Research Article

Common Fixed Point Results for Four Mappings on Partial Metric Spaces

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1. Introduction and Preliminaries

Partial metric spaces, introduced by Matthews [1, 2], are a generalization of the notion of the metric space in which in definition of metric, the condition \(d(x, x) = 0\) is replaced by the condition \(d(x, x) \leq d(x, y)\).

In [1], Matthews discussed some properties of convergence of sequence and proved the fixed point theorems for contractive mapping on partial metric spaces: any mapping \(T\) of a complete partial metric space \(X\) into itself that satisfies, where \(0 \leq k < 1\), the inequality \(d(Tx, Ty) \leq kd(x, y)\) for all \(x, y \in X\), has a unique fixed point. Recently, many authors (see [3–15]) have focused on this subject and generalized some fixed point theorems from the class of metric spaces.

The definition of partial metric space is given by Matthews (see [2]) as follows.

Definition 1.1. Let \(X\) be a nonempty set and let \(p : X \times X \to \mathbb{R}_0^+\) satisfy

\[(PM1) \quad x = y \iff p(x, x) = p(y, y) = p(x, y),\]

\[(PM2) \quad p(x, x) \leq p(x, y),\]
(PM3) $p(x, y) = p(y, x)$,
(PM4) $p(x, y) \leq p(x, z) + p(z, y) - p(z, z),$

for all $x, y$ and $z \in X$, where $\mathbb{R}_+^* = [0, \infty)$. Then the pair $(X, p)$ is called a partial metric space (in short PMS) and $p$ is called a partial metric on $X$.

Let $(X, p)$ be a PMS. Then, the functions $p^s, p^w : X \times X \to \mathbb{R}_0^+$ given by

\begin{align*}
p^s(x, y) &= 2p(x, y) - p(x, x) - p(y, y) \\
p^w(x, y) &= p(x, y) - \min\{p(x, x), p(y, y)\}
\end{align*}

are ordinary equivalent metrics on $X$. Each partial metric $p$ on $X$ generates a $T_0$ topology $\tau_p$ on $X$ with a base of the family of open $p$-balls $\{B_p(x, \varepsilon) : x \in X, \varepsilon > 0\}$, where $B_p(x, \varepsilon) = \{y \in X : p(x, y) < p(x, x) + \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

**Example 1.2** (see [1, 2]). Let $X = \{(a, b) : a, b \in \mathbb{R}, a \leq b\}$ and define

$$p((a, b), (c, d)) = \max\{b, d\} - \min\{a, c\}. \quad (1.2)$$

Then $(X, p)$ is a partial metric space.

We give same topological definitions on partial metric spaces.

**Definition 1.3** (see [1, 2, 4]).

(i) A sequence $\{x_n\}$ in a PMS $(X, p)$ converges to $x \in X$ if and only if $p(x, x) = \lim_{n \to \infty} p(x, x_n)$.

(ii) A sequence $\{x_n\}$ in a PMS $(X, p)$ is called a Cauchy sequence if and only if $\lim_{n,m \to \infty} p(x_n, x_m)$ exists (and finite).

(iii) A PMS $(X, p)$ is said to be complete if every Cauchy sequence $\{x_n\}$ in $X$ converges, with respect to $\tau_p$, to a point $x \in X$ such that $p(x, x) = \lim_{n,m \to \infty} p(x_n, x_m)$.

(iv) A mapping $f : X \to X$ is said to be continuous at $x_0 \in X$ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $f(B(x_0, \delta)) \subset B(f(x_0), \varepsilon)$.

**Lemma 1.4** (see [1, 2, 4]).

(A) A sequence $\{x_n\}$ is Cauchy in a PMS $(X, p)$ if and only if $\{x_n\}$ is Cauchy in a metric space $(X, p^s)$.

(B) A PMS $(X, p)$ is complete if and only if the metric space $(X, p^s)$ is complete. Moreover,

\begin{equation}
\lim_{n \to \infty} p^s(x, x_n) = 0 \iff p(x, x) = \lim_{n \to \infty} p(x, x_n) = \lim_{n,m \to \infty} p(x_n, x_m), \quad (1.3)
\end{equation}

where $x$ is a limit of $\{x_n\}$ in $(X, p^s)$.
Lemma 1.6 (see [10]). Assume \( x_n \to z \) as \( n \to \infty \) in a PMS \((X, p)\) such that \( p(z, z) = 0 \). Then \( \lim_{n \to \infty} p(x_n, y) = p(z, y) \) for every \( y \in X \).

