Theorem 3] for \( p = 1 \) and [14, Theorem 2.3] for \( k = 1 \).

(2) For \( \varphi(z) \equiv 1 \) and \( \phi(z) = (1 + Az)/(1 + Bz) \), \( (-1 \leq B < A \leq 1) \), Corollary 2.8 reduces to the results in [19, Theorem 4].

**Theorem 2.10.** Let \( \alpha \geq 0 \). If \( f \in \mathcal{A} \) belongs to \( \mathcal{M}_\psi(\alpha, \phi) \), then

\[
|a_2| \leq \frac{B_1}{1 + \alpha},
\]

\[
|a_3| \leq \frac{1}{2(1 + 2\alpha)} \left(B_1 + \max \left\{ B_1, \frac{1 + 3\alpha}{(1 + \alpha)^2} B_1^2 + |B_2| \right\}\right), \tag{2.40}
\]

and, for any complex number \( \mu \),

\[
|a_3 - \mu a_2^2| \leq \frac{1}{2(1 + 2\alpha)} \left(B_1 + \max \left\{ B_1, \frac{|2\mu(1 + 2\alpha) - (1 + 3\alpha)|}{(1 + \alpha)^2} B_1^2 + |B_2| \right\}\right). \tag{2.41}
\]

**Proof.** If \( f \in \mathcal{M}_\psi(\alpha, \phi) \), for \( \alpha \geq 0 \) then there are analytic functions \( \varphi \) and \( w \), with \( |\varphi(z)| \leq 1 \), \( w(0) = 0 \) and \( |w(z)| < 1 \) such that

\[
(1 - \alpha) \frac{z^f(z)}{f(z)} + \alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right) - 1 = \varphi(z)(\phi(w(z)) - 1). \tag{2.42}
\]

A computation shows that

\[
(1 - \alpha) \frac{z^f(z)}{f(z)} = (1 - \alpha) + (1 - \alpha)a_2z + (1 - \alpha)\left(-a_2^2 + 2a_3\right)z^2 + \cdots, \tag{2.43}
\]

\[
\alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right) = \alpha + 2\alpha a_2z + 2\alpha \left(-2a_2^2 + 3a_3\right)z^2 + \cdots.
\]

Hence from (2.43), we have

\[
(1 - \alpha) \frac{z^f(z)}{f(z)} + \alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right) - 1 = (1 + \alpha)a_2z + \left(-(1 + 3\alpha)a_2^2 + 2(1 + 2\alpha)a_3\right)z^2 + \cdots, \tag{2.44}
\]

It then follows from relation (2.42) and (2.5) that

\[
a_2 = \frac{B_1c_0w_1}{1 + \alpha},
\]

\[
a_3 = \frac{1}{2(1 + 2\alpha)} \left( B_1c_1w_1 + B_1c_0w_2 + \left( B_2c_0 + \frac{1 + 3\alpha}{(1 + \alpha)^2} B_1^2c_0^2 \right) w_1^2 \right). \tag{2.45}
\]

We can then conclude the proof by proceeding similarly as previous theorems. \( \square \)
Remark 2.11. (1) When \( \alpha = 0 \), Theorem 2.10 reduces to Theorem 2.1.
(2) When \( \alpha = 1 \), Theorem 2.10 reduces to Theorem 2.4.
(3) For \( \varphi(z) = 1 \), Theorem 2.10 gives a particular case of the estimates in [14, Theorem 2.9] for \( k = 1 \).

Theorem 2.12. Let \( \alpha \geq 0 \). If \( f \in \mathcal{A} \) satisfies

\[
(1 - \alpha) \frac{zf'(z)}{f(z)} + \alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right) - 1 \ll \varphi(z) - 1,
\]

then the following inequalities hold:

\[
|a_2| \leq \frac{B_1}{1 + \alpha},
\]

\[
|a_3| \leq \frac{1}{2(1 + 2\alpha)} \left( B_1 + \frac{1 + 3\alpha}{(1 + \alpha)^2} B_1^2 + |B_2| \right),
\]

and, for any complex number \( \mu \),

\[
|a_3 - \mu a_2^2| \leq \frac{1}{2(1 + 2\alpha)} \left( B_1 + \frac{2\mu(1 + 2\alpha) - (1 + 3\alpha)|\beta|}{(1 + \alpha)^2} B_1^2 + |B_2| \right).
\]