On the other hand, Kannan [16] proved a fixed point theorem for a map satisfying a contractive condition that did not require continuity at each point. Afterward Sessa [17] introduced the notion of weakly commuting maps, which generalized the concept of commuting maps. Then Jungck generalized this idea, first to compatible mappings [18] and then to weakly compatible mappings [19].

A pair \((f, T)\) of self-mappings on \( X \) is said to be weakly compatible if they commute at their coincidence point (i.e., \( fT x = T f x \) whenever \( f x = T x \)). A point \( y \in X \) is called point of coincidence of a family \( T_j, j \in J \), of self-mappings on \( X \) if there exists a point \( x \in X \) such that \( y = T_j x \) for all \( j \in J \).

The concept of almost contraction property was given to as follows by Berinde.

Definition 1.7 (see [20, 21]). Let \((X, d)\) be a metric space. A map \( f : X \to X \) is called an almost contraction if there exist a constant \( \delta \in [0, 1] \) and some \( L \geq 0 \) such that for all \( x, y \in X \)

\[
d(f x, f y) \leq \delta d(x, y) + L d(f x, y). \tag{1.4}
\]

Berinde called this as “weak contraction” in [20], then he renamed it as “almost contraction” in [21, 22], also Berinde [21] proved some fixed point theorems for almost contraction in complete metric space. Definition 1.7 is a special case of the following definition (choose \( g = I_X, I_X \) is the identity map on \( X \)).

Definition 1.8 (see [7]). Let \((X, d)\) be a metric space. A map \( f : X \to X \) is called an almost contraction with respect to a mapping \( g : X \to X \) if there exist a constant \( \delta \in [0, 1] \) and some \( L \geq 0 \) such that for all \( x, y \in X \)

\[
d(f x, f y) \leq \delta d(g x, g y) + L d(f x, g y). \tag{1.5}
\]

Babu et al. [23] considered the class of mappings that satisfy “condition (B).”

Let \((X, d)\) be a metric space. A map \( T : X \to X \) is said to satisfy “condition (B)” if there exist a constant \( \delta \in [0, 1] \) and some \( L \geq 0 \) such that for all \( x, y \in X \),

\[
d(f x, f y) \leq \delta d(x, y) + L \min\{p(x, f x), p(y, f y), p(x, f y), p(y, f x)\}. \tag{1.6}
\]

Afterward, Berinde [21], Abbas and Ilić [24], and Ćirić et al. [7] generalized the above definition and proved some fixed point results.

In recent paper, Altun and Acar [25] introduced the notion of \((\delta, L)\) weak contraction in the sense of Berinde in partial metric space.
Definition 1.9 (see [25]). Let \((X, p)\) be a partial metric space. A map \(T : X \to X\) is called \((\delta, L)\)-weak contraction if there exist a \(\delta \in [0, 1)\) and some \(L \geq 0\) such that

\[
p(Tx, Ty) \leq \delta p(x, y) + Lp^w(y, Tx),
\]

for all \(x, y \in X\).

In this paper, we give a fixed point theorem for four mappings satisfying almost generalized contractive condition in [26] on partial metric spaces.

2. Main Results

Theorem 2.1. Let \((X, p)\) be a complete partial metric space and \(f, g, S\) and \(T\) be self maps on \(X\), with \(f(X) \subseteq T(X)\) and \(g(X) \subseteq S(X)\). If there exists \(\delta \in [0, 1]\) and \(L \geq 0\) with such that

\[
p(fx, gy) \leq \delta M(x, y) + LN(x, y),
\]

for any \(x, y \in X\), where,

\[
M(x, y) = \max \left\{ p(Sx, Ty), p(fx, Sx), p(gy, Ty), \frac{p(Sx, gy) + p(fx, Ty)}{2} \right\},
\]

\[
N(x, y) = \min \{ p^w(fx, Sx), p^w(gy, Ty), p^w(Sx, gy), p^w(fx, Ty) \}.
\]

If \(\{f, S\}\) and \(\{g, T\}\) are weakly compatible and one of \(f(X), g(X), S(X),\) and \(T(X)\) is a complete subspace of \(X\), then \(f, g, S,\) and \(T\) have a common fixed point.

Proof. Let \(x_0\) be an arbitrary point in \(X\). Since \(f(X) \subseteq T(X)\), we can find \(x_1 \in X\) such that \(fx_0 = Tx_1\) and also, as \(gx_1 \in S(X)\), there exist \(x_2 \in X\) such that \(gx_1 = Sx_2\). In general, \(x_{2n+1} \in X\) is chosen such that \(fx_{2n} = Tx_{2n+1}\) and \(x_{2n+2} \in X\) such that \(gx_{2n+1} = Sx_{2n+2}\), we obtain a sequences \(\{y_n\}\) in \(X\) such that

\[
y_{2n} = fx_{2n} = Tx_{2n+1}, \quad y_{2n+1} = gx_{2n+1} = Sx_{2n+2}, \quad \forall n \geq 0.
\]

Suppose \(y_{2m} = y_{2m+1}\) for some \(m\). Thus, \(g\) and \(T\) have a coincidence point. Due to (2.1), we have

\[
p(y_{2m+2}, y_{2m+1}) = p(fx_{2m+2}, gx_{2m+1}) \leq \delta M(x_{2m+2}, x_{2m+1}) + LN(x_{2m+2}, x_{2m+1}),
\]

for all \(m \geq 0\).
where

\[ N(x_{2m+2}, x_{2m+1}) = \min \left\{ p^w(x_{2m+2}, Sx_{2m+2}), p^w(Sx_{2m+2}, Tx_{2m+1}), \right\} \]

\[ = \min \left\{ p^w(y_{2m+2}, y_{2m+1}), p^w(y_{2m+1}, y_{2m}), \right\} \]

\[ = 0, \]

\[ M(x_{2m+2}, x_{2m+1}) = \max \left\{ p(Sx_{2m+2}, Tx_{2m+1}), p(f x_{2m+2}, Sx_{2m+2}), \right\} \]

\[ = \max \left\{ p(y_{2m+1}, y_{2m}), p(y_{2m+1}, y_{2m}), \right\} \]

\[ = p(y_{2m+1}, y_{2m}). \]

So,

\[ p(y_{2m+1}, y_{2m}) \leq \delta p(y_{2m+2}, y_{2m+1}). \] (2.6)

Therefore, by \( \delta \in [0,1] \), we have \( p(y_{2m+2}, y_{2m+1}) = 0 \), that is, \( y_{2m+1} = y_{2m+2} \). So, \( f \) and \( S \) have a coincidence point.

Suppose now that \( y_n \neq y_{n+1} \) for all \( n \geq 0 \). From (2.1), we obtain

\[ p(y_{2n}, y_{2n+1}) = p(f x_{2n}, Sx_{2n+1}) \leq \delta M(x_{2n}, x_{2n+1}) + LN(x_{2n}, x_{2n+1}), \] (2.7)

where

\[ N(x_{2n}, x_{2n+1}) = \min \left\{ p^w(x_{2n}, Sx_{2n}), p^w(Sx_{2n}, Tx_{2n+1}), \right\} \]

\[ = \min \left\{ p^w(y_{2n}, y_{2n-1}), p^w(y_{2n-1}, y_{2n}), \right\} \]

\[ = 0, \]

\[ M(x_{2n}, x_{2n+1}) = \max \left\{ p(Sx_{2n}, Tx_{2n+1}), p(f x_{2n}, Sx_{2n}), \right\} \]

\[ = \max \left\{ p(y_{2n-1}, y_{2n}), p(y_{2n}, y_{2n-1}), \right\} \]

\[ = p(y_{2n+1}, y_{2n}). \] (2.8)
Due to (2.7), we have

\[ p(y_{2n}, y_{2n+1}) \leq \delta M(x_{2n}, x_{2n+1}). \]  \hfill (2.9)

Due to PM4, we have

\[ p(y_{2n-1}, y_{2n+1}) + p(y_{2n}, y_{2n}) \leq p(y_{2n-1}, y_{2n}) + p(y_{2n}, y_{2n+1}). \]  \hfill (2.10)