Theorem 2.13. Let \( \alpha \geq 0 \) and \( \beta = 1 - \alpha \). If \( f \in \mathcal{A} \) belongs to \( \mathcal{L}_4(\alpha, \varphi) \), then

\[
|a_2| \leq \frac{B_1}{|\alpha + 2\beta|},
\]

\[
|a_3| \leq \frac{1}{2|\alpha + 3\beta|} \left( B_1 + \max \left\{ B_1, \frac{|(\alpha + 2\beta)^2 - 3(\alpha + 4\beta)|}{2(\alpha + 2\beta)^2} B_1^2 + |B_2| \right\} \right),
\]

and, for any complex number \( \mu \),

\[
|a_3 - \mu a_2^2| \leq \frac{1}{2|\alpha + 3\beta|} \left( B_1 + \max \left\{ B_1, \frac{|(\alpha + 2\beta)^2 - 3(\alpha + 4\beta) - 4\mu(\alpha + 3\beta)|}{2(\alpha + 2\beta)^2} B_1^2 + |B_2| \right\} \right).
\]

Proof. If \( f \in \mathcal{L}_4(\alpha, \varphi) \), for \( \alpha \geq 0 \) and \( \beta = 1 - \alpha \) then there are analytic functions \( \varphi \) and \( \omega \), with \( |\varphi(z)| \leq 1 \), \( \varphi(0) = 0 \) and \( |\omega(z)| < 1 \) such that

\[
\left( \frac{zf'(z)}{f(z)} \right)^\alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right)^\beta - 1 = \varphi(z)(\varphi(\omega(z)) - 1).
\]
A computation shows that
\[
\left( \frac{zf'(z)}{f(z)} \right)^{\alpha} = 1 + \alpha a_2 z + \frac{1}{2} \left( \left( \alpha^2 - 3\alpha \right) a_2^2 + 4\alpha a_3 \right) z^2 + \cdots ,
\]
\[
\left( 1 + \frac{zf''(z)}{f'(z)} \right)^{\beta} = 1 + 2\beta a_2 z + \left( 2\left( \beta^2 - 3\beta \right) a_2^2 + 6\beta a_3 \right) z^2 + \cdots .
\]  
(2.52)

Thus (2.52) give
\[
\left( \frac{zf'(z)}{f(z)} \right)^{\alpha} \left( 1 + \frac{zf''(z)}{f'(z)} \right)^{\beta} - 1
\]
\[
= (\alpha + 2\beta) a_2 z + \frac{1}{2} \left( \left( \alpha + 2\beta \right)^2 - 3(\alpha + 4\beta) \right) a_2^2 + 4(\alpha + 3\beta) a_3 \right) z^2 + \cdots ,
\]
By using the above equation and (2.5) in (2.51) we have
\[
a_2 = \frac{B_1 c_0 w_1}{\alpha + 2\beta}
\]
\[
a_3 = \frac{B_1}{2(\alpha + 3\beta)} \left( B_1 - \frac{(\alpha + 2\beta)^2 - 3(\alpha + 4\beta)}{2(\alpha + 2\beta)^2} B_1^2 c_0^2 \right) w_1^2 .
\]
(2.54)

We can proceed similarly as previous theorems and proof the hypothesis. \( \square \)

**Remark 2.14.** (1) When \( \alpha = 0 \), Theorem 2.13 reduces to Theorem 2.4.
(2) When \( \alpha = 1 \), Theorem 2.13 reduces to Theorem 2.1.
(3) For \( \varphi(z) = 1 \), Theorem 2.13 gives a particular case of the estimates in [14, Theorem 2.7] for \( k = 1 \).

**Theorem 2.15.** Let \( \alpha \geq 0 \) and \( \beta = 1 - \alpha \). If \( f \in \mathcal{A} \) satisfies
\[
\frac{zf'(z)}{f(z)} \left( 1 + \frac{zf''(z)}{f'(z)} \right)^{1-\alpha} - 1 \ll \varphi(z) - 1,
\]
then the following inequalities hold:
\[
|a_2| \leq \frac{B_1}{|\alpha + 2\beta|},
\]
\[
|a_3| \leq \frac{1}{2|\alpha + 3\beta|} \left( B_1 + \frac{|(\alpha + 2\beta)^2 - 3(\alpha + 4\beta)|}{2(\alpha + 2\beta)^2} B_1^2 + |B_2| \right),
\]
(2.56)
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and, for any complex number $\mu$,

$$|a_3 - \mu a_2^2| \leq \frac{1}{2|a + 3\beta|} \left( B_1 + \frac{[(\alpha + 2\beta)^2 - 3(\alpha + 4\beta) - 4\mu(\alpha + 3\beta)]}{2(\alpha + 2\beta)^2} B_1^2 + |B_2| \right).$$  \hspace{1cm} (2.57)

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References