Hence, \( M(x_{2n}, x_{2n+1}) = \max \{p(y_{2n}, y_{2n-1}), p(y_{2n+1}, y_{2n})\}. \) If \( M(x_{2n}, x_{2n+1}) = p(y_{2n+1}, y_{2n}) \), then by (2.7)

\[ p(y_{2n+1}, y_{2n}) \leq \delta p(y_{2n+1}, y_{2n}). \]  \hfill (2.11)

Since \( \delta \in [0, 1] \), the inequality (2.9) yields a contradiction. Hence, \( M(x_{2n}, x_{2n+1}) = p(y_{2n}, y_{2n-1}) \), then by (2.7) we have

\[ p(y_{2n+1}, y_{2n}) \leq \delta p(y_{2n}, y_{2n-1}). \]  \hfill (2.12)

Thus, one can observe that

\[ p(y_{n+1}, y_n) \leq \delta^n p(y_0, y_1), \quad \forall n = 0, 1, 2, \ldots \]  \hfill (2.13)

Consider now

\[ p^*(y_{n+2}, y_{n+1}) = 2p(y_{n+2}, y_{n+1}) - p(y_{n+2}, y_{n+2}) - p(y_{n+1}, y_{n+1}) \]
\[ \leq 2p(y_{n+2}, y_{n+1}) \]
\[ \leq \delta^{n+1} p(y_0, y_1). \]  \hfill (2.14)

Hence, regarding (2.13), we have

\[ \lim_{n \to \infty} p^*(y_{n+2}, y_{n+1}) = 0. \]  \hfill (2.15)

Moreover,

\[ p^*(y_{n+1}, y_{n+k}) \leq p^*(y_{n+k-1}, y_{n+k}) + \cdots + p^*(y_{n+1}, y_{n+2}) \]
\[ \leq 2\delta^{n+k-1} p(y_0, y_1) + \cdots + 2\delta^{n+1} p(y_0, y_1). \]  \hfill (2.16)

After standard calculation, we obtain that \( \{y_n\} \) is a Cauchy sequence in \( (X, p^*) \), that is, \( p^*(y_n, y_m) \to 0 \) as \( n, m \to \infty \). Since \( (X, p) \) is complete, by Lemma 1.4, \( (X, p^*) \) is complete and sequence \( \{y_n\} \) is convergent in \( (X, p^*) \) to say \( z \in X \). From Lemma 1.4,

\[ p(z, z) = \lim_{n \to \infty} p(y_n, z) = \lim_{n, m \to \infty} p(y_n, y_m). \]  \hfill (2.17)
Since \( \{y_n\} \) is a Cauchy sequence in \((X,p^*\rangle\), we have
\[
\lim_{n,m \to \infty} p^*(y_n, y_m) = 0.
\] (2.18)

We assert that \( \lim_{n,m \to \infty} p(y_n, y_m) = 0 \). Without loss of generality, we assume that \( n > m \),
\[
p(y_{n+2}, y_n) \leq p(y_{n+2}, y_{n+1}) + p(y_{n+1}, y_n) - p(y_{n+1}, y_{n+1})
\leq p(y_{n+2}, y_{n+1}) + p(y_{n+1}, y_n).
\] (2.19)

Similarly,
\[
p(y_{n+3}, y_n) \leq p(y_{n+3}, y_{n+2}) + p(y_{n+2}, y_n) - p(y_{n+2}, y_{n+2})
\leq p(y_{n+3}, y_{n+2}) + p(y_{n+2}, y_n).
\] (2.20)

Taking into account (2.20), the expression (2.19) yields
\[
p(y_{n+3}, y_n) \leq p(y_{n+3}, y_{n+2}) + p(y_{n+2}, y_{n+1}) + p(y_{n+1}, y_n).
\] (2.21)

Inductively, we obtain
\[
p(y_m, y_n) \leq p(y_m, y_{m+1}) + \cdots + p(y_{n-2}, y_{n-1}) + p(y_{n-1}, y_n).
\] (2.22)

Due to (2.13),
\[
p(y_m, y_n) \leq \delta^m p(y_0, y_1) + \cdots + \delta^{n-2} p(y_0, y_1) + \delta^{n-1} p(y_0, y_1)
\leq \delta^m \left( 1 + \delta + \cdots + \delta^{n-m-1} \right) p(y_0, y_1).
\] (2.23)

Regarding \( \delta \in [0, 1[ \), we can observe that \( \lim_{n,m \to \infty} p(y_n, y_m) = 0 \).
Since \( y_n \to z \) in \( X \), \( \{fx_{2n}\}, \{Tx_{2n+1}\}, \{gx_{2n+1}\}, \{Sx_{2n+2}\} \) converge to \( z \).

Now we show that \( z \) is the fixed point for maps \( g \) and \( T \). Assume that \( T(X) \) is complete, there exists \( u \in X \) such that \( z = Tu \). We will show that \( gu = z \). On the contrary, assume that \( gu \neq z \).

From (2.1) we have
\[
p(fx_{2n}, gu) \leq \delta M(x_{2n}, u) + LN(x_{2n}, u),
\] (2.24)
where

\[
N(x_{2n}, u) = \min \{ p^u(f_{x_{2n}}, Sx_{2n}), p^u(gu, Tu), p^u(Sx_{2n}, gu), p^u(f_{x_{2n}}, Tu) \}
\]

\[
= \min \{ p^u(f_{x_{2n}}, Sx_{2n}), p^u(gu, z), p^u(Sx_{2n}, gu), p^u(f_{x_{2n}}, z) \},
\]

\[
M(x_{2n}, u) = \max \left\{ \frac{p(Sx_{2n}, Tu), p(f_{x_{2n}}, Sx_{2n}), p(gu, Tu),}{p(Sx_{2n}, gu) + p(f_{x_{2n}}, Tu)} \right\}
\]

\[
= \max \left\{ \frac{p(Sx_{2n}, z), p(f_{x_{2n}}, Sx_{2n}), p(gu, z),}{p(Sx_{2n}, gu) + p(f_{x_{2n}}, z)} \right\}.
\] (2.25)

Since \( \lim_{n \to \infty} M(x_{2n}, u) = p(gu, z) \) and \( \lim_{n \to \infty} N(x_{2n}, u) = 0 \). We get

\[
p(z, gu) \leq \delta p(gu, z).
\] (2.26)

Since \( \delta \in [0, 1] \), we get \( p(z, gu) = 0 \). Therefore, \( gu = Tu = z \). Since the maps \( g \) and \( T \) are weakly compatible, we have \( gz = gTu = Tgu = Tz \). We will also show that \( gz = z \). From (2.1), we have

\[
p(f_{x_{2n}}, gz) \leq \delta M(x_{2n}, z) + LN(x_{2n}, z),
\] (2.27)

where

\[
N(x_{2n}, z) = \min \{ p^u(f_{x_{2n}}, Sx_{2n}), p^u(gz, Tz), p^u(Sx_{2n}, gz), p^u(f_{x_{2n}}, Tz) \},
\]

\[
M(x_{2n}, z) = \max \left\{ \frac{p(Sx_{2n}, Tz), p(f_{x_{2n}}, Sx_{2n}),}{p(gz, Tz), p(Sx_{2n}, gz) + p(f_{x_{2n}}, Tz)} \right\}
\]

\[
= \max \left\{ \frac{p(Sx_{2n}, gz), p(f_{x_{2n}}, Sx_{2n}),}{p(gz, gz), p(Sx_{2n}, gz) + p(f_{x_{2n}}, Tz)} \right\}.
\] (2.28)

Since \( \lim_{n \to \infty} M(x_{2n}, z) = p(z, gz) \) and \( \lim_{n \to \infty} N(x_{2n}, z) = 0 \), then

\[
p(z, gz) = \lim_{n \to \infty} p(f_{x_{2n}}, gz) \leq \delta p(z, gz).
\] (2.29)

Since \( \delta \in [0, 1] \), \( p(z, gz) = 0 \). By Remark 1.5, we get \( z = gz \).

Similarly, we show that \( z \) is also fixed point of \( f \) and \( S \). Hence, \( fz = gz = Tz = Sz = z \).

The proofs for the cases in which \( S(X), f(X), \) or \( g(X) \) is complete are similar.

Last, we show \( z \) is unique. Suppose on the contrary that there is another common fixed point \( t \) of \( f, g, S, \) and \( T \). Then

\[
p(z, t) = p(fz, gt) \leq \delta M(z, t) + LN(z, t),
\] (2.30)
where
\[
N(z,t) = \min \{ p^w(fz,Sz), p^w(gt,Tt), p^w(Sz,gt), p^w(fz,Tt) \} = 0,
\]
\[
M(z,t) = \max \left\{ \frac{p(Sz,Tt), p(fz,Sz), p(gt,Tt)}{p(Sz,gt) + p(fz,Tt)} \right\}
\]
\begin{equation}
(2.31)
\end{equation}
\[
= p(Sz,Tt)
\]
\[
= p(z,t).
\]
Thus,
\[
p(z,t) \leq \delta p(z,t).
\]
\begin{equation}
(2.32)
\end{equation}

Therefore, \( p(z,t) = 0 \) and Remark 1.5 \( z = t \). So, \( z \) is the unique common fixed point of \( f, g, S, \) and \( T \).

Example 2.2. Let \( X = \{0,1,2\} \) endowed with the partial metric \( p \) given by \( p(x,y) = \max \{x,y\} \) for all \( x,y \in X \). It is clear that \((X,p)\) is a complete partial metric space. Define the mappings \( f, g, S, T : X \to X \) by
\[
f = g, \quad S = T,
\]
\[
f0 = f2 = 0, \quad f1 = 1
\]
\[
T0 = 0, \quad T1 = 2, \quad T2 = 1.
\]

We have \( f(X) \subseteq T(X) = X \). For \( \delta = 1/2, L = 1, \)
\[
p(f0,f1) = 1 \leq \delta.2 + L.1,
\]
\[
p(f2,f1) = 1 \leq \delta.2 + L.0,
\]
\[
p(f2,f2) = p(f0,f0) = 0 \leq \delta.0 + L.1,
\]
\[
p(f1,f1) = 1 \leq \delta.2 + L.0.
\]

Then, the contractive condition (2.1) is satisfied for every \( x, y \in X \). Moreover, \( \{f,T\} \) is weakly compatible. So all conditions of Theorem 2.1 are satisfied. We deduce the existence and uniqueness of a common fixed point of \( f \) and \( T \). Here, \( 0 \) is the unique common fixed point.

Corollary 2.3. Let \((X,p)\) is complete PMS and \( f \) and \( T \) be self maps on \( X \), with \( f(X) \subseteq T(X) \). If there exists \( \delta \in [0,1[ \) and \( L \geq 0 \) such that
\[
p(fx, fy) \leq \delta M(x,y) + LN(x,y),
\]
\begin{equation}
(2.35)
\end{equation}
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where,

\[
M(x, y) = \max \left\{ p(Tx, Ty), p(fx, Tx), p(fy, Ty), \frac{p(Tx, fy) + p(fx, Ty)}{2} \right\},
\]

\[
N(x, y) = \min \left\{ p^\alpha(fx, Tx), p^\alpha(fy, Ty), p^\alpha(Tx, fy), p^\alpha(x, Ty) \right\},
\]

for every \( x, y \in X \). If \( \{f, T\} \) is weakly compatible and one of \( f(X) \) and \( T(X) \) is a complete subspace of \( X \), then \( f \) and \( T \) have a common fixed point.

**Remark 2.4.** It is easy to see that for every map \( T : X \rightarrow X \), \( \{T, I_X\} \) is weakly compatible, where \( I_X \) is identity map on \( X \), so by taking \( f = g = I_X \) in Theorem 2.1 we have the following results.

**Corollary 2.5.** Let \( (X, p) \) is complete PMS and \( S \) and \( T \) be self maps on \( X \). If there exists \( \delta \in [0, 1[ \) and \( L \geq 0 \) such that

\[
p(x, y) \leq \delta M(x, y) + LN(x, y),
\]

for every \( x, y \in X \), where

\[
M(x, y) = \max \left\{ p(Sx, Ty), p(x, Sx), p(y, Ty), \frac{p(Sx, y) + p(x, Ty)}{2} \right\},
\]

\[
N(x, y) = \min \left\{ p^\alpha(x, Sx), p^\alpha(y, Ty), p^\alpha(Sx, y), p^\alpha(x, Ty) \right\}.
\]

Then \( S \) and \( T \) have a common fixed point.

**References**


